

## Chapter 5

# Cycle stability

We owe it to a book to withhold judgment until we reach page 100.

—Henrietta McNutt, George Johnson’s seventh-grade English teacher

**T**HOPOLOGICAL FEATURES of a dynamical system –singularities, periodic orbits, and the ways in which the orbits intertwine– are invariant under a general continuous change of coordinates. Equilibria and periodic orbits are *flow-invariant* sets, in the sense that the flow only shifts points along a periodic orbit, but the periodic orbit as the set of periodic points remains unchanged in time. Surprisingly, there also exist quantities that depend on the notion of metric distance between points, but nevertheless do not change value under a smooth change of coordinates. Local quantities such as the eigenvalues of equilibria and periodic orbits, and global quantities such as Lyapunov exponents, metric entropy, and fractal dimensions are examples of properties of dynamical systems independent of coordinate choice.

We now turn to the first, local class of such invariants, linear stability of equilibria and periodic orbits of flows and maps. This will give us metric information about local dynamics, as well as the key concept, the concept of a *neighborhood* of a point  $x$ : its size is primarily determined by the number of expanding directions, and the rates of expansion along them: contracting directions play only a secondary role (see sect. 5.6).

If you already know that the eigenvalues of periodic orbits are invariants of a flow, skip this chapter.



fast track:  
chapter 8, p. 135

As noted on page 41, a trajectory can be stationary, periodic or aperiodic. For chaotic systems almost all trajectories are aperiodic–nevertheless, equilibria and

periodic orbits turn out to be the key to unraveling chaotic dynamics. Here we note a few of the properties that make them so precious to a theorist.

## 5.1 Equilibria

At the still point, there the dance is.

—T. S. Eliot, *Four Quartets - Burnt Norton* [00:15:30]



For a start, consider the case where  $x_q$  is an equilibrium point (2.8). Expanding around the equilibrium point  $x_q$ , using the fact that the stability matrix  $A = A(x_q)$  in (4.2) is constant, and integrating,  $f^t(x) = x_q + e^{At}(x - x_q) + \dots$ , we verify that the simple formula (4.15) applies also to the Jacobian matrix of an equilibrium point,

$$J_q^t = e^{A_q t}, \quad J_q^t = J^t(x_q), \quad A_q = A(x_q). \quad (5.1)$$

As an equilibrium point is stationary, time plays no role. The eigenvalues and the eigenvectors of the stability matrix  $A_q$  evaluated at the equilibrium point  $x_q$ ,

$$A_q \mathbf{e}^{(j)} = \lambda_q^{(j)} \mathbf{e}^{(j)}, \quad (5.2)$$

describe the linearized neighborhood of the equilibrium point, with stability exponents  $\lambda_q^{(j)} = \mu_q^{(j)} \pm i\omega_q^{(j)}$  independent of any particular coordinate choice. Assume that these eigenvalues are non-degenerate,  $\lambda^{(j)} \neq \lambda^{(k)}$  for any pair of eigenvalues.

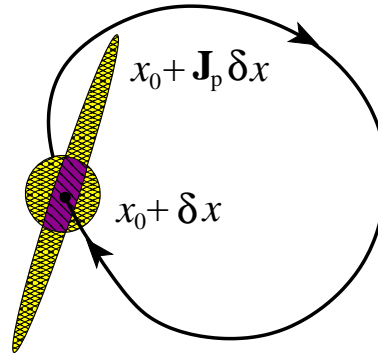
- If all  $\mu^{(j)} < 0$ , then the equilibrium is *stable*, or a *sink*. For  $\omega^{(j)} = 0$ , it is an *in node*; for  $\omega^{(j)} \neq 0$ , it is an *in spiral*.
- If some  $\mu^{(j)} < 0$ , and other  $\mu^{(j)} > 0$ , the equilibrium is *hyperbolic*, or a *saddle*.
- If all  $\mu^{(j)} > 0$ , then the equilibrium is *repelling*, or a *source*. For  $\omega^{(j)} = 0$ , it is an *out node*; for  $\omega^{(j)} \neq 0$ , it is an *out spiral*.
- If some  $\mu^{(j)} = 0$ , think again (you have a symmetry or a bifurcation).

For 2-dimensional flows, these types of equilibrium stabilities are illustrated in figures 4.4 and 4.3. The stability matrix eigenvectors (5.2) are the eigenvectors of the Jacobian matrix as well,  $J_q^t \mathbf{e}^{(j)} = \exp(t\lambda_q^{(j)}) \mathbf{e}^{(j)}$ .

## 5.2 Periodic orbits

An obvious virtue of periodic orbits is that they are *topological* invariants: a fixed point remains a fixed point for any choice of coordinates, and similarly a

**Figure 5.1:** For a prime cycle  $p$ , Floquet matrix  $J_p$  returns an infinitesimal spherical neighborhood of  $x_0 \in \mathcal{M}_p$  stretched into an ellipsoid, with overlap ratio along the eigendirection  $e^{(j)}$  of  $J_p(x)$  given by the Floquet multiplier  $|\Lambda_j|$ . These ratios are invariant under smooth nonlinear reparametrizations of state space coordinates, and are intrinsic property of cycle  $p$ .



periodic orbit remains periodic in any representation of the dynamics. Any reparametrization of a dynamical system that preserves its topology has to preserve topological relations between periodic orbits, such as their relative inter-windings and knots. So the mere existence of periodic orbits suffices to partially organize the spatial layout of a non-wandering set. No less important, as we shall now show, is the fact that cycle eigenvalues are *metric* invariants: they determine the relative sizes of neighborhoods in a non-wandering set.

We start by noting that due to the multiplicative structure (4.20) of Jacobian matrices, the Jacobian matrix for the  $r$ th repeat of a prime cycle  $p$  of period  $T$  is

$$J^{rT}(x) = J^T(f^{(r-1)T}(x)) \cdots J^T(f^T(x))J^T(x) = J_p(x)^r, \quad (5.3)$$

where  $J_p(x) = J^T(x)$  is the Jacobian matrix for a single traversal of the prime cycle  $p$ ,  $x \in \mathcal{M}_p$  is any point on the cycle, and  $f^{rT}(x) = x$  as  $f^T(x)$  returns to  $x$  every multiple of the period  $T$ . Hence, it suffices to restrict our considerations to the stability of prime cycles.



fast track:  
sect. 5.3, p. 107

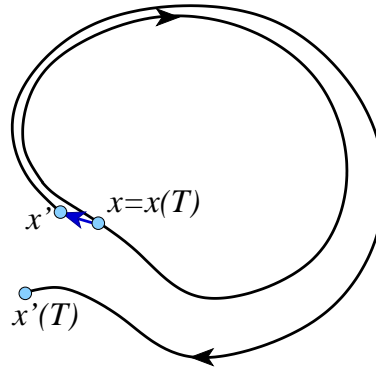
### 5.2.1 Cycle stability

The time-dependent  $T$ -periodic vector fields, such as the flow linearized around a periodic orbit, are described by Floquet theory. Hence from now on we shall refer to a Jacobian matrix evaluated on a periodic orbit  $p$  either as a  $[d \times d]$  Floquet matrix  $J_p$  or a  $[(d-1) \times (d-1)]$  monodromy matrix  $M_p$ , to its eigenvalues  $\Lambda_j$  as Floquet multipliers (4.7), and to  $\lambda_p^{(j)} = \mu_p^{(j)} + i\omega_p^{(j)}$  as Floquet exponents. The stretching/contraction rates per unit time are given by the real parts of Floquet exponents

appendix A4.2.1

$$\mu_p^{(j)} = \frac{1}{T_p} \ln |\Lambda_{p,j}|. \quad (5.4)$$

The factor  $1/T_p$  in the definition of the Floquet exponents is motivated by its form for the linear dynamical systems, for example (4.32). (Parenthetically, a Floquet



**Figure 5.2:** An unstable periodic orbit repels every neighboring trajectory  $x'(t)$ , except those on its center and stable manifolds.

exponent is not a Lyapunov exponent (6.11) evaluated on one period the prime cycle  $p$ ; read chapter 6). When  $\Lambda_j$  is real, we do care about  $\sigma^{(j)} = \Lambda_j/|\Lambda_j| \in \{+1, -1\}$ , the sign of the  $j$ th Floquet multiplier. If  $\sigma^{(j)} = -1$  and  $|\Lambda_j| \neq 1$ , the corresponding eigen-direction is said to be *inverse hyperbolic*. Keeping track of this by case-by-case enumeration is an unnecessary nuisance, so most of our formulas will be stated in terms of the Floquet multipliers  $\Lambda_j$  rather than in the terms of the multiplier signs  $\sigma^{(j)}$ , exponents  $\mu^{(j)}$  and phases  $\omega^{(j)}$ . section 8.3

In dynamics the expanding directions,  $|\Lambda_e| > 1$ , have to be taken care of first, while the contracting directions  $|\Lambda_c| < 1$  tend to take care of themselves, hence we always order multipliers  $\Lambda_k$  in order of decreasing magnitude  $|\Lambda_1| \geq |\Lambda_2| \geq \dots \geq |\Lambda_d|$ . Since  $|\Lambda_j| = e^{t\mu^{(j)}}$ , this is the same as ordering by  $\mu^{(1)} \geq \mu^{(2)} \geq \dots \geq \mu^{(d)}$ . We sort the Floquet multipliers  $\{\Lambda_{p,1}, \Lambda_{p,2}, \dots, \Lambda_{p,d}\}$  of the Floquet matrix evaluated on the  $p$ -cycle into three sets  $\{e, m, c\}$

$$\begin{aligned}
 \text{expanding:} \quad \{\Lambda\}_e &= \{\Lambda_{p,j} : |\Lambda_{p,j}| > 1\} \\
 &\{\lambda\}_e = \{\lambda_p^{(j)} : \mu_p^{(j)} > 0\} \\
 \text{marginal:} \quad \{\Lambda\}_m &= \{\Lambda_{p,j} : |\Lambda_{p,j}| = 1\} \\
 &\{\lambda\}_m = \{\lambda_p^{(j)} : \mu_p^{(j)} = 0\} \\
 \text{contracting:} \quad \{\Lambda\}_c &= \{\Lambda_{p,j} : |\Lambda_{p,j}| < 1\} \\
 &\{\lambda\}_c = \{\lambda_p^{(j)} : \mu_p^{(j)} < 0\}.
 \end{aligned} \tag{5.5}$$

In what follows, the volume of expanding manifold will play an important role. We denote by  $\Lambda_p$  (no  $j$ th eigenvalue index) the product of *expanding* Floquet multipliers

$$\Lambda_p = \prod_e \Lambda_{p,e}. \tag{5.6}$$

As  $J_p$  is a real matrix, complex eigenvalues always come in complex conjugate pairs,  $\Lambda_{p,i+1} = \Lambda_{p,i}^*$ , so the product (5.6) is always real.

A periodic orbit of a continuous-time flow, or of a map, or a fixed point of a map is

- *stable*, a *sink* or a *limit cycle* if all  $|\Lambda_j| < 1$  (real parts of all of its Floquet exponents, other than the vanishing longitudinal exponent for perturbations tangent to the cycle, see sect. 5.3.1, are strictly negative,  $0 > \mu^{(1)} \geq \mu^{(j)}$ ).
- *hyperbolic* or a *saddle*, unstable to perturbations outside its stable manifold if some  $|\Lambda_j| > 1$ , and other  $|\Lambda_j| < 1$  (a set of  $\mu^{(j)} \geq \mu_{min} > 0$  is strictly positive, the rest is strictly negative).
- *elliptic*, *neutral* or *marginal* if all  $|\Lambda_j| = 1$  ( $\mu^{(j)} = 0$ ).
- *partially hyperbolic*, if  $\mu^{(j)} = 0$  for a subset of exponents (other than the longitudinal one).
- *repelling*, or a *source*, unstable to any perturbation if *all*  $|\Lambda_j| > 1$  (all Floquet exponents, other than the vanishing longitudinal exponent, are strictly positive,  $\mu^{(j)} \geq \mu^{(d)} > 0$ ).

The region of system parameter values for which a periodic orbit  $p$  is stable is called the *stability window* of  $p$ . The set of initial points that are asymptotically attracted to  $M_p$  as  $t \rightarrow +\infty$  (for a fixed set of system parameter values) is called the *basin of attraction* of limit cycle  $p$ . Repelling and hyperbolic cycles are unstable to generic perturbations, and thus said to be *unstable*, see figure 5.2.

section 8.4

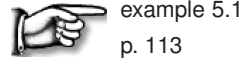
If *all* Floquet exponents (other than the vanishing longitudinal exponent) of *all* periodic orbits of a flow are strictly bounded away from zero, the flow is said to be *hyperbolic*. Otherwise the flow is said to be *nonhyperbolic*. A confined smooth flow or map is generically nonhyperbolic, with partial ellipticity or marginality expected only in presence of continuous symmetries, or for bifurcation parameter values. As we shall see in chapter 12, in presence of continuous symmetries equilibria and periodic orbits are not likely solutions, and their role is played by higher-dimensional tori, relative equilibria and relative periodic orbits. For Hamiltonian flows the symplectic  $\text{Sp}(d)$  symmetry (Liouville phase-space volume conservation, Poincaré invariants) leads to a proliferation of elliptic and partially hyperbolic tori.

section 8.5

Henriette Roux: In my 61,506-dimensional computation of a Navier-Stokes equilibrium I generated about 30 eigenvectors before I wanted to move on. How many of these eigenvectors are worth generating for a particular solution and why?

chapter 30

A: A rule of the thumb is that you need all equilibrium eigenvalues / periodic orbit Floquet exponents with positive real parts, and at least those negative exponents whose magnitude is less or comparable to the largest expanding eigenvalue. More precisely; keep adding the next least contracting eigenvalue to the sum of the preceding ones as long as the sum is positive (Kaplan-Yorke criterion). Then, just to be conservative, double the number of eigenvalues you keep. You do not need to worry about the remaining (60 thousand!) eigen-directions for which the negative eigenvalues are of larger magnitude, as they always contract: nonlinear terms cannot mix them up in such a way that expansion in some directions overwhelms the strongly contracting ones.



example 5.1  
p. 113

### 5.3 Floquet multipliers are invariant



The 1-dimensional map Floquet multiplier (5.21) is a product of derivatives over all points around the cycle, and is therefore independent of which periodic point is chosen as the initial one. In higher dimensions the form of the Floquet matrix  $J_p(x_0)$  in (5.3) does depend on the choice of coordinates and the initial point  $x_0 \in \mathcal{M}_p$ . Nevertheless, as we shall now show, the cycle *Floquet multipliers* are intrinsic property of a cycle in any dimension. Consider the  $i$ th eigenvalue, eigenvector pair  $(\Lambda_j, \mathbf{e}^{(j)})$  computed from  $J_p$  evaluated at a periodic point  $x$ ,

$$J_p(x) \mathbf{e}^{(j)}(x) = \Lambda_j \mathbf{e}^{(j)}(x), \quad x \in \mathcal{M}_p. \quad (5.7)$$

Consider another point on the cycle at time  $t$  later,  $x' = f^t(x)$  whose Floquet matrix is  $J_p(x')$ . By the semigroup property (4.20),  $J^{T+t} = J^{t+T}$ , and the Jacobian matrix at  $x'$  can be written either as

$$J^{T+t}(x) = J^T(x') J^t(x) = J_p(x') J^t(x),$$

or  $J^t(x) J_p(x)$ . Multiplying (5.7) by  $J^t(x)$ , we find that the Floquet matrix evaluated at  $x'$  has the same Floquet multiplier,

$$J_p(x') \mathbf{e}^{(j)}(x') = \Lambda_j \mathbf{e}^{(j)}(x'), \quad \mathbf{e}^{(j)}(x') = J^t(x) \mathbf{e}^{(j)}(x), \quad (5.8)$$

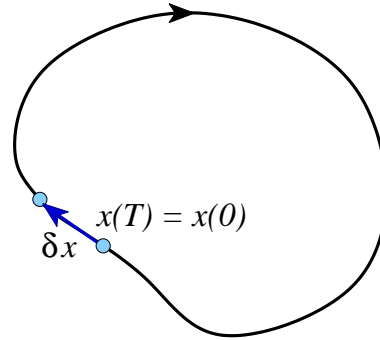
but with the eigenvector  $\mathbf{e}^{(j)}$  transported along the flow  $x \rightarrow x'$  to  $\mathbf{e}^{(j)}(x') = J^t(x) \mathbf{e}^{(j)}(x)$ . Hence, in the spirit of the Floquet theory (appendix A4.2.1) one can define time-periodic eigenvectors (in a co-moving ‘Lagrangian frame’)

$$\mathbf{e}^{(j)}(t) = e^{-\lambda^{(j)}t} J^t(x) \mathbf{e}^{(j)}(0), \quad \mathbf{e}^{(j)}(t) = \mathbf{e}^{(j)}(x(t)), \quad x(t) \in \mathcal{M}_p. \quad (5.9)$$

$J_p$  evaluated anywhere along the cycle has the same set of Floquet multipliers  $\{\Lambda_1, \Lambda_2, \dots, 1, \dots, \Lambda_{d-1}\}$ . As quantities such as  $\text{tr } J_p(x)$ ,  $\det J_p(x)$  depend only on the eigenvalues of  $J_p(x)$  and not on the starting point  $x$ , in expressions such as  $\det(\mathbf{1} - J_p^r(x))$  we may omit reference to  $x$ ,

$$\det(\mathbf{1} - J_p^r) = \det(\mathbf{1} - J_p^r(x)) \quad \text{for any } x \in \mathcal{M}_p. \quad (5.10)$$

We postpone the proof that the cycle Floquet multipliers are smooth conjugacy invariants of the flow to sect. 5.4; time-forward map (5.8) is the special case of this general property of smooth manifolds and their tangent spaces.



**Figure 5.3:** Any two points along a periodic orbit  $p$  are mapped into themselves after one cycle period  $T$ , hence a longitudinal deviation vector  $\delta x = v(x_0)\delta t$  is mapped into itself by the cycle Jacobian matrix  $J_p$ .

### 5.3.1 Marginal eigenvalues

The presence of marginal eigenvalues signals either a continuous symmetry of the flow (which one should immediately exploit to simplify the problem), or a non-hyperbolicity of a flow (a source of much pain, hard to avoid). In that case (typical of parameter values for which bifurcations occur) one has to go beyond linear stability, deal with Jordan type subspaces (see example 4.3), and sub-exponential growth rates, such as  $t^\alpha$ . For flow-invariant solutions such as periodic orbits, the time evolution is itself a continuous symmetry, hence a periodic orbit of a flow always has a *marginal Floquet multiplier*, as we now show.

chapter 29  
exercise 5.1

The Jacobian matrix  $J^t(x)$  transports the velocity field  $v(x)$  by (4.9),  $v(x(t)) = J^t(x_0)v(x_0)$ . In general the velocity at point  $x(t)$  does not point in the same direction as the velocity at point  $x_0$ , so this is not an eigenvalue condition for  $J^t$ ; the Jacobian matrix computed for an arbitrary segment of an arbitrary trajectory has no invariant meaning. However, if the orbit is periodic,  $x(T_p) = x(0)$ , after a complete period

$$J_p(x)v(x) = v(x), \quad x \in \mathcal{M}_p. \tag{5.11}$$

Two successive points on the cycle initially distance  $\delta x = x'(0) - x(0)$  apart, are separated by the exactly same distance after a completed period  $\delta x(T) = \delta x$ , see figure 5.3, hence for a periodic orbit of a *flow* the velocity field  $v$  at any point along cycle is an eigenvector  $\mathbf{e}^{(ll)}(x) = v(x)$  of the Jacobian matrix  $J_p$  with the unit Floquet multiplier, zero Floquet exponent

$$\Lambda_{||} = 1, \quad \lambda^{(ll)} = 0. \tag{5.12}$$

The continuous invariance that gives rise to this marginal Floquet multiplier is the invariance of a cycle (the set  $\mathcal{M}_p$ ) under a time translation of its points along the cycle. As we shall see in sect. 5.5, this marginal stability direction can be eliminated by cutting the cycle by a Poincaré section and replacing the continuous flow Floquet matrix by the Floquet matrix of the Poincaré return map.

exercise A2.3

If the flow is governed by a time-independent Hamiltonian, the energy is conserved, and that leads to an additional marginal Floquet multiplier (we shall show in sect. 8.4 that due to the symplectic invariance (8.27) real eigenvalues come in pairs). Further marginal eigenvalues arise in presence of continuous symmetries, as discussed in chapter 12.

## 5.4 Floquet multipliers are metric invariants



In sect. 5.3 we established that for a given flow, the Floquet multipliers are intrinsic to a given cycle, independent of the starting point along the cycle. Now we prove a much stronger statement: cycle Floquet multipliers are *smooth conjugacy* or *metric invariants* of the flow, the same in *any* representation of the dynamical system. That follows by elementary differential geometry considerations:

If the same dynamics is given by a map  $f$  in  $x$  coordinates, and a map  $g$  in the  $y = h(x)$  coordinates, then  $f$  and  $g$  (or any other good representation) are related by a *smooth conjugacy*, a reparameterization and a coordinate transformation  $g = h \circ f \circ h^{-1}$  which maps nearby points of  $f$  into nearby points of  $g$ . As both  $f$  and  $g$  are arbitrary representations of the dynamical system, the explicit form of the conjugacy  $h$  is of no interest, only the properties invariant under any transformation  $h$  are of general import. Furthermore, a good representation should not mutilate the data; the mapping  $h$  must be a *smooth conjugacy* which maps nearby points of  $f$  into nearby points of  $g$ .

This smoothness guarantees that the cycles are not only topological invariants, but that their linearized neighborhoods are also metric invariants. For a fixed point  $f(x) = x$  of a 1-dimensional map this follows from the chain rule for derivatives,

$$\begin{aligned} g'(y) &= h'(f \circ h^{-1}(y))f'(h^{-1}(y))\frac{1}{h'(x)} \\ &= h'(x)f'(x)\frac{1}{h'(x)} = f'(x). \end{aligned} \tag{5.13}$$

In  $d$  dimensions the relationship between the maps in different coordinate representations is again  $g \circ h = h \circ f$ . The chain rule now relates  $J'$ , the Jacobian matrix of the map  $g$ , to the Jacobian matrix of map  $f$ :

$$J'(y)_{ij} = \Gamma(f(x))_{ik}J(x)_{kl}\Gamma(x)^{-1}_{lj}, \tag{5.14}$$

where the coordinate transformation Jacobian matrices are

$$\Gamma(x)_{ik} = \frac{\partial y_i}{\partial x_k} \quad \text{and} \quad \Gamma(x)^{-1}_{ik} = \frac{\partial x_i}{\partial y_k}. \tag{5.15}$$

(Here, as elsewhere in this book, a repeated index implies summation.) If  $x$  is an equilibrium point,  $x = f(x)$ ,  $\Gamma$  is the matrix inverse of  $\Gamma^{-1}$ , and (5.14) is a *similarity* transformation and thus preserves eigenvalues. It is easy to verify that in the case of period  $n_p$  cycle  $J'_p(y)$  and  $J_p(x)$  are again related by a similarity transformation. (Note, though, that this is not true for  $J^r(x)$  with  $r \neq n_p$ ). As stability of a flow can always be reduced to stability of a Poincaré return map, a Floquet multiplier of any cycle, for a flow or a map in arbitrary dimension, is a metric invariant of the dynamical system.

exercise A2.3

The  $i$ th Floquet (multiplier, eigenvector) pair  $(\Lambda_i, \mathbf{e}^{(i)})$  are computed from  $J$  evaluated at a periodic point  $x$ ,  $J(x)\mathbf{e}^{(i)}(x) = \Lambda_i\mathbf{e}^{(i)}(x)$ ,  $x \in \mathcal{M}_p$ . Multiplying



by  $\Gamma(x)$  from the left, and inserting  $\mathbf{1} = \Gamma(x)^{-1}\Gamma(x)$ , we find that the  $J$  evaluated at  $y = h(x)$  has the same Floquet multiplier,

$$J'_p(y) \mathbf{e}^{(i)}(y)' = \Lambda_i \mathbf{e}^{(i)}(y)', \quad (5.16)$$

but with the eigenvector  $\mathbf{e}^{(i)}(x)$  mapped to  $\mathbf{e}^{(i)}(y)' = \Gamma(x) \mathbf{e}^{(i)}(x)$ .

## 5.5 Stability of Poincaré map cycles



(R. Paškauskas and P. Cvitanović)

If a continuous flow periodic orbit  $p$  pierces the Poincaré section  $\mathcal{P}$  once, the section point is a fixed point of the Poincaré return map  $P$  with stability (4.25)

$$\hat{J}_{ij} = \left( \delta_{ik} - \frac{v_i U_k}{(v \cdot U)} \right) J_{kj}, \quad (5.17)$$

with all primes dropped, as the initial and the final points coincide,  $x' = f^T(x) = x$ . If the periodic orbit  $p$  pierces the Poincaré section  $n$  times, the same observation applies to the  $n$ th iterate of  $P$ .

We have already established in (4.26) that the velocity  $v(x)$  is a zero eigenvector of the Poincaré section Floquet matrix,  $\hat{J}v = 0$ . Consider next  $(\Lambda_\alpha, \mathbf{e}^{(\alpha)})$ , the full state space  $\alpha$ th (eigenvalue, eigenvector) pair (5.7), evaluated at a periodic point on a Poincaré section,

$$J(x) \mathbf{e}^{(\alpha)}(x) = \Lambda_\alpha \mathbf{e}^{(\alpha)}(x), \quad x \in \mathcal{P}. \quad (5.18)$$

Multiplying (5.17) by  $\mathbf{e}^{(\alpha)}$  and inserting (5.18), we find that the full state space Floquet matrix and the Poincaré section Floquet matrix  $\hat{J}$  have the same Floquet multiplier

$$\hat{J}(x) \hat{\mathbf{e}}^{(\alpha)}(x) = \Lambda_\alpha \hat{\mathbf{e}}^{(\alpha)}(x), \quad x \in \mathcal{P}, \quad (5.19)$$

where  $\hat{\mathbf{e}}^{(\alpha)}$  is a projection of the full state space eigenvector onto the Poincaré section:

$$(\hat{\mathbf{e}}^{(\alpha)})_i = \left( \delta_{ik} - \frac{v_i U_k}{(v \cdot U)} \right) (\mathbf{e}^{(\alpha)})_k. \quad (5.20)$$

Hence,  $\hat{J}_p$  evaluated on any Poincaré section point along the cycle  $p$  has the same set of Floquet multipliers  $\{\Lambda_1, \Lambda_2, \dots, \Lambda_d\}$  as the full state space Floquet matrix  $J_p$ , except for the marginal unit Floquet multiplier (5.12).

As established in (4.26), due to the continuous symmetry (time invariance)  $\hat{J}_p$  is a rank  $d-1$  matrix. We shall refer to the rank  $[(d-1-N) \times (d-1-N)]$  submatrix with  $N-1$  continuous symmetries quotiented out as the *monodromy matrix*  $M_p$  (from Greek *mono-* = alone, single, and *dromo* = run, racecourse, meaning a single run around the stadium). Quotienting continuous symmetries is discussed in chapter 12 below.

## 5.6 There goes the neighborhood



In what follows, our task will be to determine the size of a *neighborhood* of  $x(t)$ , and that is why we care about the Floquet multipliers, and especially the unstable (expanding) ones.

Nearby points aligned along the stable (contracting) directions remain in the neighborhood of the trajectory  $x(t) = f^t(x_0)$ ; the ones to keep an eye on are the points which leave the neighborhood along the unstable directions: all chaos arises from flights along these these directions. The sub-volume  $|\mathcal{M}_{x_0}| = \prod_i^e \Delta x_i$  of the set of points which get no further away from  $f^t(x_0)$  than  $L$ , the typical size of the system, is fixed by the condition that  $\Delta x_i \Lambda_i = O(L)$  in each expanding direction  $i$ . Hence the neighborhood size scales as  $|\mathcal{M}_{x_0}| \propto O(L^{d_e})/|\Lambda_p| \propto 1/|\Lambda_p|$  where  $\Lambda_p$  is the product of expanding Floquet multipliers (5.6) only; contracting ones play a secondary role. Discussion of sect. 1.5.1, figure 1.9, and figure 5.1 illustrate intersection of initial volume with its return, and chapters 15 and 21 illustrate the key role that the unstable directions play in systematically partitioning the state space of a given dynamical system. The contracting directions are so secondary that even infinitely many of them (for example, the infinity of contracting eigen-directions of the spatiotemporally chaotic dynamics described by a PDE will not matter.

So the dynamically important information is carried by the expanding sub-volume, not the total volume computed so easily in (4.29). That is also the reason why the dissipative and the Hamiltonian chaotic flows are much more alike than one would have naively expected for ‘compressible’ vs. ‘incompressible’ flows. In hyperbolic systems what matters are the expanding directions. Whether the contracting eigenvalues are inverses of the expanding ones or not is of secondary importance. As long as the number of unstable directions is finite, the same theory applies both to the finite-dimensional ODEs and infinite-dimensional PDEs.

## Résumé

Periodic orbits play a central role in any invariant characterization of the dynamics, because (a) their existence and inter-relations are a *topological*, coordinate-independent property of the dynamics, and (b) their Floquet multipliers form an infinite set of *metric invariants*: The Floquet multipliers of a periodic orbit remain invariant under any smooth nonlinear change of coordinates  $f \rightarrow h \circ f \circ h^{-1}$ . Let us summarize the linearized flow notation used throughout the ChaosBook.

**Stability of invariant solutions:** The linear stability of an equilibrium  $v(x_q) = 0$  is described by the eigenvalues and eigenvectors  $\{\lambda^{(j)}, \mathbf{e}^{(j)}\}$  of the stability matrix  $A$  evaluated at the equilibrium point, and the linear stability of a periodic orbit  $f^T(x) = x, x \in \mathcal{M}_p$ ,

$$J_p(x) \mathbf{e}^{(j)}(x) = \Lambda_j \mathbf{e}^{(j)}(x), \quad \Lambda_j = \sigma^{(j)} e^{\lambda^{(j)} T},$$

by its Floquet multipliers, vectors and exponents  $\{\Lambda_j, \mathbf{e}^{(j)}\}$ , where  $\lambda^{(j)} = \mu^{(j)} \pm i\omega^{(j)}$ . For every continuous symmetry there is a marginal eigen-direction, with  $\Lambda_j = 1$ ,  $\lambda^{(j)} = 0$ . With all  $1 + N$  continuous symmetries quotiented out (Poincaré sections for time, slices for continuous symmetries of dynamics, see chapter 13) linear stability of a periodic orbit (and, more generally, of a partially hyperbolic torus) is described by the  $[(d-1-N) \times (d-1-N)]$  monodromy matrix, all of whose Floquet multipliers  $|\Lambda_j| \neq 1$  are generically strictly hyperbolic,

$$M_p(x) \mathbf{e}^{(j)}(x) = \Lambda_j \mathbf{e}^{(j)}(x), \quad x \in \mathcal{M}_p/G.$$

We shall show in chapter 14 that extending the linearized stability hyperbolic eigen-directions into stable and unstable manifolds yields important global information about the topological organization of state space. What matters most are the expanding directions. The physically important information is carried by the unstable manifold, and the expanding sub-volume characterized by the product of expanding Floquet multipliers of  $J_p$ . As long as the number of unstable directions is finite, the theory can be applied to flows of arbitrarily high dimension.



in depth:  
appendix A4, p. 843



fast track:  
chapter 11, p. 175

## Commentary

**Remark 5.1** Periodic orbits vs. ‘cycles’. Throughout this text, the terms ‘periodic orbit’ and ‘cycle’ (which has many other uses in mathematics) are used interchangeably; while ‘periodic orbit’ is more precise, ‘pseudo-cycle’ is easier on the ear than ‘pseudo-periodic-orbit’. In Soviet times obscure abbreviations were a rage, but here we shy away from acronyms such as UPOs (Unstable Periodic Orbits). How Kadanoff and Tang [A1.32] felt about the matter they let on by referring to these as ‘repulsive cycles’. We refer to unstable periodic orbits simply as ‘periodic orbits’, and the stable ones ‘limit cycles’. Strogatz [1.9] refers to periodic orbits as ‘closed orbits’ if they are isolated, in order to distinguish limit cycles and unstable orbits from the continuous family of the harmonic oscillator periodic orbits. Lost in the mists of time is the excitement experienced by the first physicist to discover that there are periodic orbits other than the limit cycles reached by mindless computation forward in time; but once one understands that there are at most several stable limit cycles (SPOs?) as opposed to the Smale horseshoe infinities of unstable cycles (UPOs?), what is gained by prefix ‘U’? A bit like calling all bicycles ‘unstable bicycles’.

**Remark 5.2** Periodic orbits and Floquet theory. Study of time-dependent and  $T$ -periodic vector fields is a classical subject in the theory of differential equations [5.3, ?]. In physics literature Floquet exponents often assume different names according to the context where the theory is applied: they are called Bloch phases in the discussion of Schrödinger equation with a periodic potential [5.1], or quasi-momenta in the quantum theory of time-periodic Hamiltonians. Here a discussion of Floquet theory is given in appendix A4.2.1.

For further reading on periodic orbits, consult Moehlis and K. Josić [5.2] Scholarpedia.org article.

## 5.7 Examples

The reader is urged to study the examples collected here. To return back to the main text, click on [click to return] pointer on the margin.

**Example 5.1 Stability of cycles of 1-dimensional maps:** *The stability of a prime cycle  $p$  of a 1-dimensional map follows from the chain rule (4.43) for stability of the  $n_p$ th iterate of the map*

$$\Lambda_p = \frac{d}{dx_0} f^{n_p}(x_0) = \prod_{m=0}^{n_p-1} f'(x_m), \quad x_m = f^m(x_0). \quad (5.21)$$

$\Lambda_p$  is a property of the cycle, not the initial periodic point, as taking any periodic point in the  $p$  cycle as the initial one yields the same  $\Lambda_p$ .

A critical point  $x_c$  is a value of  $x$  for which the mapping  $f(x)$  has vanishing derivative,  $f'(x_c) = 0$ . A periodic orbit of a 1-dimensional map is stable if

$$|\Lambda_p| = |f'(x_{n_p})f'(x_{n_p-1}) \cdots f'(x_2)f'(x_1)| < 1,$$

and superstable if the orbit includes a critical point, so that the above product vanishes. For a stable periodic orbit of period  $n$  the slope  $\Lambda_p$  of the  $n$ th iterate  $f^n(x)$  evaluated on a periodic point  $x$  (fixed point of the  $n$ th iterate) lies between  $-1$  and  $1$ . If  $|\Lambda_p| > 1$ ,  $p$ -cycle is unstable.

**Example 5.2 Stability of cycles for maps:** *No matter what method one uses to determine unstable cycles, the theory to be developed here requires that their Floquet multipliers be evaluated as well. For maps a Floquet matrix is easily evaluated by picking any periodic point as a starting point, running once around a prime cycle, and multiplying the individual periodic point Jacobian matrices according to (4.22). For example, the Floquet matrix  $M_p$  for a prime cycle  $p$  of length  $n_p$  of the Hénon map (3.17) is given by (4.44),*

$$M_p(x_0) = \prod_{k=n_p}^1 \begin{bmatrix} -2ax_k & b \\ 1 & 0 \end{bmatrix}, \quad x_k \in M_p,$$

and the Floquet matrix  $M_p$  for a 2-dimensional billiard prime cycle  $p$  of length  $n_p$

$$M_p = (-1)^{n_p} \prod_{k=n_p}^1 \begin{bmatrix} 1 & \tau_k \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ r_k & 1 \end{bmatrix}$$

follows from (9.11) of chapter 9 below. The decreasing order in the indices of the products in above formulas is a reminder that the successive time steps correspond to multiplication from the left,  $M_p(x_1) = M(x_{n_p}) \cdots M(x_1)$ . We shall compute Floquet multipliers of Hénon map cycles once we learn how to find their periodic orbits, see exercise 16.11.

click to return: p. ??

## Exercises

### 5.1. A limit cycle with analytic Floquet exponent.

There are only two examples of nonlinear flows for which the Floquet multipliers can be evaluated analytically. Both are cheats. One example is the 2-dimensional flow

$$\begin{aligned}\dot{q} &= p + q(1 - q^2 - p^2) \\ \dot{p} &= -q + p(1 - q^2 - p^2).\end{aligned}$$

Determine all periodic solutions of this flow, and determine analytically their Floquet exponents. Hint: go to polar coordinates  $(q, p) = (r \cos \theta, r \sin \theta)$ .

G. Bard Ermentrout

5.2. **The other example of a limit cycle with analytic Floquet exponent.** What is the other example of a nonlinear flow for which the Floquet multipliers can be evaluated analytically? Hint: email G.B. Ermentrout.

5.3. **Yet another example of a limit cycle with analytic Floquet exponent.** Prove G.B. Ermentrout wrong by solving a third example (or more) of a nonlinear flow for which the Floquet multipliers can be evaluated analytically.

## References

- [5.1] N. W. Ashcroft and N. D. Mermin, *Solid State Physics* (Holt, Rinehart and Winston, New York 1976).
- [5.2] J. Moehlis and K. Josić, “Periodic orbit,” [www.scholarpedia.org/article/Periodic\\_orbit](http://www.scholarpedia.org/article/Periodic_orbit).
- [5.3] Floquet, G., “Sur les équations différentielles linéaires à coefficients périodiques”, *Ann. Sci. Ec. Norm. Sér.* **12** 47–88 (1883)