Chapter 5

Cycle stability

TOPOLOGICAL FEATURES of a dynamical system –singularities, periodic orbits, and the ways in which the orbits intertwine– are invariant under a general continuous change of coordinates. Surprisingly, there also exist quantities that depend on the notion of metric distance between points, but nevertheless do not change value under a smooth change of coordinates. Local quantities such as the eigenvalues of equilibria and periodic orbits, and global quantities such as Lyapunov exponents, metric entropy, and fractal dimensions are examples of properties of dynamical systems independent of coordinate choice.

We now turn to the first, local class of such invariants, linear stability of periodic orbits of flows and maps. This will give us metric information about local dynamics. If you already know that the eigenvalues of periodic orbits are invariants of a flow, skip this chapter.

fast track:
chapter 7, p. 121

5.1 Stability of periodic orbits

As noted on page 40, a trajectory can be stationary, periodic or aperiodic. For chaotic systems almost all trajectories are aperiodic–nevertheless, equilibria and periodic orbits turn out to be the key to unraveling chaotic dynamics. Here we note a few of the properties that make them so precious to a theorist.

An obvious virtue of periodic orbits is that they are *topological* invariants: a fixed point remains a fixed point for any choice of coordinates, and similarly a periodic orbit remains periodic in any representation of the dynamics. Any reparametrization of a dynamical system that preserves its topology has to preserve topological relations between periodic orbits, such as their relative inter-windings

and knots. So the mere existence of periodic orbits suffices to partially organize the spatial layout of a non–wandering set. No less important, as we shall now show, is the fact that cycle eigenvalues are *metric* invariants: they determine the relative sizes of neighborhoods in a non–wandering set.

We start by noting that due to the multiplicative structure (4.44) of Jacobian matrices, the Jacobian matrix for the rth repeat of a prime cycle p of period T_p is

$$J^{rT_p}(x) = J^{T_p}(f^{(r-1)T_p}(x)) \cdots J^{T_p}(f^{T_p}(x)) J^{T_p}(x) = J_p(x)^r \,, \tag{5.1}$$

where $J_p(x) = J^{T_p}(x)$ is the Jacobian matrix for a single traversal of the prime cycle p, $x \in \mathcal{M}_p$ is any point on the cycle, and $f^{rT_p}(x) = x$ as $f^t(x)$ returns to x every multiple of the period T_p. Hence, it suffices to restrict our considerations to the stability of prime cycles.

fast track:
sect. 5.2, p. 99

5.1.1 Floquet vectors

When dealing with periodic orbits, some of the quantities already introduced inherit names from the Floquet theory of differential equations with time-periodic coefficients. Consider the equation of variations (4.2) evaluated on a periodic orbit p,

$$\dot{\delta x} = A(t)\,\delta x \,, \qquad A(t) = A(x(t)) = A(t + T_p)\,. \tag{5.2}$$

The T_p periodicity of the stability matrix implies that if $\delta x(t)$ is a solution of (5.2) then also $\delta x(t + T_p)$ satisfies the same equation: moreover the two solutions are related by (4.6)

$$\delta x(t + T_p) = J_p(x)\,\delta x(t)\,. \tag{5.3}$$

Even though the Jacobian matrix $J_p(x)$ depends upon x (the 'starting' point of the periodic orbit), we shall show in sect. 5.2 that its eigenvalues do not, so we may write for its eigenvectors $\mathbf{e}^{(j)}$ (sometimes referred to as 'covariant Lyapunov vectors,' or, for periodic orbits, as 'Floquet vectors')

$$J_p(x)\,\mathbf{e}^{(j)}(x) = \Lambda_{p,j}\,\mathbf{e}^{(j)}(x)\,, \qquad \Lambda_{p,j} = \sigma_p^{(j)} e^{\lambda_p^{(j)} T_p}\,. \tag{5.4}$$

where $\lambda_p^{(j)} = \mu_p^{(j)} \pm i\omega_p^{(j)}$ and $\sigma_p^{(j)}$ are independent of x. When $\Lambda_{p,j}$ is real, we do care about $\sigma_p^{(j)} = \Lambda_{p,j}/|\Lambda_{p,j}| \in \{+1, -1\}$, the sign of the jth Floquet multiplier.

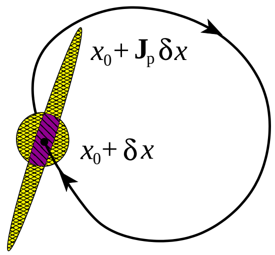

Figure 5.1: For a prime cycle p, Floquet matrix J_p returns an infinitesimal spherical neighborhood of $x_0 \in \mathcal{M}_p$ stretched into an ellipsoid, with overlap ratio along the eigdirection $\mathbf{e}^{(i)}$ of $J_p(x)$ given by the Floquet multiplier $|\Lambda_{p,i}|$. These ratios are invariant under smooth nonlinear reparametrizations of state space coordinates, and are intrinsic property of cycle p.

If $\sigma_p^{(j)} = -1$ and $\lambda_p^{(j)} \neq 0$, the corresponding eigen-direction is said to be *inverse hyperbolic*. Keeping track of this by case-by-case enumeration is an unnecessary nuisance, so most of our formulas will be stated in terms of the Floquet multipliers Λ_j rather than in the terms of the multiplier signs $\sigma^{(j)}$, exponents $\mu^{(j)}$ and phases $\omega^{(j)}$. section 7.2

Expand δx in the (5.4) eigenbasis, $\delta x(t) = \sum \delta x_j(t)\,\mathbf{e}^{(j)}$, $\mathbf{e}^{(j)} = \mathbf{e}^{(j)}(x(0))$. Taking into account (5.3), we get that $\delta x_j(t)$ is multiplied by $\Lambda_{p,j}$ per each period

$$\delta x(t+T_p) = \sum_j \delta x_j(t+T_p)\,\mathbf{e}^{(j)} = \sum_j \Lambda_{p,j}\,\delta x_j(t)\,\mathbf{e}^{(j)} .$$

We can absorb this exponential growth / contraction by rewriting the coefficients $\delta x_j(t)$ as

$$\delta x_j(t) = e^{\lambda_p^{(j)} t}\, u_j(t) , \qquad u_j(0) = \delta x_j(0) ,$$

with $u_j(t)$ *periodic* with period T_p. Thus each solution of the equation of variations (4.2) may be expressed in the Floquet form

$$\delta x(t) = \sum_j e^{\lambda_p^{(j)} t}\, u_j(t)\,\mathbf{e}^{(j)} , \qquad u_j(t+T_p) = u_j(t) . \tag{5.5}$$

The continuous time t appearing in (5.5) does not imply that eigenvalues of the Jacobian matrix enjoy any multiplicative property for $t \neq rT_p$: $\lambda_p^{(j)} = \mu_p^{(j)} \pm i\omega_p^{(j)}$ refer to a full traversal of the periodic orbit. Indeed, while $u_j(t)$ describes the variation of $\delta x(t)$ with respect to the stationary eigen-frame fixed by eigenvectors at the point $x(0)$, the object of real interest is the co-moving eigen-frame defined below in (5.13).

5.1.2 Floquet matrix eigenvalues and exponents

The time-dependent T-periodic vector fields, such as the flow linearized around the periodic orbit, are described by Floquet theory. Hence from now on we shall

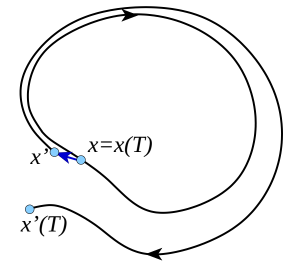

Figure 5.2: An unstable periodic orbit repels every neighboring trajectory $x'(t)$, except those on its center and unstable manifolds.

refer to a Jacobian matrix evaluated on a periodic orbit as a Floquet matrix, to its eigenvalues $\Lambda_{p,j}$ as Floquet multipliers (5.4), and to $\lambda_p^{(j)} = \mu_p^{(j)} + i\omega_p^{(j)}$ as Floquet or characteristic exponents. We sort the *Floquet multipliers* $\{\Lambda_{p,1}, \Lambda_{p,2}, \ldots, \Lambda_{p,d}\}$ of the $[d\times d]$ Floquet matrix J_p evaluated on the p-cycle into sets $\{e, m, c\}$

$$\begin{aligned}
\text{expanding:}\quad & \{\Lambda\}_e = \{\Lambda_{p,j} : \left|\Lambda_{p,j}\right| > 1\} \\
\text{marginal:}\quad & \{\Lambda\}_m = \{\Lambda_{p,j} : \left|\Lambda_{p,j}\right| = 1\} \\
\text{contracting:}\quad & \{\Lambda\}_c = \{\Lambda_{p,j} : \left|\Lambda_{p,j}\right| < 1\} .
\end{aligned} \tag{5.6}$$

and denote by Λ_p (no jth eigenvalue index) the product of *expanding* Floquet multipliers

$$\Lambda_p = \prod_e \Lambda_{p,e} . \tag{5.7}$$

As J_p is a real matrix, complex eigenvalues always come in complex conjugate pairs, $\Lambda_{p,i+1} = \Lambda_{p,i}^*$, so the product (5.7) is always real.

The stretching/contraction rates per unit time are given by the real parts of Floquet exponents

$$\mu_p^{(i)} = \frac{1}{T_p} \ln \left|\Lambda_{p,i}\right| . \tag{5.8}$$

The factor $1/T_p$ in the definition of the Floquet exponents is motivated by its form for the linear dynamical systems, for example (4.16), as well as the fact that exponents so defined can be interpreted as Lyapunov exponents (17.33) evaluated on the prime cycle p. As in the three cases of (5.6), we sort the Floquet exponents $\lambda = \mu \pm i\omega$ into three sets section 17.3

$$\begin{aligned}
\text{expanding:}\quad & \{\lambda\}_e = \{\lambda_p^{(i)} : \mu_p^{(i)} > 0\} \\
\text{marginal:}\quad & \{\lambda\}_m = \{\lambda_p^{(i)} : \mu_p^{(i)} = 0\} \\
\text{contracting:}\quad & \{\lambda\}_c = \{\lambda_p^{(i)} : \mu_p^{(i)} < 0\} .
\end{aligned} \tag{5.9}$$

A periodic orbit p of a d-dimensional flow or a map is *stable* if real parts of all of its Floquet exponents (other than the vanishing longitudinal exponent, explained in sect. 5.2.1) are strictly negative, $\mu_p^{(i)} < 0$. The region of system parameter values for which a periodic orbit p is stable is called the *stability window* of p. The set $\mathcal{M}_p$ of initial points that are asymptotically attracted to p as $t \to +\infty$ (for a fixed set of system parameter values) is called the *basin of attraction* of p. If *all* Floquet exponents (other than the vanishing longitudinal exponent) are strictly positive, $\mu^{(i)} \geq \mu_{min} > 0$, the cycle is *repelling*, and unstable to any perturbation. If some are strictly positive, and rest strictly negative, $-\mu^{(i)} \geq \mu_{min} > 0$, the cycle is said to be *hyperbolic* or a *saddle*, and unstable to perturbations outside its stable manifold. Repelling and hyperbolic cycles are unstable to generic perturbations, and thus said to be *unstable*, see figure 5.2. If all $\mu^{(i)} = 0$, the orbit is said to be *elliptic*, and if $\mu^{(i)} = 0$ for a subset of exponents (other than the longitudinal one), the orbit is said to be *partially hyperbolic*. Such orbits proliferate in Hamiltonian flows. section 7.3

If *all* Floquet exponents (other than the vanishing longitudinal exponent) of *all* periodic orbits of a flow are strictly bounded away from zero, the flow is said to be *hyperbolic*. Otherwise the flow is said to be *nonhyperbolic*.

Example 5.1 ***Stability of cycles of 1-dimensional maps:*** *The stability of a prime cycle p of a 1-dimensional map follows from the chain rule (4.51) for stability of the n_pth iterate of the map*

$$\Lambda_p = \frac{d}{dx_0} f^{n_p}(x_0) = \prod_{m=0}^{n_p-1} f'(x_m)\,, \quad x_m = f^m(x_0)\,. \tag{5.10}$$

Λ_p is a property of the cycle, not the initial periodic point, as taking any periodic point in the p cycle as the initial one yields the same Λ_p.

A critical point *x_c is a value of x for which the mapping $f(x)$ has vanishing derivative, $f'(x_c) = 0$. A periodic orbit of a 1-dimensional map is* stable *if*

$$\left|\Lambda_p\right| = \left|f'(x_{n_p}) f'(x_{n_p-1}) \cdots f'(x_2) f'(x_1)\right| < 1\,,$$

and superstable *if the orbit includes a critical point, so that the above product vanishes. For a stable periodic orbit of period n the slope Λ_p of the nth iterate $f^n(x)$ evaluated on a periodic point x (fixed point of the nth iterate) lies between -1 and 1. If $\left|\Lambda_p\right| > 1$, p-cycle is* unstable.

Example 5.2 ***Stability of cycles for maps:*** *No matter what method we use to determine the unstable cycles, the theory to be developed here requires that their Floquet multipliers be evaluated as well. For maps a Floquet matrix is easily evaluated by picking any periodic point as a starting point, running once around a prime cycle, and multiplying the individual periodic point Jacobian matrices according to (4.52). For example, the Floquet matrix M_p for a Hénon map (3.19) prime cycle p of length n_p is given by (4.53),*

$$M_p(x_0) = \prod_{k=n_p}^{1} \begin{pmatrix} -2ax_k & b \\ 1 & 0 \end{pmatrix}, \qquad x_k \in \mathcal{M}_p\,,$$

and the Floquet matrix M_p for a 2-dimensional billiard prime cycle p of length n_p

$$M_p = (-1)^{n_p} \prod_{k=n_p}^{1} \begin{pmatrix} 1 & \tau_k \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ r_k & 1 \end{pmatrix}$$

follows from (8.11) of chapter 8 below. The decreasing order in the indices of the products in above formulas is a reminder that the successive time steps correspond to multiplication from the left, $M_p(x_1) = M(x_{n_p}) \cdots M(x_1)$. We shall compute Floquet multipliers of Hénon map cycles once we learn how to find their periodic orbits, see exercise 13.13.

5.2 Floquet multipliers are invariant

The 1-dimensional map Floquet multiplier (5.10) is a product of derivatives over all points around the cycle, and is therefore independent of which periodic point is chosen as the initial one. In higher dimensions the form of the Floquet matrix $J_p(x_0)$ in (5.1) does depend on the choice of coordinates and the initial point $x_0 \in \mathcal{M}_p$. Nevertheless, as we shall now show, the cycle *Floquet multipliers* are intrinsic property of a cycle in any dimension. Consider the ith eigenvalue, eigenvector pair $(\Lambda_{p,i}, \mathbf{e}^{(i)})$ computed from J_p evaluated at a periodic point x,

$$J_p(x)\,\mathbf{e}^{(i)}(x) = \Lambda_{p,i}\,\mathbf{e}^{(i)}(x)\,, \quad x \in \mathcal{M}_p\,. \tag{5.11}$$

Consider another point on the cycle at time t later, $x' = f^t(x)$ whose Floquet matrix is $J_p(x')$. By the group property (4.44), $J^{T_p+t} = J^{t+T_p}$, and the Jacobian matrix at x' can be written either as

$$J^{T_p+t}(x) = J^{T_p}(x')\,J^t(x) = J_p(x')\,J^t(x)\,,$$

or $J^t(x)\,J_p(x)$. Multiplying (5.11) by $J^t(x)$, we find that the Floquet matrix evaluated at x' has the same Floquet multiplier,

$$J_p(x')\,\mathbf{e}^{(i)}(x') = \Lambda_{p,i}\,\mathbf{e}^{(i)}(x')\,, \quad \mathbf{e}^{(i)}(x') = J^t(x)\,\mathbf{e}^{(i)}(x)\,, \tag{5.12}$$

but with the eigenvector $\mathbf{e}^{(i)}$ transported along the flow $x \to x'$ to $\mathbf{e}^{(i)}(x') = J^t(x)\,\mathbf{e}^{(i)}(x)$. Hence, in the spirit of the Floquet theory (5.5) one can define time-periodic unit eigenvectors (in a co-moving 'Lagrangian frame')

$$\mathbf{e}^{(j)}(t) = e^{-\lambda_p^{(j)} t}\,J^t(x)\,\mathbf{e}^{(j)}(0)\,, \qquad \mathbf{e}^{(j)}(t) = \mathbf{e}^{(j)}(x(t))\,, \quad x(t) \in \mathcal{M}_p\,. \tag{5.13}$$

J_p evaluated anywhere along the cycle has the same set of Floquet multipliers $\{\Lambda_{p,1}, \Lambda_{p,2}, \cdots, 1, \cdots, \Lambda_{p,d-1}\}$. As quantities such as $\operatorname{tr} J_p(x)$, $\det J_p(x)$ depend only on the eigenvalues of $J_p(x)$ and not on the starting point x, in expressions such as $\det\left(\mathbf{1} - J_p^r(x)\right)$ we may omit reference to x,

$$\det\left(\mathbf{1} - J_p^r\right) = \det\left(\mathbf{1} - J_p^r(x)\right) \quad \text{for any } x \in \mathcal{M}_p\,. \tag{5.14}$$

We postpone the proof that the cycle Floquet multipliers are smooth conjugacy invariants of the flow to sect. 6.6.

5.2.1 Marginal eigenvalues

The presence of marginal eigenvalues signals either a continuous symmetry of the flow (which one should immediately exploit to simplify the problem), or a non-hyperbolicity of a flow (a source of much pain, hard to avoid). In that case (typical of parameter values for which bifurcations occur) one has to go beyond linear stability, deal with Jordan type subspaces (see example 4.4), and sub-exponential growth rates, such as t^α.

chapter 24
exercise 5.1

For flow-invariant solutions such as periodic orbits, the time evolution is itself a continuous symmetry, hence a periodic orbit of a flow always has a *marginal Floquet multiplier*:

As $J^t(x)$ transports the velocity field $v(x)$ by (4.7), after a complete period

$$J_p(x)\, v(x) = v(x)\,, \tag{5.15}$$

so for a periodic orbit of a *flow* the local velocity field is always has an eigenvector $\mathbf{e}^{(\|)}(x) = v(x)$ with the unit Floquet multiplier,

$$\Lambda_{p,\|} = 1\,, \qquad \lambda_p^{(\|)} = 0\,. \tag{5.16}$$

exercise 6.3

The continuous invariance that gives rise to this marginal Floquet multiplier is the invariance of a cycle (the set $\mathcal{M}_p$) under a translation of its points along the cycle: two points on the cycle (see figure 4.3) initially distance δx apart, $x'(0) - x(0) = \delta x(0)$, are separated by the exactly same δx after a full period T_p. As we shall see in sect. 5.3, this marginal stability direction can be eliminated by cutting the cycle by a Poincaré section and eliminating the continuous flow Floquet matrix in favor of the Floquet matrix of the Poincaré return map.

If the flow is governed by a time-independent Hamiltonian, the energy is conserved, and that leads to an additional marginal Floquet multiplier (we shall show in sect. 7.3 that due to the symplectic invariance (7.19) real eigenvalues come in pairs). Further marginal eigenvalues arise in presence of continuous symmetries, as discussed in chapter 10 below.

5.3 Stability of Poincaré map cycles

(R. Paškauskas and P. Cvitanović)

If a continuous flow periodic orbit p pierces the Poincaré section $\mathcal{P}$ once, the section point is a fixed point of the Poincaré return map P with stability (4.57)

$$\hat{J}_{ij} = \left(\delta_{ik} - \frac{v_i\, U_k}{(v \cdot U)}\right) J_{kj}\,, \tag{5.17}$$

with all primes dropped, as the initial and the final points coincide, $x' = f^{T_p}(x) = x$. If the periodic orbit p pierces the Poincaré section n times, the same observation applies to the nth iterate of P.

We have already established in (4.58) that the velocity $v(x)$ is a zero eigenvector of the Poincaré section Floquet matrix, $\hat{J}\, v = 0$. Consider next $(\Lambda_{p,\alpha}, \mathbf{e}^{(\alpha)})$, the full state space αth (eigenvalue, eigenvector) pair (5.11), evaluated at a periodic point on a Poincaré section,

$$J(x)\, \mathbf{e}^{(\alpha)}(x) = \Lambda_\alpha\, \mathbf{e}^{(\alpha)}(x)\,, \quad x \in \mathcal{P}\,. \tag{5.18}$$

Multiplying (5.17) by $\mathbf{e}^{(\alpha)}$ and inserting (5.18), we find that the full state space Floquet matrix and the Poincaré section Floquet matrix $\hat{J}$ have the same Floquet multiplier

$$\hat{J}(x)\, \hat{\mathbf{e}}^{(\alpha)}(x) = \Lambda_\alpha\, \hat{\mathbf{e}}^{(\alpha)}(x)\,, \quad x \in \mathcal{P}\,, \tag{5.19}$$

where $\hat{\mathbf{e}}^{(\alpha)}$ is a projection of the full state space eigenvector onto the Poincaré section:

$$(\hat{\mathbf{e}}^{(\alpha)})_i = \left(\delta_{ik} - \frac{v_i\, U_k}{(v \cdot U)}\right) (\mathbf{e}^{(\alpha)})_k\,. \tag{5.20}$$

Hence, $\hat{J}_p$ evaluated on any Poincaré section point along the cycle p has the same set of Floquet multipliers $\{\Lambda_{p,1}, \Lambda_{p,2}, \cdots \Lambda_{p,d}\}$ as the full state space Floquet matrix J_p, except for the marginal unit Floquet multiplier (5.16).

As established in (4.58), due to the continuous symmetry (time invariance) $\hat{J}_p$ is a rank $d-1$ matrix. We shall refer to any such rank $[(d-1-N)\times(d-1-N)]$ submatrix with $N-1$ continuous symmetries quotiented out as the *monodromy matrix* M_p (from Greek *mono-* = alone, single, and *dromo* = run, racecourse, meaning a single run around the stadium). Quotienting continuous symmetries is discussed in chapter 10 below.

5.4 There goes the neighborhood

In what follows, our task will be to determine the size of a *neighborhood* of $x(t)$, and that is why we care about the Floquet multipliers, and especially the unstable (expanding) ones. Nearby points aligned along the stable (contracting) directions remain in the neighborhood of the trajectory $x(t) = f^t(x_0)$; the ones to keep an eye on are the points which leave the neighborhood along the unstable directions. The sub-volume $|\mathcal{M}_i| = \prod_i^e \Delta x_i$ of the set of points which get no further away from $f^t(x_0)$ than L, the typical size of the system, is fixed by the condition that $\Delta x_i \Lambda_i = O(L)$ in each expanding direction i. Hence the neighborhood size scales as $\propto 1/|\Lambda_p|$ where Λ_p is the product of expanding Floquet multipliers (5.7) only; contracting ones play a secondary role.

So the dynamically important information is carried by the expanding sub-volume, not the total volume computed so easily in (4.47). That is also the reason why the dissipative and the Hamiltonian chaotic flows are much more alike than one would have naively expected for 'compressible' *vs.* 'incompressible' flows. In hyperbolic systems what matters are the expanding directions. Whether the contracting eigenvalues are inverses of the expanding ones or not is of secondary importance. As long as the number of unstable directions is finite, the same theory applies both to the finite-dimensional ODEs and infinite-dimensional PDEs.

Résumé

Periodic orbits play a central role in any invariant characterization of the dynamics, because (a) their existence and inter-relations are a *topological*, coordinate-independent property of the dynamics, and (b) their Floquet multipliers form an infinite set of *metric invariants*: The Floquet multipliers of a periodic orbit remain invariant under any smooth nonlinear change of coordinates $f \to h \circ f \circ h^{-1}$. Let us summarize the linearized flow notation used throughout the ChaosBook. section 6.6

Differential formulation, flows:

$$\dot{x} = v\,, \qquad \dot{\delta x} = A\,\delta x$$

governs the dynamics in the tangent bundle $(x, \delta x) \in \mathbf{T}\mathcal{M}$ obtained by adjoining the d-dimensional tangent space $\delta x \in T\mathcal{M}_x$ to every point $x \in \mathcal{M}$ in the d-dimensional state space $\mathcal{M} \subset \mathbb{R}^d$. The *stability matrix* $A = \partial v/\partial x$ describes the instantaneous rate of shearing of the infinitesimal neighborhood of $x(t)$ by the flow.

Finite time formulation, maps: A discrete sets of trajectory points $\{x_0, x_1, \cdots, x_n, \cdots\} \in \mathcal{M}$ can be generated by composing finite-time maps, either given as $x_{n+1} = f(x_n)$, or obtained by integrating the dynamical equations

$$x_{n+1} = f(x_n) = x_n + \int_{t_n}^{t_{n+1}} d\tau\, v(x(\tau))\,, \tag{5.21}$$

for a discrete sequence of times $\{t_0, t_1, \cdots, t_n, \cdots\}$, specified by some criterion such as strobing or Poincaré sections. In the discrete time formulation the dynamics in the tangent bundle $(x, \delta x) \in \mathbf{T}\mathcal{M}$ is governed by

$$x_{n+1} = f(x_n)\,, \quad \delta x_{n+1} = J(x_n)\,\delta x_n\,, \qquad J(x_n) = J^{t_{n+1}-t_n}(x_n)\,,$$

where $J(x_n) = \partial x_{n+1}/\partial x_n = \int d\tau\, \exp(A\,\tau)$ is the Jacobian matrix.

Stability of invariant solutions: The linear stability of an equilibrium $v(x_EQ) = 0$ is described by the eigenvalues and eigenvectors $\{\lambda^{(j)}, \mathbf{e}^{(j)}\}$ of the stability matrix A evaluated at the equilibrium point, and the linear stability of a periodic orbit $f^T(x) = x$, $x \in \mathcal{M}_p$,

$$J_p(x)\,\mathbf{e}^{(j)}(x) = \Lambda_{p,j}\,\mathbf{e}^{(j)}(x)\,, \qquad \Lambda_{p,j} = \sigma_p^{(j)} e^{\lambda_p^{(j)} T_p}\,,$$

by its Floquet multipliers, vectors and exponents $\{\Lambda_j, \mathbf{e}^{(j)}\}$, where $\lambda_p^{(j)} = \mu_p^{(j)} \pm i\omega_p^{(j)}$ For every continuous symmetry there is a marginal eigen-direction, with $\Lambda_{p,j} = 1$, $\lambda_p^{(j)} = 0$. With all $1+N$ continuous symmetries quotiented out (Poincaré sections for time, slices for continuous symmetries of dynamics, see sect. 10.4) linear stability of a periodic orbit (and, more generally, of a partially hyperbolic torus) is described by the $[(d\text{-}1\text{-}N)\times(d\text{-}1\text{-}N)]$ monodromy matrix, all of whose Floquet multipliers $|\Lambda_{p,j}| \neq 1$ are generically strictly hyperbolic,

$$M_p(x)\,\mathbf{e}^{(j)}(x) = \Lambda_{p,j}\,\mathbf{e}^{(j)}(x)\,, \qquad x \in \mathcal{M}_p/G\,.$$

We shall show in chapter 11 that extending the linearized stability hyperbolic eigen-directions into stable and unstable manifolds yields important global information about the topological organization of state space. What matters most are the expanding directions. The physically important information is carried by the unstable manifold, and the expanding sub-volume characterized by the product of expanding Floquet multipliers of J_p. As long as the number of unstable directions is finite, the theory can be applied to flows of arbitrarily high dimension.

in depth:
appendix B, p. 752

fast track:
chapter 9, p. 143

Commentary

Remark 5.1 Floquet theory. Study of time-dependent and T-periodic vector fields is a classical subject in the theory of differential equations [5.1, 5.2]. In physics literature Floquet exponents often assume different names according to the context where the theory is applied: they are called Bloch phases in the discussion of Schrödinger equation with a periodic potential [5.3], or quasi-momenta in the quantum theory of time-periodic Hamiltonians.

Exercises

5.1. **A limit cycle with analytic Floquet exponent.** There are only two examples of nonlinear flows for which the Floquet multipliers can be evaluated analytically. Both are cheats. One example is the 2 − *dimensional* flow

$$\begin{aligned}\dot{q} &= p + q(1 - q^2 - p^2)\\ \dot{p} &= -q + p(1 - q^2 - p^2)\,.\end{aligned}$$

Determine all periodic solutions of this flow, and determine analytically their Floquet exponents. Hint: go to polar coordinates $(q, p) = (r\cos\theta, r\sin\theta)$. G. Bard Ermentrout

5.2. **The other example of a limit cycle with analytic Floquet exponent.** What is the other example of a nonlinear flow for which the Floquet multipliers can be evaluated analytically? Hint: email G.B. Ermentrout.

5.3. **Yet another example of a limit cycle with analytic Floquet exponent.** Prove G.B. Ermentrout wrong by solving a third example (or more) of a nonlinear flow for which the Floquet multipliers can be evaluated analytically.

References

[5.1] G. Floquet, “Sur les equations differentielles lineaires à coefficients periodique,” *Ann. Ecole Norm. Ser. 2*, **12**, 47 (1883).

[5.2] E. L. Ince, *Ordinary Differential Equations* (Dover, New York 1953).

[5.3] N. W. Ashcroft and N. D. Mermin, *Solid State Physics* (Holt, Rinehart and Winston, New York 1976).