Appendix A4

Linear stability

> Mopping up operations are the activities that engage most scientists throughout their careers.
>
> — Thomas Kuhn, *The Structure of Scientific Revolutions*

THE SUBJECT OF LINEAR ALGEBRA generates innumerable tomes of its own, and is way beyond what we can exhaustively cover. Here we recapitulate a few essential concepts that ChaosBook relies on. The punch line is Eq. (A4.25):

Hamilton-Cayley equation $\prod(\mathbf{M} - \lambda_i \mathbf{1}) = 0$ associates with each distinct root λ_i of a matrix $\mathbf{M}$ a projection onto ith vector subspace

$$\mathbf{P}_i = \prod_{j \neq i} \frac{\mathbf{M} - \lambda_j \mathbf{1}}{\lambda_i - \lambda_j} \,.$$

A4.1 Linear algebra

In this section we collect a few basic definitions. The reader might prefer going straight to sect. A4.2.

Vector space. A set V of elements $\mathbf{x}, \mathbf{y}, \mathbf{z}, \ldots$ is called a *vector* (or *linear*) *space* over a field $\mathbb{F}$ if

(a) *vector addition* "+" is defined in V such that V is an abelian group under addition, with identity element $\mathbf{0}$;

(b) the set is *closed* with respect to *scalar multiplication* and vector addition

$$\begin{aligned} a(\mathbf{x}+\mathbf{y}) &= a\mathbf{x}+a\mathbf{y}\,, \qquad a,b\in\mathbb{F}\,, \quad \mathbf{x},\mathbf{y}\in V \\ (a+b)\mathbf{x} &= a\mathbf{x}+b\mathbf{x} \\ a(b\mathbf{x}) &= (ab)\mathbf{x} \\ 1\,\mathbf{x} &= \mathbf{x}\,, \qquad 0\,\mathbf{x}=\mathbf{0}\,. \end{aligned} \tag{A4.1}$$

Here the field $\mathbb{F}$ is either $\mathbb{R}$, the field of reals numbers, or $\mathbb{C}$, the field of complex numbers. Given a subset $V_0 \subset V$, the set of all linear combinations of elements of V_0, or the *span* of V_0, is also a vector space.

A basis. $\{\mathbf{e}^{(1)},\cdots,\mathbf{e}^{(d)}\}$ is any linearly independent subset of V whose span is V. The number of basis elements d is the *dimension* of the vector space V.

Dual space, dual basis. Under a general linear transformation $g \in GL(n,\mathbb{F})$, the row of basis vectors transforms by right multiplication as $\mathbf{e}^{(j)} = \sum_k (\mathbf{g}^{-1})^j{}_k\, \mathbf{e}^{(k)}$, and the column of x_a's transforms by left multiplication as $x' = \mathbf{g}x$. Under left multiplication the column (row transposed) of basis vectors $\mathbf{e}_{(k)}$ transforms as $\mathbf{e}_{(j)} = (\mathbf{g}^\dagger)_j{}^k \mathbf{e}_{(k)}$, where the *dual rep* $\mathbf{g}^\dagger = (\mathbf{g}^{-1})^\top$ is the transpose of the inverse of $\mathbf{g}$. This observation motivates introduction of a *dual* representation space $\bar{V}$, the space on which $GL(n,\mathbb{F})$ acts via the dual rep $\mathbf{g}^\dagger$.

Definition. If V is a vector representation space, then the *dual space* $\bar{V}$ is the set of all linear forms on V over the field $\mathbb{F}$.

If $\{\mathbf{e}^{(1)},\cdots,\mathbf{e}^{(d)}\}$ is a basis of V, then $\bar{V}$ is spanned by the *dual basis* $\{\mathbf{e}_{(1)},\cdots,\mathbf{e}_{(d)}\}$, the set of d linear forms $\mathbf{e}_{(k)}$ such that

$$\mathbf{e}_{(j)} \cdot \mathbf{e}^{(k)} = \delta^k_j\,,$$

where δ^k_j is the Kronecker symbol, $\delta^k_j = 1$ if $j = k$, and zero otherwise. The components of dual representation space vectors $\bar{y} \in \bar{V}$ will here be distinguished by upper indices

$$(y^1, y^2, \ldots, y^n)\,. \tag{A4.2}$$

They transform under $GL(n,\mathbb{F})$ as

$$y'^a = (\mathbf{g}^\dagger)^a{}_b y^b\,. \tag{A4.3}$$

For $GL(n,\mathbb{F})$ no complex conjugation is implied by the † notation; that interpretation applies only to unitary subgroups $U(n) \subset GL(n,\mathbb{C})$. In the index notation, $\mathbf{g}$ can be distinguished from $\mathbf{g}^\dagger$ by keeping track of the relative ordering of the indices,

$$(\mathbf{g})^b_a \to g_a{}^b\,, \qquad (\mathbf{g}^\dagger)^b_a \to g^b{}_a\,. \tag{A4.4}$$

Algebra. A set of r elements $\mathbf{t}_\alpha$ of a vector space $\mathcal{T}$ forms an algebra if, in addition to the vector addition and scalar multiplication,

(a) the set is *closed* with respect to multiplication $\mathcal{T} \cdot \mathcal{T} \to \mathcal{T}$, so that for any two elements $\mathbf{t}_\alpha, \mathbf{t}_\beta \in \mathcal{T}$, the product $\mathbf{t}_\alpha \cdot \mathbf{t}_\beta$ also belongs to $\mathcal{T}$:

$$\mathbf{t}_\alpha \cdot \mathbf{t}_\beta = \sum_{\gamma=0}^{r-1} \tau_{\alpha\beta}{}^\gamma \mathbf{t}_\gamma \,, \qquad \tau_{\alpha\beta}{}^\gamma \in \mathbb{C} \,; \tag{A4.5}$$

(b) the multiplication operation is *distributive*:

$$\begin{aligned} (\mathbf{t}_\alpha + \mathbf{t}_\beta) \cdot \mathbf{t}_\gamma &= \mathbf{t}_\alpha \cdot \mathbf{t}_\gamma + \mathbf{t}_\beta \cdot \mathbf{t}_\gamma \\ \mathbf{t}_\alpha \cdot (\mathbf{t}_\beta + \mathbf{t}_\gamma) &= \mathbf{t}_\alpha \cdot \mathbf{t}_\beta + \mathbf{t}_\alpha \cdot \mathbf{t}_\gamma \,. \end{aligned}$$

The set of numbers $\tau_{\alpha\beta}{}^\gamma$ are called the *structure constants*. They form a matrix rep of the algebra,

$$(\mathbf{t}_\alpha)_\beta{}^\gamma \equiv \tau_{\alpha\beta}{}^\gamma \,, \tag{A4.6}$$

whose dimension is the dimension r of the algebra itself.

Depending on what further assumptions one makes on the multiplication, one obtains different types of algebras. For example, if the multiplication is associative

$$(\mathbf{t}_\alpha \cdot \mathbf{t}_\beta) \cdot \mathbf{t}_\gamma = \mathbf{t}_\alpha \cdot (\mathbf{t}_\beta \cdot \mathbf{t}_\gamma) \,,$$

the algebra is *associative*. Typical examples of products are the *matrix product*

$$(\mathbf{t}_\alpha \cdot \mathbf{t}_\beta)^c_a = (t_\alpha)^b_a (t_\beta)^c_b \,, \qquad \mathbf{t}_\alpha \in V \otimes \bar{V} \,, \tag{A4.7}$$

and the *Lie product*

$$(\mathbf{t}_\alpha \cdot \mathbf{t}_\beta)^c_a = (t_\alpha)^b_a (t_\beta)^c_b - (t_\alpha)^b_c (t_\beta)^a_b \,, \qquad \mathbf{t}_\alpha \in V \otimes \bar{V} \tag{A4.8}$$

which defines a *Lie algebra*.

A4.2 Eigenvalues and eigenvectors

Eigenvalues of a $[d \times d]$ matrix $\mathbf{M}$ are the roots of its characteristic polynomial

$$\det(\mathbf{M} - \lambda \mathbf{1}) = \prod (\lambda_i - \lambda) = 0 \,. \tag{A4.9}$$

Given a nonsingular matrix $\mathbf{M}$, with all $\lambda_i \neq 0$, acting on d-dimensional vectors $\mathbf{x}$, we would like to determine *eigenvectors* $\mathbf{e}^{(i)}$ of $\mathbf{M}$ on which $\mathbf{M}$ acts by scalar multiplication by eigenvalue λ_i

$$\mathbf{M}\, \mathbf{e}^{(i)} = \lambda_i \mathbf{e}^{(i)} \,. \tag{A4.10}$$

If $\lambda_i \neq \lambda_j$, $\mathbf{e}^{(i)}$ and $\mathbf{e}^{(j)}$ are linearly independent. There are at most d distinct eigenvalues and eigenspaces, which we assume have been computed by some method, and ordered by their real parts, $\mathrm{Re}\, \lambda_i \geq \mathrm{Re}\, \lambda_{i+1}$.

If all eigenvalues are distinct $\mathbf{e}^{(j)}$ are d linearly independent vectors which can be used as a (non-orthogonal) basis for any d-dimensional vector $\mathbf{x} \in \mathbb{R}^d$

$$\mathbf{x} = x_1\,\mathbf{e}^{(1)} + x_2\,\mathbf{e}^{(2)} + \cdots + x_d\,\mathbf{e}^{(d)}\,. \tag{A4.11}$$

From (A4.10) it follows that

$$(\mathbf{M} - \lambda_i\mathbf{1})\,\mathbf{e}^{(j)} = (\lambda_j - \lambda_i)\,\mathbf{e}^{(j)}\,,$$

matrix $(\mathbf{M} - \lambda_i\mathbf{1})$ annihilates $\mathbf{e}^{(i)}$, the product of all such factors annihilates any vector, and the matrix $\mathbf{M}$ satisfies its characteristic equation (A4.9),

$$\prod_{i=1}^{d}(\mathbf{M} - \lambda_i\mathbf{1}) = 0\,. \tag{A4.12}$$

This humble fact has a name: the Hamilton-Cayley theorem. If we delete one term from this product, we find that the remainder projects $\mathbf{x}$ onto the corresponding eigenspace:

$$\prod_{j\neq i}(\mathbf{M} - \lambda_j\mathbf{1})\mathbf{x} = \prod_{j\neq i}(\lambda_i - \lambda_j)x_i\mathbf{e}^{(i)}\,.$$

Dividing through by the $(\lambda_i - \lambda_j)$ factors yields the *projection operators*

$$\mathbf{P}_i = \prod_{j\neq i}\frac{\mathbf{M} - \lambda_j\mathbf{1}}{\lambda_i - \lambda_j}\,, \tag{A4.13}$$

which are *orthogonal* and *complete*:

$$\mathbf{P}_i\mathbf{P}_j = \delta_{ij}\mathbf{P}_j\,, \quad \text{(no sum on } j)\,, \qquad \sum_{i=1}^{r}\mathbf{P}_i = \mathbf{1}\,. \tag{A4.14}$$

It follows from the characteristic equation (A4.12) that λ_i is the eigenvalue of $\mathbf{M}$ on $\mathbf{P}_i$ subspace:

$$\mathbf{M}\,\mathbf{P}_i = \lambda_i\mathbf{P}_i \qquad \text{(no sum on } i)\,. \tag{A4.15}$$

Using $\mathbf{M} = \mathbf{M}\,\mathbf{1}$ and completeness relation (A4.14) we can rewrite $\mathbf{M}$ as

$$\mathbf{M} = \lambda_1\mathbf{P}_1 + \lambda_2\mathbf{P}_2 + \cdots + \lambda_d\mathbf{P}_d\,. \tag{A4.16}$$

Any matrix function $f(\mathbf{M})$ takes the scalar value $f(\lambda_i)$ on the $\mathbf{P}_i$ subspace, $f(\mathbf{M})\,\mathbf{P}_i = f(\lambda_i)\,\mathbf{P}_i$, and is thus easily evaluated through its *spectral decomposition*

$$f(\mathbf{M}) = \sum_i f(\lambda_i)\mathbf{P}_i\,. \tag{A4.17}$$

This, of course, is the reason why anyone but a fool works with irreducible reps: they reduce matrix (AKA "operator") evaluations to manipulations with numbers.

By (A4.10) every column of $\mathbf{P}_i$ is proportional to a right eigenvector $\mathbf{e}^{(i)}$, and its every row to a left eigenvector $\mathbf{e}_{(i)}$. In general, neither set is orthogonal, but by the idempotence condition (A4.14), they are mutually orthogonal,

$$\mathbf{e}_{(i)} \cdot \mathbf{e}^{(j)} = c\,\delta_i^j\,. \tag{A4.18}$$

The non-zero constant c is convention dependent and not worth fixing, unless you feel nostalgic about Clebsch-Gordan coefficients. We shall set $c = 1$. Then it is convenient to collect all left and right eigenvectors into a single matrix as follows.

Fundamental matrix (take 1). As the system is a linear, a superposition of any two solutions to $x(t) = J^t x(0)$ is also a solution. One can take any d independent initial states, $x^{(1)}(0)$, $x^{(2)}(0)$, $\ldots$, $x^{(d)}(0)$, assemble them as columns of a matrix $\Phi(0)$, and formally write the solution for an arbitrary initial condition projected onto this basis,

$$x(t) = \Phi(t)\Phi(0)^{-1} x(0) \tag{A4.19}$$

where $\Phi(t) = [x^{(1)}(t), x^{(2)}(t), \cdots, x^{(d)}(t)]$. $\Phi(t)$ is called the *fundamental matrix* of the system, and the Jacobian matrix $J^t = \Phi(t)\Phi(0)^{-1}$ can thus be fashioned out of d trajectories $\{x^{(j)}(t)\}$. Numerically this works for sufficiently short times.

Fundamental matrix (take 2). The set of solutions $x(t) = J^t(x_0)x_0$ for a system of homogeneous linear differential equations $\dot{x}(t) = A(t)x(t)$ of order 1 and dimension d forms a d-dimensional vector space. A basis $\{\mathbf{e}^{(1)}(t), \ldots, \mathbf{e}^{(d)}(t)\}$ for this vector space is called a *fundamental system*. Every solution $x(t)$ can be written as

$$x(t) = \sum_{i=1}^{d} c_i \, \mathbf{e}^{(i)}(t) \, .$$

The $[d\times d]$ matrix $\mathbf{F}^{-1}_{ij} = \mathbf{e}^{(j)}_i$ whose columns are the right eigenvectors of J^t

$$\mathbf{F}(t)^{-1} = (\mathbf{e}^{(1)}(t), \ldots, \mathbf{e}^{(d)}(t)) \, , \qquad \mathbf{F}(t)^T = (\mathbf{e}_{(1)}(t), \ldots, \mathbf{e}_{(d)}(t)) \tag{A4.20}$$

is the inverse of a *fundamental matrix*.

Jacobian matrix. The Jacobian matrix $J^t(x_0)$ is the linear approximation to a differentiable function $f^t(x_0)$, describing the orientation of a tangent plane to the function at a given point and the amount of local rotation and shearing caused by the transformation. The inverse of the Jacobian matrix of a function is the Jacobian matrix of the inverse function. If f is a map from d-dimensional space to itself, the Jacobian matrix is a square matrix, whose determinant we refer to as the 'Jacobian.'

The Jacobian matrix can be written as transformation from basis at time t_0 to the basis at time t_1,

$$J^{t_1-t_0}(x_0) = \mathbf{F}(t_1)\mathbf{F}(t_0)^{-1} \, . \tag{A4.21}$$

Then the matrix form of (A4.18) is $\mathbf{F}(t)\mathbf{F}(t)^{-1} = \mathbf{1}$, i.e., for zero time the Jacobian matrix is the identity. exercise A4.1

Example A4.1 ***Fundamental matrix.*** *If A is constant in time, the system (4.2) is autonomous, and the solution is*

$$x(t) = e^{A\,t} x(0) \, ,$$

where $\exp(A\,t)$ is defined by the Taylor series for $\exp(x)$. As the system is linear, the sum of any two solutions is also a solution. Therefore, given d independent initial conditions,

$x_1(0), x_2(0), \ldots x_d(0)$ we can write the solution for an arbitrary initial condition based on its projection on to this set,

$$x(t) = \mathbf{F}(t)\,\mathbf{F}(0)^{-1}x(0) = e^{At}x(0)\,,$$

where $\mathbf{F}(t) = (x_1(t), x_2(t), \ldots, x_d(t))$ is a fundamental matrix of the system. (J. Halcrow)
exercise A4.1

Example A4.2 ***Complex eigenvalues.*** *As A has only real entries, it will in general have either real eigenvalues, or complex conjugate pairs of eigenvalues. That is not surprising, but also the corresponding eigenvectors can be either real or complex. All coordinates used in defining a dynamical flow are real numbers, so what is the meaning of a* complex *eigenvector?*

If λ_k, λ_{k+1} eigenvalues that lie within a diagonal $[2\times 2]$ sub-block $A' \subset A$ form a complex conjugate pair, $\{\lambda_k, \lambda_{k+1}\} = \{\mu + i\omega, \mu - i\omega\}$, the corresponding complex eigenvectors can be replaced by their real and imaginary parts, $\{\mathbf{e}^{(k)}, \mathbf{e}^{(k+1)}\} \to \{\mathrm{Re}\,\mathbf{e}^{(k)}, \mathrm{Im}\,\mathbf{e}^{(k)}\}$. In this 2-dimensional real representation, $A' \to N$, the block N is a sum of the rescaling-$\times$identity and the generator of $SO(2)$ rotations

$$\mathbf{N} = \begin{bmatrix} \mu & -\omega \\ \omega & \mu \end{bmatrix} = \mu \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \omega \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Trajectories of $\dot{\mathbf{x}} = \mathbf{N}\,\mathbf{x}$, given by $\mathbf{x}(t) = J^t\,\mathbf{x}(0)$, where

$$J^t = e^{t\mathbf{N}} = e^{t\mu} \begin{bmatrix} \cos\,\omega t & -\sin\,\omega t \\ \sin\,\omega t & \cos\,\omega t \end{bmatrix}, \tag{A4.22}$$

spiral in/out around $(x, y) = (0, 0)$, see figure 4.3, with the rotation period T and the radial expansion /contraction multiplier along the $\mathbf{e}^{(j)}$ eigen-direction per a turn of the spiral:

$$T = 2\pi/\omega\,, \qquad \Lambda_{radial} = e^{T\mu}\,. \tag{A4.23}$$

We learn that the typical turnover time scale in the neighborhood of the equilibrium $(x, y) = (0, 0)$ is of order $\approx T$ (and not, let us say, $1000\,T$, or $10^{-2}T$).

Degenerate eigenvalues. While for a matrix with generic real elements all eigenvalues are distinct with probability 1, that is not true in presence of symmetries, or spacial parameter values (bifurcation points). What can one say about situation where d_α eigenvalues are degenerate, $\lambda_\alpha = \lambda_i = \lambda_{i+1} = \cdots = \lambda_{i+d_\alpha-1}$? Hamilton-Cayley (A4.12) now takes form

$$\prod_{\alpha=1}^{r} (\mathbf{M} - \lambda_\alpha \mathbf{1})^{d_\alpha} = 0\,, \qquad \sum_\alpha d_\alpha = d\,. \tag{A4.24}$$

We distinguish two cases:

M can be brought to diagonal form. The characteristic equation (A4.24) can be replaced by the minimal polynomial,

$$\prod_{\alpha=1}^{r} (\mathbf{M} - \lambda_\alpha \mathbf{1}) = 0\,, \tag{A4.25}$$

where the product includes each distinct eigenvalue only once. Matrix $\mathbf{M}$ acts multiplicatively

$$\mathbf{M}\,\mathbf{e}^{(\alpha,k)} = \lambda_i \mathbf{e}^{(\alpha,k)}\,, \tag{A4.26}$$

on a d_α-dimensional subspace spanned by a linearly independent set of basis eigenvectors $\{\mathbf{e}^{(\alpha,1)}, \mathbf{e}^{(\alpha,2)}, \cdots, \mathbf{e}^{(\alpha,d_\alpha)}\}$. This is the easy case whose discussion we continue in appendix A7.2.1. Luckily, if the degeneracy is due to a finite or compact symmetry group, relevant $\mathbf{M}$ matrices can always be brought to such Hermitian, diagonalizable form.

M can only be brought to upper-triangular, Jordan form. This is the messy case, so we only illustrate the key idea in example A4.3.

Example A4.3 ***Decomposition of 2-dimensional vector spaces:*** *Enumeration of every possible kind of linear algebra eigenvalue / eigenvector combination is beyond what we can reasonably undertake here. However, enumerating solutions for the simplest case, a general* $[2\times2]$ *non-singular matrix*

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}.$$

takes us a long way toward developing intuition about arbitrary finite-dimensional matrices. The eigenvalues

$$\lambda_{1,2} = \frac{1}{2}\mathrm{tr}\,A \pm \frac{1}{2}\sqrt{(\mathrm{tr}\,A)^2 - 4\det A} \tag{A4.27}$$

are the roots of the characteristic (secular) equation (A4.9):

$$\begin{aligned} \det(A - \lambda\,\mathbf{1}) &= (\lambda_1 - \lambda)(\lambda_2 - \lambda) \\ &= \lambda^2 - \mathrm{tr}\,A\,\lambda + \det A \;=\; 0\,. \end{aligned}$$

For any linear system in $\mathbb{R}^2$*, there is a similarity transformation*

$$B = U^{-1}AU\,,$$

where the columns of U *consist of the generalized eigenvectors of* A *such that* B *has one of the following forms:*

$$B = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}, \qquad B = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}, \qquad B = \begin{bmatrix} \mu & -\omega \\ \omega & \mu \end{bmatrix}.$$

These three cases, called normal forms, correspond to A *having (1) distinct real eigenvalues, (2) degenerate real eigenvalues, or (3) a complex pair of eigenvalues. It follows that*

$$e^{Bt} = \begin{bmatrix} e^{\lambda t} & 0 \\ 0 & e^{\mu t} \end{bmatrix}, \qquad e^{Bt} = e^{\lambda t}\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}, \qquad e^{Bt} = e^{at}\begin{bmatrix} \cos bt & -\sin bt \\ \sin bt & \cos bt \end{bmatrix},$$

where the corresponding Jacobian matrix is $e^{At} = Ue^{Bt}U^{-1}$*. What we have done is classify all* $[2\times2]$ *matrices as belonging to one of three classes of geometrical transformations. The first case is scaling, the second is a shear, and the third is a combination of rotation and scaling. The generalization of these normal forms to* $\mathbb{R}^d$ *is called the Jordan normal form.*

Distinct eigenvalues case *has already been described in sect. 4.8, and in the full generality for arbitrary dimension in sect. 5.1. The left/right eigenvectors are (up to overall multiplicate factors) the rows/columns of projection operators*

$$P_1 = \frac{A-\lambda_2 \mathbf{1}}{\lambda_1-\lambda_2}\,, \qquad P_2 = \frac{A-\lambda_1 \mathbf{1}}{\lambda_2-\lambda_1}\,, \qquad \lambda_1 \neq \lambda_2\,. \tag{A4.28}$$

Complex eigenvalues pair case *is discussed in example A4.2.*

Degenerate eigenvalues. *If $\lambda_1 = \lambda_2 = \lambda$, we distinguish two cases: (a) A can be brought to diagonal form. This is the easy case whose discussion in any dimension we continue in appendix A7.2.1. (b) A can be brought to* Jordan form*, with zeros everywhere except for the diagonal, and some 1's directly above it; for a $[2\times 2]$ matrix the Jordan form is*

$$A = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}, \qquad \mathbf{e}^{(1)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}^{(2)} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

$\mathbf{v}^{(2)}$ helps span the 2-dimensional space, $(A-\lambda)^2\mathbf{v}^{(2)} = 0$, but is not an eigenvector, as $A\mathbf{v}^{(2)} = \lambda\mathbf{v}^{(2)} + \mathbf{e}^{(1)}$. For every such Jordan $[d_\alpha \times d_\alpha]$ block there is only one eigenvector per block. Noting that

$$A^m = \begin{bmatrix} \lambda^m & m\lambda^{m-1} \\ 0 & \lambda^m \end{bmatrix},$$

we see that instead of acting multiplicatively on $\mathbb{R}^2$, Jacobian matrix $J^t = \exp(t\mathbf{M})$

$$e^{tA}\begin{pmatrix} u \\ v \end{pmatrix} = e^{t\lambda}\begin{pmatrix} u+tv \\ v \end{pmatrix} \tag{A4.29}$$

picks up a power-low correction. That spells trouble (logarithmic term $\ln t$ if we bring the extra term into the exponent).

(J. Halcrow)

Example A4.4 ***Projection operator decomposition in 2 dimensions:*** *Let's illustrate how the distinct eigenvalues case works with the $[2\times 2]$ matrix*

$$\mathbf{M} = \begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}.$$

Its eigenvalues $\{\lambda_1, \lambda_2\} = \{5, 1\}$ are the roots of (A4.27):

$$\det(\mathbf{M} - \lambda\mathbf{1}) = \lambda^2 - 6\,\lambda + 5 = (5-\lambda)(1-\lambda) = 0\,.$$

That $\mathbf{M}$ satisfies its secular equation (Hamilton-Cayley theorem) can be verified by explicit calculation:

$$\begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}^2 - 6\begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix} + 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Associated with each root λ_i is the projection operator *(A4.28)*

$$P_1 = \frac{1}{4}(\mathbf{M} - \mathbf{1}) = \frac{1}{4}\begin{bmatrix} 3 & 1 \\ 3 & 1 \end{bmatrix} \tag{A4.30}$$

$$P_2 = \frac{1}{4}(\mathbf{M} - 5\cdot\mathbf{1}) = \frac{1}{4}\begin{bmatrix} 1 & -1 \\ -3 & 3 \end{bmatrix}. \tag{A4.31}$$

Matrices $\mathbf{P}_i$ are orthonormal and complete, The dimension of the ith subspace is given by $d_i = \mathrm{tr}\,\mathbf{P}_i$; in case at hand both subspaces are 1-dimensional. From the characteristic equation it follows that $\mathbf{P}_i$ satisfies the eigenvalue equation $\mathbf{M}\mathbf{P}_i = \lambda_i\mathbf{P}_i$. Two consequences are immediate. First, we can easily evaluate any function of $\mathbf{M}$ by spectral decomposition, for example

$$\mathbf{M}^7 - 3\cdot\mathbf{1} = (5^7-3)\mathbf{P}_1 + (1-3)\mathbf{P}_2 = \begin{bmatrix} 58591 & 19531 \\ 58593 & 19529 \end{bmatrix}.$$

Second, as $\mathbf{P}_i$ satisfies the eigenvalue equation, its every column is a right eigenvector, and every row a left eigenvector. Picking first row/column we get the eigenvectors:

$$\begin{aligned} \{\mathbf{e}^{(1)}, \mathbf{e}^{(2)}\} &= \{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\-3\end{bmatrix}\} \\ \{\mathbf{e}_{(1)}, \mathbf{e}_{(2)}\} &= \{\begin{bmatrix}3\\1\end{bmatrix}, \begin{bmatrix}1\\-1\end{bmatrix}\}, \end{aligned}$$

with overall scale arbitrary.The matrix is not hermitian , so $\{\mathbf{e}^{(j)}\}$ do not form an orthogonal basis. The left-right eigenvector dot products $\mathbf{e}_{(j)}\cdot\mathbf{e}^{(k)}$, however, are orthogonal as in (A4.18), by inspection. (Continued in example 15.2.)

Example A4.5 ***Computing matrix exponentials.*** *If A is diagonal (the system is uncoupled), then e^{tA} is given by*

$$\exp\begin{pmatrix} \lambda_1 t & & & \\ & \lambda_2 t & & \\ & & \ddots & \\ & & & \lambda_d t \end{pmatrix} = \begin{pmatrix} e^{\lambda_1 t} & & & \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ & & & e^{\lambda_d t} \end{pmatrix}.$$

If A is diagonalizable, $A = FDF^{-1}$, where D is the diagonal matrix of the eigenvalues of A and F is the matrix of corresponding eigenvectors, the result is simple: $A^n = (FDF^{-1})(FDF^{-1})\ldots(FDF^{-1}) = FD^nF^{-1}$. Inserting this into the Taylor series for e^x gives $e^{At} = Fe^{Dt}F^{-1}$. But A may not have d linearly independent eigenvectors, forcing us to take a different, Jordan route, explained in example A4.3.

A4.2.1 Floquet theory

When dealing with periodic orbits, some of the quantities already introduced in chapter 4 inherit names from the Floquet theory of differential equations with time-periodic coefficients. Consider the equation of variations (4.2) evaluated on a periodic orbit p of period T, at point $x(t) \in \mathcal{M}_p$,

$$\dot{\delta x} = A(t)\,\delta x\,, \qquad A(t) = A(t+T)\,,$$

with $A(t) = A(x(t))$. The periodicity of the stability matrix implies that if $\delta x(t)$ is a solution, then also $\delta x(t+T)$ satisfies the same equation: moreover the two solutions are related by (4.5)

$$\delta x(t+T) = J_p(x)\,\delta x(t)\,, \qquad x \in \mathcal{M}_p\,. \tag{A4.32}$$

Table A4.1: The first 27 least stable Floquet exponents $\lambda = \mu \pm i\,\omega$ of equilibrium EQ_5 for plane Couette flow, $Re = 400$. The exponents are ordered by the decreasing real part. The two zero exponents, to the numerical precision of our computation, arise from the two translational symmetries. For details, see ref. [13.43].

j	$\mu^{(j)}_{EQ5}$	$\omega^{(j)}_{EQ5}$	$s_1 s_2 s_3$
1,2	0.07212161	0.04074989	S S S
3	0.06209526		S A A
4	0.06162059		A S A
5,6	0.02073075	0.07355143	S S S
7	0.009925378		S A A
8,9	0.009654012	0.04551274	A A S
10,11	0.009600794	0.2302166	S A A
12,13	1.460798e-06	1.542103e-06	- - A
14,15	-0.0001343539	0.231129	A A S
16	-0.006178861		A S A
17,18	-0.007785718	0.1372092	A A S
19	-0.01064716		S A A
20,21	-0.01220116	0.2774336	S S S
22,23	-0.01539667	0.2775381	S A A
24,25	-0.03451081	0.08674062	A S A
26,27	-0.03719139	0.215319	S A A

Even though the Jacobian matrix $J_p(x)$ depends upon x (the 'starting' point of the periodic orbit), we shall show in sect. 5.3 that its eigenvalues do not, so we may write the eigenvalue equation as

$$J_p(x)\,\mathbf{e}^{(j)}(x) = \Lambda_j\,\mathbf{e}^{(j)}(x)\,, \tag{A4.33}$$

where Λ_j are independent of x, and we refer to eigenvectors $\mathbf{e}^{(j)}$ as 'covariant vectors', or, for periodic orbits, as 'Floquet vectors'.

Expand δx in the (A4.33) eigenbasis, $\delta x(t) = \sum \delta x_j(t)\,\mathbf{e}^{(j)}$, $\mathbf{e}^{(j)} = \mathbf{e}^{(j)}(x(0))$. Taking into account (A4.32), we get that $\delta x_j(t)$ is multiplied by Λ_j per each period

$$\delta x(t+T) = \sum_j \delta x_j(t+T)\,\mathbf{e}^{(j)} = \sum_j \Lambda_j\,\delta x_j(t)\,\mathbf{e}^{(j)}\,.$$

We can absorb this exponential growth / contraction by rewriting the coefficients $\delta x_j(t)$ as $\delta x_j(t) = \exp(\lambda^{(j)}t)\,u_j(t)$, $u_j(0) = \delta x_j(0)$. Thus each solution of the equation of variations (4.2) may be expressed in the Floquet form,

$$\delta x(t) = \sum_j e^{\lambda^{(j)}t}\,u_j(t)\,\mathbf{e}^{(j)}\,, \qquad u_j(t+T) = u_j(t)\,, \tag{A4.34}$$

with $u_j(t)$ periodic with period T. The $\exp(\lambda^{(j)}t)$ factor is not an eigenvalue of the Jacobian matrix J^t, it is only an interpolation between x and $f^T(x)$. The continuous time t in (A4.34) does not imply that eigenvalues of the Jacobian matrix enjoy any multiplicative property for $t \neq rT$: exponents $\lambda^{(j)}$ refer to a full traversal of the periodic orbit. Indeed, while $u_j(t)$ describes the variation of $\delta x(t)$ with respect to the stationary eigen-frame fixed by eigenvectors at the point $x(0)$, the object of dynamical significance is the co-moving eigen-frame defined below in (5.9).

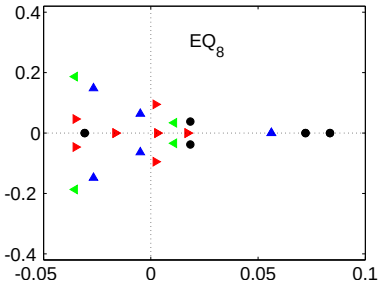

Figure A4.1: Eigenvalues of the plane Couette flow equilibrium EQ_8, plotted according to their isotropy groups: ● $+++$, the S-invariant subspace, ▶ $+--$, ◀ $-+-$, and ▲ $--+$, where $\pm$ symbols stand for symmetric/antisymmetric under symmetry operation s_1, s_2, and s_3 respectively, defined in ref. [13.42]. For tables of numerical values of stability eigenvalues see `Channelflow.org`.

A4.3 Eigenspectra: what to make out of them?

> Well Mack the Finger said to Louie the King
> I got forty red white and blue shoe strings
> And a thousand telephones that don't ring
> Do you know where I can get rid of these things?
> — Bob Dylan, *Highway 61 Revisited*

Table A4.1, taken from ref. [13.43], is an example of how to tabulate the leading Floquet eigenvalues of the stability matrix of an equilibrium or relative equilibrium. The isotropy subgroup $G_{EQ}^{(j)}$ of the corresponding eigenfunction should be indicated. If the isotropy is trivial, $G_{EQ}^{(j)} = \{e\}$, it is omitted from the table. The isotropy subgroup G_{EQ} of the solution itself needs to be noted, and for relative equilibrium (12.19) the velocity c along the group orbit. In addition, if the least stable (i.e., the most unstable) eigenvalue is complex, it is helpful to state the period of the spiral-out motion (or spiral-in, if stable), $T_{EQ} = 2\pi/\omega_{EQ}^{(1)}$.

Table A4.2, taken from ref. [**?**], is an example of how to tabulate the leading Floquet exponents of the monodromy matrix of an periodic orbit or relative periodic orbit. For a periodic orbit one states the period T_p, $\Lambda_p = \prod \Lambda_{p,e}$, and the isotropy group G_p of the orbit; for a relative periodic orbit (12.23) one states in addition the shift parameters $\phi = (\phi_1, \phi_2, \cdots \phi_N)$. Λ_p, the product of expanding Floquet multipliers (5.6) is useful, as $1/|\Lambda_p|$ is the geometric weight of cycle p in a cycle expansion (remember that each complex eigenvalue contributes twice). We often do care about $\sigma_p^{(j)} = \Lambda_{p,j}/|\Lambda_{p,j}| \in \{+1, -1\}$, the sign of the jth Floquet multiplier, or, if $\Lambda_{p,j}$ is complex, its phase $T_p\omega_p^{(j)}$.

Surveying this multitude of equilibrium and Floquet exponents is aided by a plot of the complex exponent plane (μ, ω). An example are the eigenvalues of equilibrium EQ_8 from ref. [13.42], plotted in figure A4.1. To decide how many of the these are "physical" in the PDE case (where number of exponents is always infinite, in principle), it is useful to look at the $(j, \mu^{(j)})$ plot. However, intelligent choice of the j-axis units can be tricky for high-dimensional problems. For Kuramoto-Sivashinsky system the correct choice are the wave-numbers which, due to the O(2) symmetry, come in pairs. For plane Couette flow the good choice is not known as yet; one needs to group $\mathrm{O}(2) \times \mathrm{O}(2)$ wave-numbers, as well as take care of the wall-normal node counting.

Table A4.2: The first 13 least stable Floquet exponents $\lambda = \mu \pm i\omega$ of periodic orbit p = P59.77 for plane Couette flow, $Re = 400$, together with the symmetries of corresponding eigenvectors. The eigenvalues are ordered by the decreasing real part. The one zero eigenvalue, to the numerical precision of our computation, arises from the spanwise translational SO(2) symmetry of this periodic orbit. For details, see ref. [**?**].

j	$\sigma_p^{(j)}$	$\mu_p^{(j)}$	$\omega_p^{(j)}$	$G_p^{(j)}$
1,2		0.07212161	0.04074989	D_1
3	1	0.06209526		?
4	-1	0.06162059		
5,6		0.02073075	0.07355143	
7	-1	0.009925378		
8,9		0.009654012	0.04551274	
10,11		0.009600794	0.2302166	

A4.4 Stability of Hamiltonian flows

(M.J. Feigenbaum and P. Cvitanović)

The symplectic structure of Hamilton's equations buys us much more than the incompressibility, or the phase space volume conservation alluded to in sect. 8.1. The evolution equations for any p, q dependent quantity $Q = Q(q, p)$ are given by (19.28).

In terms of the Poisson brackets, the time-evolution equation for $Q = Q(q, p)$ is given by (19.30). We now recast the symplectic condition (8.9) in a form convenient for using the symplectic constraints on M. Writing $x(t) = x' = [p', q']$ and the Jacobian matrix and its inverse

$$M = \begin{bmatrix} \frac{\partial q'}{\partial q} & \frac{\partial q'}{\partial p} \\ \frac{\partial p'}{\partial q} & \frac{\partial p'}{\partial p} \end{bmatrix}, \qquad M^{-1} = \begin{bmatrix} \frac{\partial q}{\partial q'} & \frac{\partial q}{\partial p'} \\ \frac{\partial p}{\partial q'} & \frac{\partial p}{\partial p'} \end{bmatrix}, \tag{A4.35}$$

we can spell out the symplectic invariance condition (8.9):

$$\begin{aligned} \frac{\partial q'_k}{\partial q_i}\frac{\partial p'_k}{\partial q_j} - \frac{\partial p'_k}{\partial q_i}\frac{\partial q'_k}{\partial q_j} &= 0 \\ \frac{\partial q'_k}{\partial p_i}\frac{\partial p'_k}{\partial p_j} - \frac{\partial p'_k}{\partial p_i}\frac{\partial q'_k}{\partial p_j} &= 0 \\ \frac{\partial q'_k}{\partial q_i}\frac{\partial p'_k}{\partial p_j} - \frac{\partial p'_k}{\partial q_i}\frac{\partial q'_k}{\partial p_j} &= \delta_{ij}\,. \end{aligned} \tag{A4.36}$$

From (8.26) we obtain

$$\frac{\partial q_i}{\partial q'_j} = \frac{\partial p'_j}{\partial p_i}\,, \quad \frac{\partial p_i}{\partial p'_j} = \frac{\partial q'_j}{\partial q_i}\,, \quad \frac{\partial q_i}{\partial p'_j} = -\frac{\partial q'_j}{\partial p_i}\,, \quad \frac{\partial p_i}{\partial q'_j} = -\frac{\partial p'_j}{\partial q_i}\,. \tag{A4.37}$$

Taken together, (A4.37) and (A4.36) imply that the flow conserves the $\{p, q\}$ Pois-

son brackets

$$\begin{aligned} \{q_i, q_j\} &= \frac{\partial q_i}{\partial p'_k}\frac{\partial q_j}{\partial q'_k} - \frac{\partial q_j}{\partial p'_k}\frac{\partial q_i}{\partial q'_k} = 0 \\ \{p_i, p_j\} &= 0\,, \qquad \{p_i, q_j\} = \delta_{ij}\,, \end{aligned} \tag{A4.38}$$

i.e., the transformations induced by a Hamiltonian flow are *canonical*, preserving the form of the equations of motion. The first two relations are symmetric under i, j interchange and yield $D(D-1)/2$ constraints each; the last relation yields D^2 constraints. Hence only $(2D)^2 - 2D(D-1)/2 - D^2 = d(2D+1)$ elements of M are linearly independent, as it behooves group elements of the symplectic group $Sp(2D)$.

A4.5 Monodromy matrix for Hamiltonian flows

(G. Tanner)

It is not the Jacobian matrix J of the flow (4.5), but the *monodromy matrix* M, which enters the trace formula. This matrix gives the time dependence of a displacement perpendicular to the flow on the energy manifold. Indeed, we discover some trivial parts in the Jacobian matrix J. An initial displacement in the direction of the flow $x = \omega\nabla H(x)$ transfers according to $\delta x(t) = x_t(t)\delta t$ with δt time independent. The projection of any displacement on δx on $\nabla H(x)$ is constant, i.e., $\nabla H(x(t))\delta x(t) = \delta E$. We get the equations of motion for the monodromy matrix directly choosing a suitable local coordinate system on the orbit $x(t)$ in form of the (non singular) transformation $\mathbf{U}(x(t))$:

$$\tilde{J}(x(t)) = \mathbf{U}^{-1}(x(t))\, J(x(t))\, \mathbf{U}(x(0)) \tag{A4.39}$$

These lead to

$$\begin{aligned} \dot{\tilde{J}} &= \tilde{\mathbf{L}}\,\tilde{J} \\ \text{with} \qquad \tilde{\mathbf{L}} &= \mathbf{U}^{-1}(\mathbf{L}\mathbf{U} - \dot{\mathbf{U}}) \end{aligned} \tag{A4.40}$$

Note that the properties a) – c) are only fulfilled for $\tilde{J}$ and $\tilde{\mathbf{L}}$ if $\mathbf{U}$ itself is symplectic.

Choosing $x_E = \nabla H(t)/|\nabla H(t)|^2$ and x_t as local coordinates uncovers the two trivial eigenvalues 1 of the transformed matrix in (A4.39) at any time t. Setting $\mathbf{U} = ({x_t}^\top, {x_E}^\top, {x_1}^\top, \ldots, {x_{2d-2}}^\top)$ gives

$$\tilde{J} = \begin{pmatrix} 1 & * & * & \ldots & * \\ 0 & 1 & 0 & \ldots & 0 \\ 0 & * & & & \\ \vdots & \vdots & & \mathbf{M} & \\ 0 & * & & & \end{pmatrix}; \qquad \tilde{\mathbf{L}} = \begin{pmatrix} 0 & * & * & \ldots & * \\ 0 & 0 & 0 & \ldots & 0 \\ 0 & * & & & \\ \vdots & \vdots & & \mathbf{l} & \\ 0 & * & & & \end{pmatrix}, \tag{A4.41}$$

The matrix $\mathbf{M}$ is now the monodromy matrix and the equation of motion are given by

$$\dot{\mathbf{M}} = \mathbf{l}\,\mathbf{M}. \tag{A4.42}$$

The vectors $x_1, \ldots, x_{2d-2}$ must span the space perpendicular to the flow on the energy manifold.

For a system with two degrees of freedom, the matrix $\mathbf{U}(\mathbf{t})$ can be written down explicitly, i.e.,

$$\mathbf{U}(t) = (x_t, x_1, x_E, x_2) = \begin{pmatrix} \dot{x} & -\dot{y} & -\dot{u}/q^2 & -\dot{v}/q^2 \\ \dot{y} & \dot{x} & -\dot{v}/q^2 & \dot{u}/q^2 \\ \dot{u} & \dot{v} & \dot{x}/q^2 & -\dot{y}/q^2 \\ \dot{v} & -\dot{u} & \dot{y}/q^2 & \dot{x}/q^2 \end{pmatrix} \tag{A4.43}$$

with $x^\top = (x, y; u, v)$ and $q = |\nabla H| = |\dot{x}|$. The matrix $\mathbf{U}$ is non singular and symplectic at every phase space point x, except the equilibrium points $\dot{x} = 0$. The matrix elements for $\mathbf{l}$ are given (A4.45). One distinguishes 4 classes of eigenvalues of $\mathbf{M}$.

- *stable* or *elliptic*, if $\Lambda = e^{\pm i\pi\nu}$ and $\nu \in]0, 1[$.
- *marginal*, if $\Lambda = \pm 1$.
- *hyperbolic*, *inverse hyperbolic*, if $\Lambda = e^{\pm\lambda}$, $\Lambda = -e^{\pm\lambda}$.
- *loxodromic*, if $\Lambda = e^{\pm\mu\pm i\omega}$ with μ and ω real. This is the most general case, possible only in systems with 3 or more degree of freedoms.

For 2 degrees of freedom, i.e., $\mathbf{M}$ is a [2×2] matrix, the eigenvalues are determined by

$$\lambda = \frac{\mathrm{tr}\,(\mathbf{M}) \pm \sqrt{\mathrm{tr}\,(\mathbf{M})^2 - 4}}{2}, \tag{A4.44}$$

i.e., $\mathrm{tr}\,(\mathbf{M}) = 2$ separates stable and unstable behavior.

The $\mathbf{l}$ matrix elements for the local transformation (A4.43) are

$$\begin{aligned}
\tilde{\mathbf{l}}_{11} &= \frac{1}{q}[(h_x^2 - h_y^2 - h_u^2 + h_v^2)(h_{xu} - h_{yv}) + 2(h_x h_y - h_u h_v)(h_{xv} + h_{yu}) \\
&\quad -(h_x h_u + h_y h_v)(h_{xx} + h_{yy} - h_{uu} - h_{vv})] \\
\tilde{\mathbf{l}}_{12} &= \frac{1}{q^2}[(h_x^2 + h_v^2)(h_{yy} + h_{uu}) + (h_y^2 + h_u^2)(h_{xx} + h_{vv}) \\
&\quad -2(h_x h_u + h_y h_v)(h_{xu} + h_{yv}) - 2(h_x h_y - h_u h_v)(h_{xy} - h_{uv})] \\
\tilde{\mathbf{l}}_{21} &= -(h_x^2 + h_y^2)(h_{uu} + h_{vv}) - (h_u^2 + h_v^2)(h_{xx} + h_{yy}) \\
&\quad +2(h_x h_u - h_y h_v)(h_{xu} - h_{yv}) + 2(h_x h_v + h_y h_u)(h_{xv} + h_{yu}) \\
\tilde{\mathbf{l}}_{22} &= -\tilde{\mathbf{l}}_{11},
\end{aligned} \tag{A4.45}$$

with h_i, h_{ij} is the derivative of the Hamiltonian H with respect to the phase space coordinates and $q = |\nabla H|^2$.

Exercises

A4.1. **Real representation of complex eigenvalues.** (Verification of example A4.2.) λ_k, λ_{k+1} eigenvalues form a complex conjugate pair, $\{\lambda_k, \lambda_{k+1}\} = \{\mu + i\omega, \mu - i\omega\}$. Show that

(a) corresponding projection operators are complex conjugates of each other,

$$\mathbf{P} = \mathbf{P}_k\,, \qquad \mathbf{P}^* = \mathbf{P}_{k+1}\,,$$

where we denote $\mathbf{P}_k$ by $\mathbf{P}$ for notational brevity.

(b) $\mathbf{P}$ can be written as

$$\mathbf{P} = \frac{1}{2}(\mathbf{R} + i\mathbf{Q})\,,$$

where $\mathbf{R} = \mathbf{P}_k + \mathbf{P}_{k+1}$ and $\mathbf{Q}$ are matrices with real elements.

(c)

$$\begin{bmatrix}\mathbf{P}_k & \mathbf{P}_{k+1}\end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix}\begin{bmatrix}\mathbf{R} & \mathbf{Q}\end{bmatrix}.$$

(d) $\cdots + \lambda_k\mathbf{P}_k + \lambda_k^*\mathbf{P}_{k+1} + \cdots$ complex eigenvalue pair in the spectral decomposition (A4.16) is now replaced by a real [2×2] matrix

$$\cdots + \begin{bmatrix} \mu & -\omega \\ \omega & \mu \end{bmatrix}\begin{bmatrix}\mathbf{R} & \mathbf{Q}\end{bmatrix} + \cdots$$

or whatever you find the clearest way to write this real representation.

Chapter A4 solutions: Linear stability

Solution A4.1 - Real representation of complex eigenvalues.

$$\frac{1}{2}\begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}\begin{bmatrix} \lambda & 0 \\ 0 & \lambda^* \end{bmatrix}\begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix} = \begin{bmatrix} \mu & -\omega \\ \omega & \mu \end{bmatrix}.$$

(P. Cvitanović)