Do clouds solve PDEs?

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Georgia Tech

Feigenbaum Memorial Conference
Renormalization retrospective

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ChaosBook.org/overheads/spatiotemporal

overview

1. what this talk is about
2. turbulence in spacetime
3. space is time
4. bye bye, dynamics

do clouds solve PDEs?

do clouds integrate Navier-Stokes equations?

NO!

$\Longrightarrow$ other swirls $\Longrightarrow$

do clouds obey Navier-Stokes equations?

yes!

they satisfy them locally, everywhere and at all times

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challenge : describe turbulence

use Navier-Stokes equations

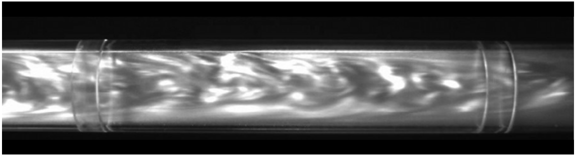
$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = \frac{1}{R}\nabla^2 \mathbf{v} - \nabla p + \mathbf{f}\,, \qquad \nabla \cdot \mathbf{v} = 0,$$

velocity field $\mathbf{v} \in \mathbb{R}^3$; pressure field p ; driving force $\mathbf{f}$

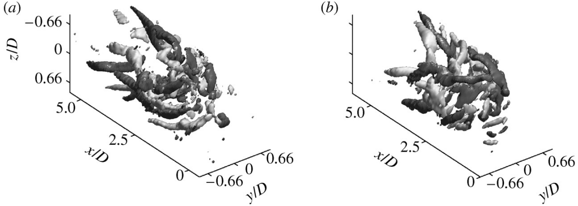
to determine the building blocks of turbulence

starting from the equations (no statistical assumptions)

challenge : experiments are amazing



T. Mullin lab



B. Hof lab

pedagogy : for plumbers we do 3D turbulence, but for this talk

Navier-Stokes equations

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = \frac{1}{R}\nabla^2 \mathbf{v} + (\cdots)$$

velocity field $\mathbf{v}(\mathbf{x}; t) \in \mathbb{R}^3$

not helpful for developing intuition

we cannot visualize 3D velocity field at every 3D spatial point

look instead at 1D ‘flame fronts’

spacetime (3+1)-dimensional Navier-Stokes

Navier-Stokes equations **(1822)**

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = \frac{1}{R}\nabla^2 \mathbf{v} + (\cdots)$$

▼

▼

▼

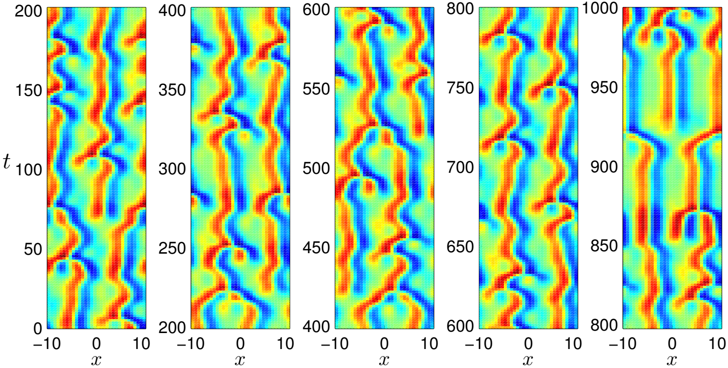
Kuramoto-Sivashinsky (1+1)-dimensional PDE **(1975)**

$$u_t + u \nabla u = -\nabla^2 u - \nabla^4 u \,, \qquad x \in \mathbb{R} \,,$$

describes spatially extended systems such as

- flame fronts in combustion
- reaction-diffusion systems
- ...

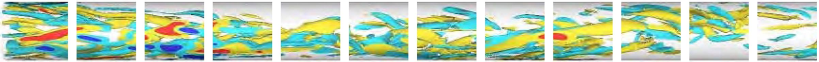
Kuramoto-Sivashinsky solutions are ‘turbulent’



horizontal: $x \in [-11, 11]$
vertical: time
color: magnitude of 1D velocity $u(x, t)$

building blocks of turbulence ?

3D Navier-Stokes flow close to the onset of turbulence[1]



we do have a detailed theory of small turbulent fluid cells

can we can we construct the infinite pipe by coupling small turbulent cells ?

what would that theory look like ?

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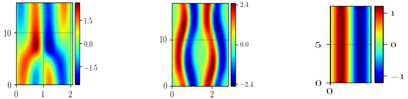
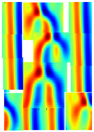
[1] M. Avila and B. Hof, Phys. Rev. E **87**, 063012 (2013).

can do : spatiotemporal turbulence building blocks

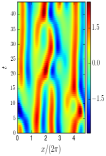
an alphabet of Kuramoto-Sivashinsky fundamental tiles :



can tile, glue - a typical resulting solution :

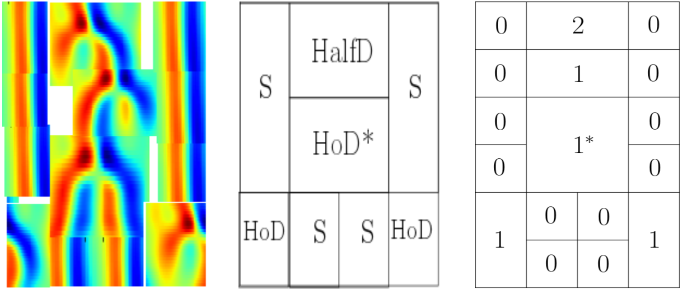


2-torus



Gudorf 2019

symbolic dynamics = 2-dimensional array



- each symbol = a spatiotemporal “rubber” tile[2]

yes, but how do you do this?

[2] M. N. Gudorf, “Spatiotemporal formulation of the Kuramoto-Sivashinsky equation”, PhD thesis (School of Physics, Georgia Inst. of Technology, Atlanta, 2019).

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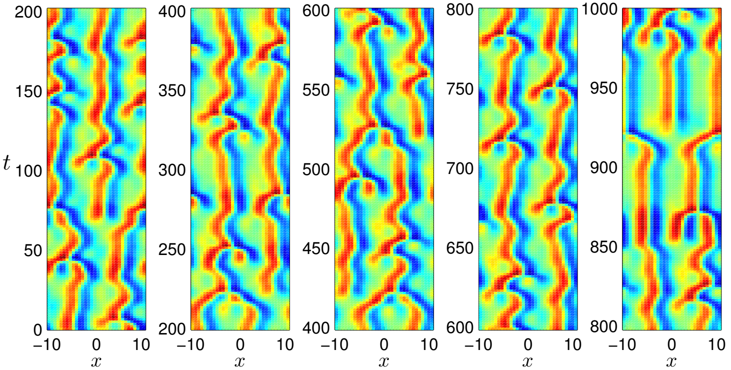
traditionally : compact space, infinite time

Integrate the PDE forward in time

Kuramoto-Sivashinsky equation

$$u_t = -(+\nabla^2 + \nabla^4)u - u\nabla u\,, \qquad x \in [-L/2, L/2]\,,$$



but, can also do : compact time, infinite space

rewrite Kuramoto-Sivashinsky

$$u_t = -uu_x - u_{xx} - u_{xxxx}$$

as 4-fields vector

$$\mathbf{u}^\top = (u, u', u'', u''')$$
$$u' \equiv u_x\,,\; u'' \equiv u_{xx}\,,\; u''' \equiv u_{xxx}$$

Kuramoto-Sivashinsky = four coupled 1st order PDEs

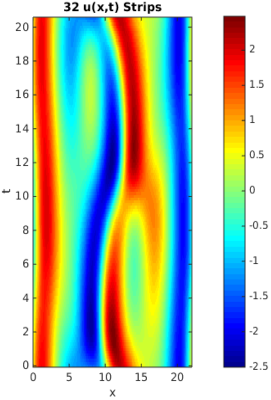
1st order in spatial derivative

$$\frac{d}{dx}\mathbf{u}(x) = \mathbf{v}(x)$$

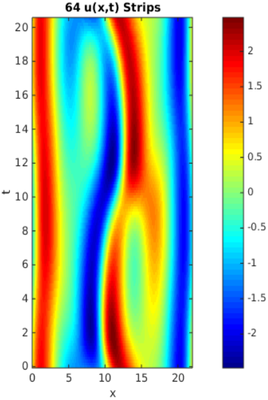
'time' is now the spatial coordinate x, integrate

can do : integrate in either time or space

(left)



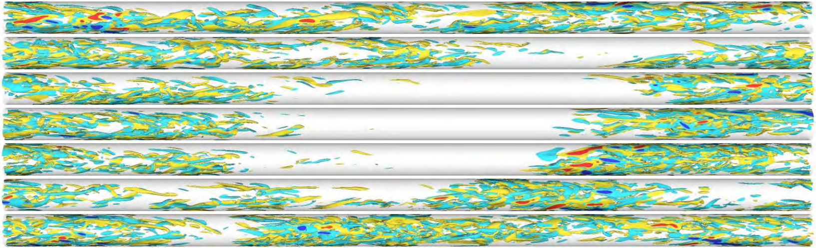
(right)



(left) old : time evolution. (right) new : space evolution
$x = [0, L]$ initial condition : time periodic line $t = [0, T]$

Gudorf 2016

but : too unstable to compute !



the same for pipe flow close to onset of turbulence

we have **hit a wall** :

'exact coherent structures' are too unstable to compute

the integrations are uncontrollably unstable

neither time nor space integration works
for large domains

rethink the formulation!

1. turbulence in spacetime
2. space is time
3. spacetime
4. bye bye, dynamics

here is a thought. Forget Newton. Instead :

build : a chaotic field theory

from : the simplest chaotic blocks

using

- time invariance
- space invariance

of the defining partial differential equations

traditionally : compact space, infinite time

Kuramoto-Sivashinsky equation

$$u_t = -(+\nabla^2 + \nabla^4)u - u\nabla u\,, \qquad x \in [-L/2, L/2]\,,$$

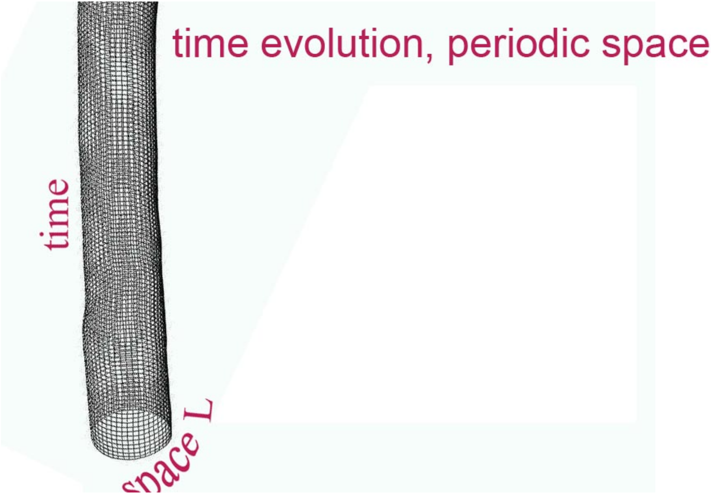
in terms of discrete spatial Fourier modes

N ordinary differential equations (ODEs) in time

$$\dot{\phi}_k(t) = (q_k^2 - q_k^4)\,\phi_k(t) - i\frac{q_k}{2}\sum_{k'=0}^{N-1}\phi_{k'}(t)\phi_{k-k'}(t)\,.$$

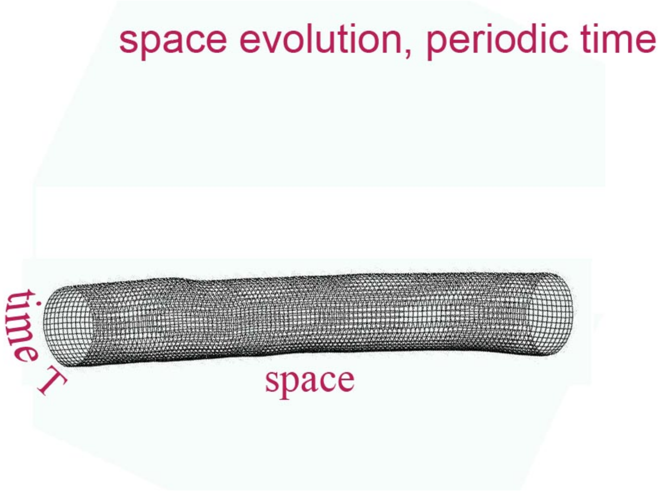
lattice formulation

always do : compact space, infinite time discrete lattice cylinder



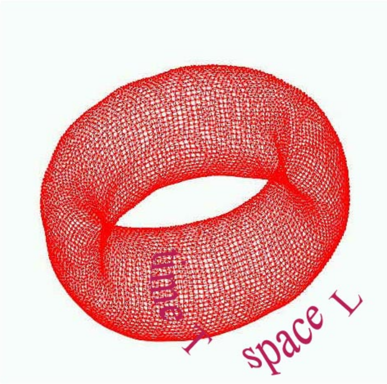
so far : Navier-Stokes on compact spatial domains, all times

can do : compact time, infinite space discrete lattice cylinder



use spatiotemporally compact solutions as spacetime ‘tiles’

periodic spacetime : 2-torus



every compact solution is a fixed point on a discrete lattice

discretize $u_{nm} = u(x_n, t_m)$ over NM points of spatiotemporal periodic lattice $x_n = nL/N$, $t_m = mT/M$, Fourier transform :

$$\phi_{k\ell} = \frac{1}{NM} \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} u_{nm}\, e^{-i(q_k x_n + \omega_\ell t_m)} \,, \quad q_k = \frac{2\pi k}{L} \,,\ \omega_\ell = \frac{2\pi \ell}{T}$$

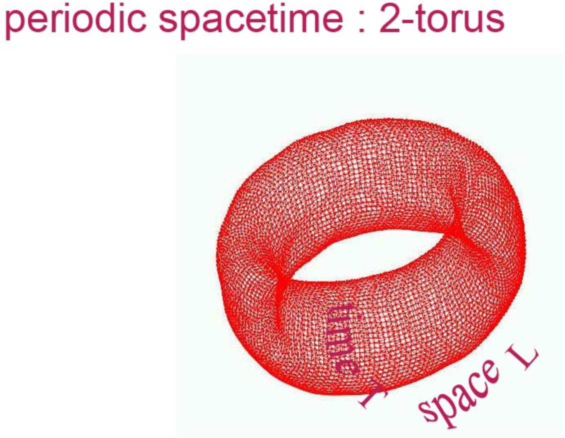
Kuramoto-Sivashinsky is no more a PDE,
but an algebraic $[N \times M]$-dimensional problem
of determining global solution Φ to

fixed point condition

$$\left(-i\omega_\ell - (q_k^2 - q_k^4)\right) \phi_{k\ell} + i\frac{q_k}{2} \sum_{k'=0}^{N-1} \sum_{m'=0}^{M-1} \phi_{k' m'} \phi_{k-k', m-m'} = 0$$

every calculation is a spatiotemporal lattice calculation

field is discretized as $\phi_{k\ell}$ values
over *NM* points of a periodic lattice



there is **no more** time or space evolution

A solution is now given as

condition that

at each lattice point $k\ell$
the tangent field at $\phi_{k\ell}$

satisfies

the global fixed point condition

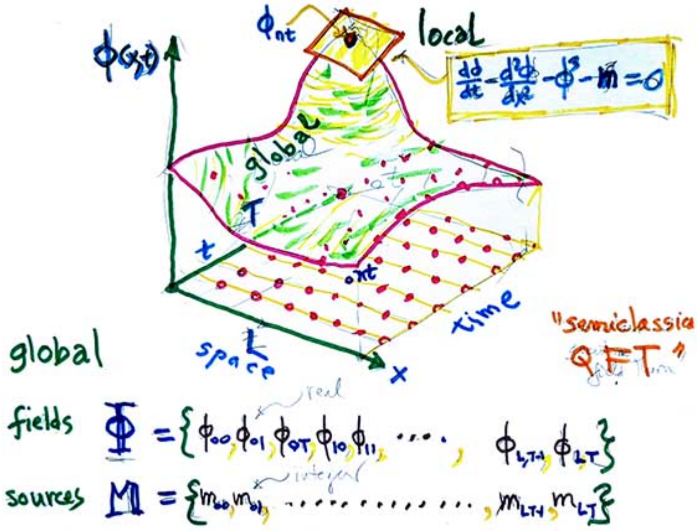
$$F[\phi] = 0$$

where for Kuramoto-Sivashinsky

$$F[\phi] = \left(-i\omega_\ell - (q_k^2 - q_k^4)\right)\phi_{k\ell} + i\frac{q_k}{2}\sum_{k'=0}^{N-1}\sum_{m'=0}^{M-1}\phi_{k'm'}\phi_{k-k',m-m'}$$

this is a local tangent field constraint on a global solution

think globally, act locally



for each symbol array M, a periodic lattice state Φ_M

Mephistopheles knocks at Faust's door and says, "Du mußt es dreimal sagen!"

. *"You have to say it three times"*

— Johann Wolfgang von Goethe

. *Faust I - Studierzimmer 2. Teil*

1. temporal cat
2. spatiotemporal cat
3. bye bye, dynamics

what ? We need a simple, pencil & paper example !

we now illustrate the approach with

the cat map in 1 spacetime dimension

then we generalize to

d-dimensional spatiotemporal cat

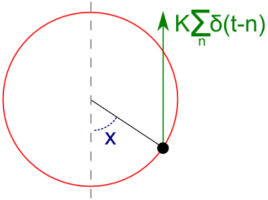
- traditional cat map (a recap, then)
- modern, temporal lattice cat
  (so much more elegant!)

think of turbulence as herding cats

evolution in time

take the simplest mechanical system : a single kicked rotor

an electron circling an atom, subject to
a discrete time sequence of angle-dependent kicks $F(x_t)$



Taylor, Chirikov and Greene standard map

$$\begin{array}{lrcll} \text{velocity} & x_{t+1} - x_t & = & p_{t+1} & \mod 1 \\ \text{acceleration} & p_{t+1} - p_t & = & F(x_t) & \end{array}$$

$\rightarrow$ chaos in Hamiltonian systems

the simplest example : a cat map evolving in time

if force $F(x) = Kx$ linear in the displacement x (Hooke's law), the equations are linear, and can be written in a matrix form, or Continuous Automorphism of the Torus, or

cat map

a linear, area preserving map of a 2-torus onto itself

$$\begin{bmatrix} \phi_t \\ \phi_{t+1} \end{bmatrix} = J \begin{bmatrix} \phi_{t-1} \\ \phi_t \end{bmatrix} - \begin{bmatrix} 0 \\ m_t \end{bmatrix}, \qquad J = \begin{bmatrix} 0 & 1 \\ -1 & s \end{bmatrix}$$

for integer 'stretching' $s > 2$
the map is beloved by ergodicists :
hyperbolic $\Rightarrow$ perfect chaotic Hamiltonian dynamical system

a cat is literally Hooke's wild, 'anti-harmonic' sister

for stretch $s < 2$ Hooke rules

local restoring oscillations
around the sleepy z-z-z-zzz resting state

for stretch $s > 2$ cats rule

exponential runaway
wrapped globally around a phase space torus

cat is to chaos what harmonic oscillator is to order

there is no more fundamental example of chaos in mechanics temporal lattice formulation

a modern cat lives on the temporal lattice

replace momentum by velocity

$$p_{t+1} = (\phi_{t+1} - \phi_t)/\Delta t$$

obtain

$$\begin{bmatrix} \phi_t \\ \phi_{t+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & s \end{bmatrix} \begin{bmatrix} \phi_{t-1} \\ \phi_t \end{bmatrix} - \begin{bmatrix} 0 \\ m_t \end{bmatrix}$$

rewrite as

2-step difference equation

$$\phi_{t+1} - s\,\phi_t + \phi_{t-1} = -m_t$$

temporal lattice formulation[3] is pretty !

[3] I. Percival and F. Vivaldi, Physica D **27**, 373–386 (1987).

think globally, act locally

temporal cat law at every instant t, local in time

$$\phi_{t+1} - s\,\phi_t + \phi_{t-1} = -m_t$$

is enforced by the global equation

$$\mathcal{J}\,\Phi = -\mathsf{M}$$

with

$$\Phi = (\phi_{t+1}, \cdots, \phi_{t+n})\,, \quad \mathsf{M} = (m_{t+1}, \cdots, m_{t+n})$$

a lattice state, a symbol block and $[n\times n]$ orbit Jacobian matrix

$$\mathcal{J} = \sigma - s\,\mathbb{1} + \sigma^{-1}$$

orbit Jacobian matrix

solving a

$$F[\Phi] = 0 \quad \text{fixed point condition}$$

requires evaluation of the $[n \times n]$

orbit Jacobian matrix

$$\mathcal{J}_{ij} = \frac{\delta F[\Phi]_i}{\delta \phi_j}$$

what does this global orbit Jacobian matrix do?

1. global stability of lattice state Φ, perturbed everywhere

the meaning of Hill determinant

Hill determinant[4] $\mathrm{Det}\,\mathcal{J}_\mathsf{M}$ determines the size of the phase space neighborhood[5] of a periodic lattice state M

in periodic orbit theory

this is known as the flow conservation sum rule :

$$\sum_\mathsf{M} \frac{1}{|\mathrm{Det}\,\mathcal{J}_\mathsf{M}|} = 1$$

sum over periodic lattice states Φ_M of period n

phase space is divided into
neighborhoods of periodic lattice states of period n

[4] G. W. Hill, Acta Math. **8**, 1–36 (1886).

[5] P. Cvitanović, “Why cycle?”, in *Chaos: Classical and Quantum*, edited by P. Cvitanović et al. (Niels Bohr Inst., Copenhagen, 2020).

old : chaos for ergodicists, 20th century

definition : chaos is

positive Lyapunov - positive entropy

- Lyapunov : how fast is local escape?
- entropy : how many ways of getting back?

$\Rightarrow$ ergodicity

modern : field theorist's chaos, 3rd millennium

definition : chaos is

expanding	Hill determinants	$\mathrm{Det}\,\mathcal{J}_{\mathsf{M}}$
exponential $\#$	periodic lattice states	Φ_{M}

the precise sense in which a
(discretized) field theory is deterministically chaotic

note : there is no 'time' in this definition

think globally, act locally - summary

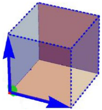
the problem of enumerating and determining all global solutions stripped to its essentials :

1. each solution is a zero of the global fixed point condition

$$F[\Phi] = 0$$

   here the entire global lattice state Φ is

   a single fixed point $\Phi = (\phi_1, \phi_2, \cdots, \phi_n)$



   in the n-dimensional unit hyper-cube $\quad \Phi \in [0,1)^n$

2. global stability : the orbit Jacobian matrix

$$\mathcal{J}_{ij} = \frac{\delta F[\Phi]_i}{\delta \phi_j}$$

Du mußt es dreimal sagen!

— Mephistopheles

1. temporal cat
2. **spatiotemporal cat**
3. bye bye, dynamics

herding cats in d spacetime dimensions

start with

a cat at each lattice site

talk to neighbors :

spacetime d-dimensional   spatiotemporal cat

spatiotemporal cat

spatiotemporal cat

consider a 1 spatial dimension lattice, with field ϕ_{nt}
(the angle of a kicked rotor "particle" at instant t, at site n)

require

- each site couples to its nearest neighbors $\phi_{n\pm1,t}$
- invariance under spatial translations
- invariance under spatial reflections
- invariance under the space-time exchange

Gutkin & Osipov[6] obtain

2-dimensional coupled cat map lattice

$$\phi_{n,t+1} + \phi_{n,t-1} - 2s\,\phi_{nt} + \phi_{n+1,t} + \phi_{n-1,t} = -m_{nt}$$

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[6] B. Gutkin and V. Osipov, Nonlinearity **29**, 325–356 (2016).

herding cats : a discrete Euclidean space-time field theory

write the spatial-temporal differences as discrete derivatives

Laplacian in $d = 2$ dimensions

$$\Box\,\phi_{nt} \;=\; \phi_{n,t+1} + \phi_{n,t-1} - 4\,\phi_{nt} + \phi_{n+1,t} + \phi_{n-1,t}$$

subtract 2-dimensional coupled cat map lattice equation

$$-m_{nt} = \phi_{n,t+1} + \phi_{n,t-1} - 2s\,\phi_{nt} + \phi_{n+1,t} + \phi_{n-1,t}$$

cat herd is thus governed by the law of the

spatiotemporal cat

$$(-\Box + \mu^2)\,\Phi = \mathsf{M}\,, \qquad \mu^2 = d(s-2)$$

where d is the spacetime dimension, s is local ‘stretching’, and μ is the Klein-Gordon scalar particle mass

that's it! for spacetime of any dimension

spatiotemporal cat is the Klein-Gordon equation

$$(-\Box + \mu^2)\,\Phi = \mathsf{M}$$

can work out completely and analytically!

did you know that a cat map can be so cool?

discretized linear PDE

d-dimensional spatiotemporal cat

$$(-\Box + \mu^2)\,\Phi = \mathsf{M}$$

is known as

- tight-binding model or Helmholtz equation
  if stretching is weak, $s < 2$
  [oscillatory sine, cosine solutions]
- Euclidean Klein-Gordon or (damped Poisson)
  if stretching is strong, $s > 2$
  [hyperbolic sinches, coshes, 'mass' $\mu^2 = d(s-2)$]

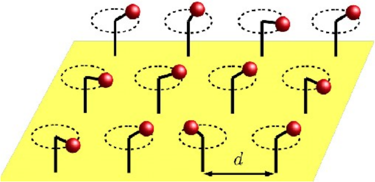
nonlinearity is hidden in the 'sources' M

spring mattress vs field of rotors

traditional field theory

Helmholtz

chaotic field theory



damped Klein-Gordon

think globally, act locally

solving the spatiotemporal cat equation

$$\mathcal{J}\Phi = -\mathsf{M}\,,$$

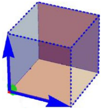
with the $[n\times n]$ matrix $\qquad \mathcal{J} = \sum_{j=1}^{2}\left(\sigma_j - s\mathbf{1} + \sigma_j^{-1}\right)$

can be viewed as a search for zeros of the function

$$F[\Phi] = \mathcal{J}\Phi + \mathsf{M} = 0$$

where the entire global lattice state Φ_M is

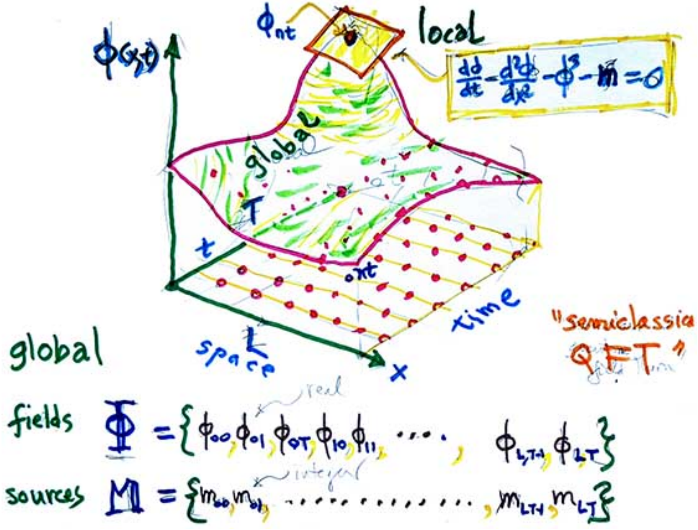
a single fixed point $\Phi_\mathsf{M} = \{\phi_z\}$



in the LT-dimensional unit hyper-cube $\Phi \in [0,1)^{LT}$

L is the 'spatial', T the 'temporal' lattice period

think globally, act locally



for each symbol array M, a periodic lattice state Φ_M

our song of chaos has been sang – what next ?

1. temporal cat
2. spatiotemporal cat
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what have we learned about spatiotemporal chaos?

spatiotemporal cat

insight 1 : how is turbulence described?

not by the evolution of an initial state

exponentially unstable system have finite (Lyapunov) time and space prediction horizons

but

by enumeration of admissible field configurations

and their natural weights

insight 2 : symbolic dynamics for turbulent flows

applies to all PDEs with *d* translational symmetries

a *d*-dimensional spatiotemporal field configuration

$$\{\phi_z\} = \{\phi_z, z \in \mathbb{Z}^d\}$$

is labelled by a *d*-dimensional spatiotemporal block of symbols

$$\{m_z\} = \{m_z, z \in \mathbb{Z}^d\}\,,$$

rather than a single temporal symbol sequence

(as is done when describing a small coupled few-“body” system, or a small computational domain).

insight 3 : description of turbulence by invariant *d*-tori

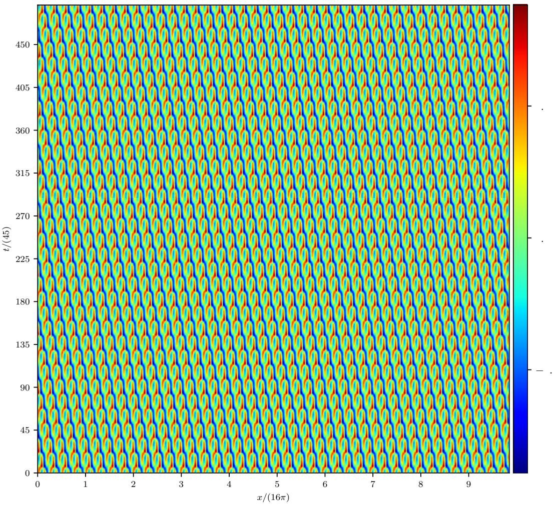
1 time, 0 space dimensions

a *temporal* periodic orbit returns after a time T
tiles the time axis by infinitely many repeats

1 time, *d*-1 space dimensions

a *spatiotemporally periodic orbit* $=$ invariant *d*-torus
tiles the lattice state with period ℓ_j in jth lattice direction

Kuramoto-Sivashinsky tiled by a small tile



tiling by relative periodic invariant 2-torus
$(L, T) = (13.02, 15)$

bye bye, dynamics

1. now can describe states of turbulence in infinite spatiatemporal domains
2. theory : classify, enuremate all spatiotemporal tilings
3. example : spatiotemporal cat, the simplest model of “turbulence”

there is no more time

there is only enumeration of
admissible spacetime field configurations

Verbrechen des Jahrhunderts : das Ende der Zeit

die Zeit ist tot !

time is dead !

goodbye

to long time and/or space integrators

they never worked and could never work

how do clouds solve Navier-Stokes ?

clouds do not integrate Navier-Stokes equations

$\Longleftarrow$ all possible clouds $\Longrightarrow$

do clouds obey Navier-Stokes equations?

yes!

they satisfy them locally, everywhere and at all times