a chaotic field theory

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ChaosBook.org/overheads/spatiotemporal
$\rightarrow$ chaotic field theory talks, papers

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Rutgers

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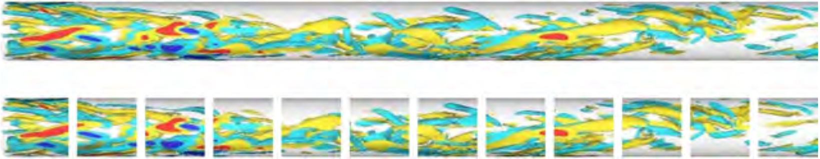
then this happens

Q. why a "chaotic" field theory?

turbulence !

a motivation : need a theory of large turbulent domains

a turbulent pipe flow [1]



we have a detailed theory of small turbulent fluid cells

can we can we construct the infinite pipe by coupling small turbulent cells ?

what would that theory look like ?

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[1] M. Avila and B. Hof, Phys. Rev. **E 87** (2013)

Q. why a "chaotic" field theory?

statistical mechanics

sum over all configurations !

Q. why a "chaotic" field theory?

quantum gravity partition functions

sum over all geometries !

Q. why a "chaotic" field theory?

many-body chaos !

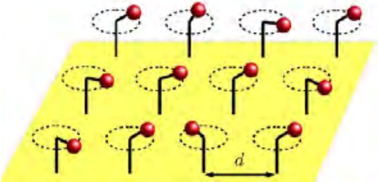
needed : a theory of $(N \to \infty)$-body chaos

take-home :

traditional field theory

Helmholtz

chaotic field theory

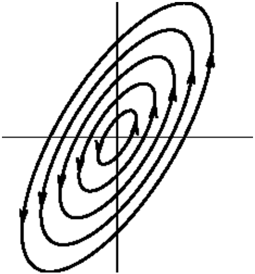


damped Poisson, Yukawa

now you may space out on the rest of the talk :)

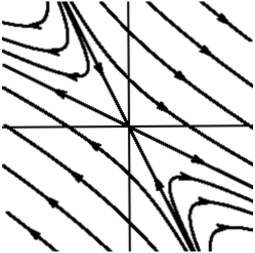
take-home :

harmonic field theory



oscillatory eigenmodes

chaotic field theory



hyperbolic instabilities

build
a chaotic field theory

from
the simplest chaotic blocks

using

- time invariance
- space invariance

of system's defining equations

1. what this is about
2. semiclassical field theory
3. deterministic field theory
4. spatiotemporal cat
5. bye bye, dynamics

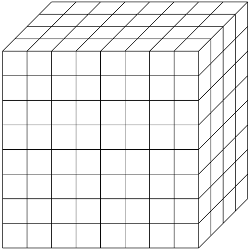
lattice field theory

here - “lattice” for pedagogy - continuum essentially the same

lattice field theory

field $\phi(x)$

evaluated on lattice $\mathcal{L}$ sites z



$\phi_z = \phi(x)$
$x = a\,z =$ lattice site
$z \in \mathbb{Z}^d$

discretized scalar field configuration

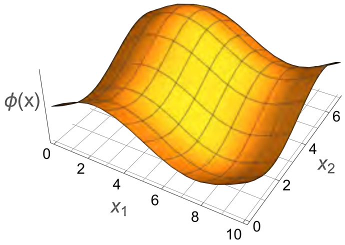
example : two spatiotemporal dimensions

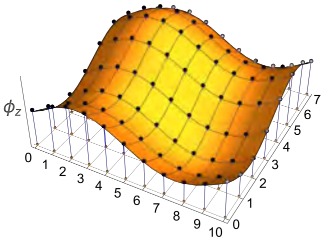
$$\Phi = \begin{array}{ccccccc} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \phi_{-2,1} & \phi_{-1,1} & \phi_{0,1} & \phi_{1,1} & \phi_{2,1} & \cdots \\ \cdots & \phi_{-2,0} & \phi_{-1,0} & \phi_{0,0} & \phi_{1,0} & \phi_{2,0} & \cdots \\ \cdots & \phi_{-2,-1} & \phi_{-1,-1} & \phi_{0,-1} & \phi_{1,-1} & \phi_{2,-1} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{array} .$$

a field configuration = any set of values $\phi_z \in \mathbb{R}$ in system's ∞-dimensional state space.

discretization of a $2D$ field

scalar field evaluated on lattice points





field $\phi(x)$
over continuous
coordinates (x_1, x_2)

discretized field ϕ_z
over lattice $\mathcal{L}$, integer
coordinates (z_1, z_2)

horizontal: spatiotemporal coordinate,
lattice sites marked by $\circ$, labelled by $z \in \mathbb{Z}^2$

vertical: value of the lattice site field $\phi_z \in \mathbb{R}$
plotted as a bar centred at lattice site (z_1, z_2)

how likely is a ‘tornado’?

semiclassical field theory

quantum field theory

path integral

field configuration Φ occurs with probability amplitude

$$p(\Phi) \; = \; \frac{1}{Z} \, e^{\frac{i}{\hbar} S[\Phi]} \, , \qquad Z = Z[0]$$

partition sum $=$ integral over all configurations

$$Z[\mathrm{J}] \; = \; \int [d\phi] \, e^{\frac{i}{\hbar} (S[\Phi] + \Phi \cdot \mathrm{J})} \, , \qquad [d\phi] = \prod_{z}^{\mathcal{L}} \frac{d\phi_z}{\sqrt{2\pi}}$$

evaluate how ? here : WKB or semiclassical approximation

method of stationary phase : on 1 lattice site

path integral : Laplace integral of form

$$I = \int \frac{d\phi}{\sqrt{2\pi}} A(\phi)\, e^{\frac{i}{\hbar} S(\phi)}$$

extremal point ϕ_c given by the stationary phase condition

$$\frac{d}{d\phi} S(\phi_c) = 0\,,$$

semiclassical approximation

$$\int \frac{d\phi}{\sqrt{2\pi}} A(\phi)\, e^{\frac{i}{\hbar} S(\phi)} \approx A(\phi_c) \frac{e^{\frac{i}{\hbar} S(\phi_c) \pm i\frac{\pi}{4}}}{|S''(\phi_c)/\hbar|^{1/2}}\,,$$

method of stationary phase

Euler–Lagrange equation

$$\frac{\delta S[\Phi_c]}{\delta \phi_z} = 0$$

a global deterministic solution Φ_c satisfies this local extremal condition on every lattice site z

WKB or semiclassical approximation

$$S[\Phi] = S[\Phi_c] + \frac{1}{2}(\Phi - \Phi_c)^\top \mathcal{J}_c (\Phi - \Phi_c) + \cdots$$

orbit Jacobian matrix

$$(\mathcal{J}_c)_{z'z} = \left.\frac{\delta^2 S[\Phi]}{\delta \phi_{z'} \delta \phi_z}\right|_{\Phi=\Phi_c}$$

semiclassical field theory

deterministic solution Φ_c probability amplitude

$$p(\Phi_c) = \frac{1}{Z} \frac{e^{iS[\Phi_c]+im_c}}{|\mathrm{Det}\, \mathcal{J}_c|^{1/2}}, \qquad Z = Z[0]$$

partition sum : support on deterministic solutions over $\mathcal{L}$

$$Z_{\mathcal{L}}[\mathrm{J}] = \sum_c \frac{e^{i(S[\Phi_c]+m_c+\Phi_c\cdot \mathrm{J})}}{|\mathrm{Det}\, \mathcal{J}_c|^{1/2}}$$

example : Gutzwiller trace formula
continuous time (limit of a $1D$ temporal lattice $\mathcal{L}$)

$$\int_{\mathcal{L}} [d\Phi]\, A[\Phi]\, e^{iS[\Phi]} \approx \sum_c A[\Phi_c] \frac{e^{iS[\Phi_c]+im_c}}{|\mathrm{Det}\, \mathcal{J}_c|^{1/2}}$$

quantum mechanics, not field theory[2]

[2] M. C. Gutzwiller, J. Math. Phys. **8**, 1979–2000 (1967).

bird's eye view : semiclassical field theory

WKB backbone is the

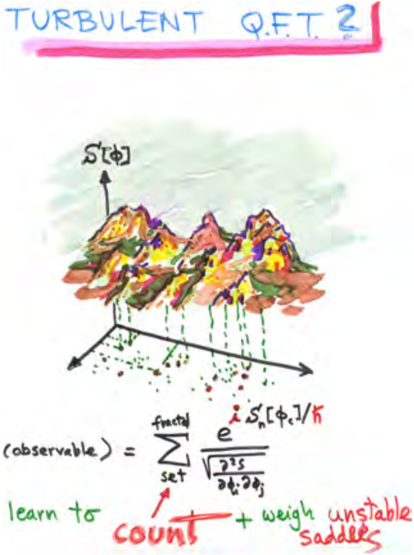
deterministic field theory

extremal condition $\Rightarrow$

$$\frac{\delta S[\Phi_c]}{\delta \phi_z} = 0$$

deterministic solution Φ_c satisfies the local extremal condition on every lattice site

a fractal set of saddles



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3. **deterministic field theory**
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fluid turbulence is described by

deterministic field theory

deterministic partition function is given by sum over the saddles

wisdom of statistical mechanicians

partition function

field configuration Φ occurs with probability

$$p(\Phi) \;=\; \frac{1}{Z}\, e^{-S[\Phi]}\,, \qquad Z = Z[0]$$

partition function $=$ sum over all field configurations

$$Z_{\mathcal{L}}[\mathrm{J}] \;=\; \int [d\phi]\, e^{-S[\Phi]+\Phi\cdot\mathrm{J}}\,, \qquad [d\phi] = \prod_{z}^{\mathcal{L}} \frac{d\phi_z}{\sqrt{2\pi}}$$

definition : deterministic field theory

deterministic partition function has support only on the solutions Φ_c to saddle-point condition

$$F[\Phi_c]_z = \frac{\delta S[\Phi_c]}{\delta \phi_z} = 0$$

which we refer to as

Euler–Lagrange equation

$$F[\Phi_c]_z = 0$$

note : works both for dissipative and Hamiltonian systems

deterministic partition function

Φ_c is a deterministic solution, so its

probability density

is $|\mathcal{L}|$-dimensional Dirac delta function (determinism !!!)

$$p_c(\Phi) = \frac{1}{Z}\delta(F[\Phi])$$

and

deterministic partition function

$$Z_{\mathcal{L}}[\mathsf{J}] = \sum_c \int_{\mathcal{M}_c} [d\Phi]\,\delta(F[\Phi])e^{\Phi\cdot\mathsf{J}} = \sum_c \frac{e^{\Phi_c\cdot\mathsf{J}}}{|\mathrm{Det}\,\mathcal{J}_c|}$$

$\mathcal{M}_c$ = small neighborhood of periodic state Φ_c

is the sum over probabilities of all periodic states over lattice $\mathcal{L}$

think globally, act locally

definition : periodic state is

a global deterministic solution

$$\Phi_c = \{\phi_{c,z}\}$$
$$= \text{set of lattice site field values}$$

periodic along each translationally invariant direction

that satisfies the

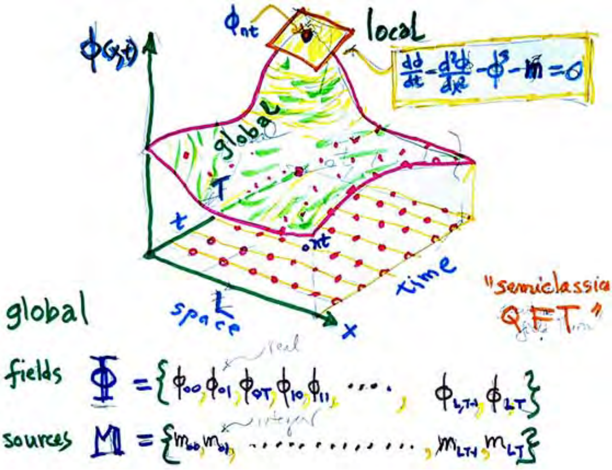
local condition : Euler–Lagrange equation

$$F[\Phi_c]_z = 0$$

on every lattice site z of multi-periodic primitive cell $\mathbb{A}$

think globally, act locally

a global periodic state field configuration



satisfies the local Euler–Lagrange equation at each lattice site

example : spacetime tiled by a finite primitive cell $\mathbb{A}$

spatial (horizontal) period $L = 3$
temporal (vertical) period $T = 2$

$[3\times2]$ lattice field configuration

$$\Phi = \begin{bmatrix} \phi_{01} & \phi_{11} & \phi_{21} \\ \phi_{00} & \phi_{10} & \phi_{20} \end{bmatrix}$$

$|\mathcal{L}| = 6$ lattice sites in a tile, (relative) periodic bc's

WKB integral over $|\mathcal{L}| = 6$ lattice sites

neighborhood of a deterministic solution Φ_c

$$\int_c [d\Phi]\, A(\Phi)\, e^{\frac{i}{\hbar}\frac{1}{2}\Phi^\top \mathcal{J}_c \Phi} = \frac{A(\Phi_c)}{\left|\mathrm{Det}\,\mathcal{J}_c\right|^{1/2}}$$

(drop Maslov index, $\hbar = 1$ in what follows)

$[L \times T]_S$ lattice tiling, visualized as brick wall

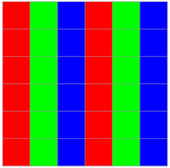
for example,

ϕ_{11}	ϕ_{21}	ϕ_{01}	ϕ_{11}	ϕ_{21}	ϕ_{01}
ϕ_{10}	ϕ_{20}	ϕ_{00}	ϕ_{10}	ϕ_{20}	ϕ_{00}
ϕ_{21}	ϕ_{01}	ϕ_{11}	ϕ_{21}	ϕ_{01}	ϕ_{11}
ϕ_{20}	ϕ_{00}	ϕ_{10}	ϕ_{20}	ϕ_{00}	ϕ_{10}
ϕ_{01}	ϕ_{11}	ϕ_{21}	ϕ_{01}	ϕ_{11}	ϕ_{21}
ϕ_{00}	ϕ_{10}	ϕ_{20}	ϕ_{00}	ϕ_{10}	ϕ_{20}

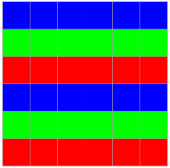
spacetime brick wall tiled by a $[3 \times 2]_1$ 'brick'

examples of field configurations : spatiotemporal mosaics

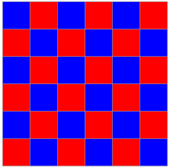
site field values color-coded



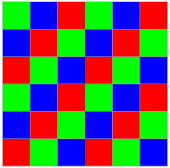
$[3 \times 1]_0$



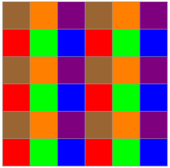
$[1 \times 3]_0$



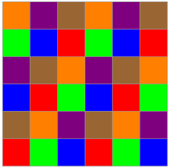
$[2 \times 1]_1$



$[3 \times 1]_1$



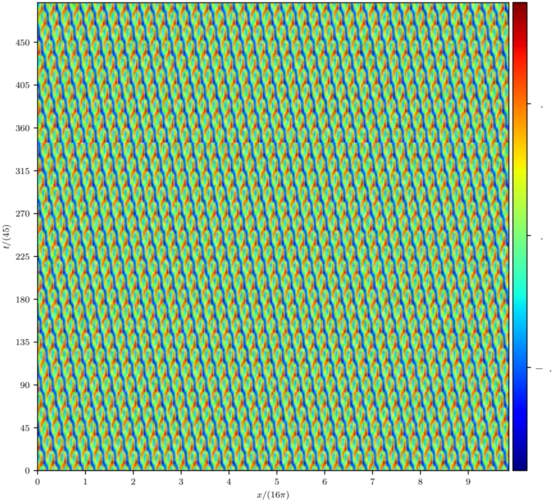
$[3 \times 2]_0$



$[3 \times 2]_1$

tilings of $[6 \times 6]$ domain by smaller primitive cells

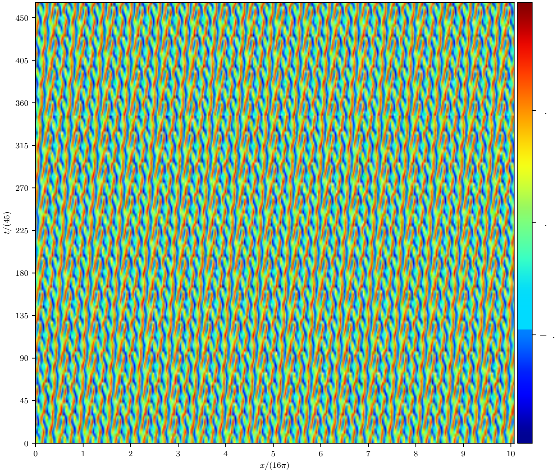
example : a small '1D Navier-Stokes' doubly-periodic state



tiling by Kuramoto-Sivashinsky relative periodic state
$(L, T) = (13.02, 15)$

Gudorf 2018

continuum example : spacetime tiled by a larger tile



tiling by relative periodic invariant 2-torus
$(L, T) = (33.73, 35)$

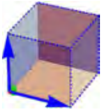
Gudorf 2018

computing periodic states

search for zeros of function

$$F[\Phi_c]_z = 0$$

the entire global periodic state Φ_c over primitive cell $\mathbb{A}$ is a single point $(\phi_1, \phi_2, \cdots, \phi_n)$



in the n-dimensional state space $\Phi \in [0,1)^n$

periodic states : think globally, act locally

the problem of determining all periodic states stripped to its essentials :

1. each solution is a zero of the global fixed point condition

$$F[\Phi_c]_z = 0$$

2. global stability : the orbit Jacobian matrix

$$(\mathcal{J}_c)_{z'z} = \frac{\delta F[\Phi_c]_{z'}}{\delta \phi_z}$$

3. orbit weight : the Hill determinant

$$1/|\mathrm{Det}\,\mathcal{J}_c|$$

4. partition function $Z_{\mathcal{L}}[\mathrm{J}]$: all predictions of the theory

bird's eye view : quantum vs. deterministic chaos

support for both are deterministic solutions to the Euler–Lagrange equations, but with different weights

'Ruelle' deterministic theory

$$Z_{\mathcal{L}}[\mathsf{J}] = \sum_c \frac{e^{\Phi\cdot\mathsf{J}}}{|\mathrm{Det}\,\mathcal{J}_c|}$$

'Gutzwiller' semiclassical theory

$$Z_{\mathcal{L}}[\mathsf{J}] = \sum_c \frac{e^{i(S[\Phi_c]+m_c+\Phi_c\cdot\mathsf{J})}}{|\mathrm{Det}\,\mathcal{J}_c|^{1/2}}$$

antecedents

Gutzwiller trace formula[a] is a continuous time limit of a $1D$ temporal lattice : quantum mechanics, a few degrees of freedom, no spatial extent

Ruelle dynamical zeta function[b] : $1D$ temporal lattice, iterated maps, no spatial extent

[a] M. C. Gutzwiller, J. Math. Phys. **8**, 1979–2000 (1967).

[b] D. Ruelle, Bull. Amer. Math. Soc **82**, 153–157 (1976).

why deterministic field theory ?

1. QFT / ∞-body motivation : semi-classical WKB has support on saddle-point, deterministic solutions[3]

   $\Rightarrow$ deterministic field theory
2. chaos, maps : $1D$ temporal lattice

   discrete time dynamics[4]
3. chaos, ODEs : $1D$ temporal lattice continuum limit

   the dynamical systems theory
4. spatiotemporal chaos, PDEs : D-dimensional deterministic field theory is a theory of turbulence[5,6]

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[3] M. C. Gutzwiller, *Chaos in Classical and Quantum Mechanics*, (Springer, New York, 1990).

[4] P. Cvitanović et al., *Chaos: Classical and Quantum*, (Niels Bohr Inst., Copenhagen, 2023).

[5] J. F. Gibson et al., J. Fluid Mech. **611**, 107–130 (2008).

[6] M. N. Gudorf, "Spatiotemporal Tiling of the Kuramoto-Sivashinsky Equation", PhD thesis (School of Physics, Georgia Inst. of Technology, Atlanta, 2020).

1. what this is about
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4. **spatiotemporal chaos**
5. bye bye, dynamics

orbit stability

the most important thing : Hill determinant

- find a deterministic solution

$$F[\Phi_c]_z = 0 \qquad \text{fixed point condition}$$

- evaluate $\mathrm{Det}\, \mathcal{J}_c$ of

orbit Jacobian matrix

$$\mathcal{J}_{z'z} = \frac{\delta F[\Phi_c]_{z'}}{\delta \phi_z}$$

what does this global orbit Jacobian matrix do?

global stability

of periodic state Φ_c, perturbed everywhere

orbit Jacobian (Hill, Hessian, ...) matrix

each periodic state Φ_c has its own

$$\mathcal{J}[\Phi_c] = \begin{pmatrix} -s_0 & 1 & 0 & 0 & \cdots & 0 & 0 & 1 \\ 1 & -s_1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & -s_2 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & -s_{n-2} & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 1 & -s_{n-1} \end{pmatrix},$$

stretching factor $s_z = S''[\phi_z]$
is function of site fields ϕ_z of a given periodic state Φ_c

1. can compute Hill determinant $\mathrm{Det}\,\mathcal{J}$
2. Hill-Lindstedt-Poincaré :
   all calculations should be done on reciprocal lattice
3. toolbox : discrete Fourier transforms

the most critical thing

functional ‘fluctuation’ determinant

$$\mathrm{Det}\,\mathcal{J}_c$$

must be computed in the

infinite volume limit

new !

exact deterministic weight

$$\frac{1}{|\mathrm{Det}\,\mathcal{J}_c|}$$

in any spacetime dimension

vastly preferable to the
dynamical systems forward-in-time formulation

wisdom of solid state physicists

exact Hill determinant

is given by bands over the Brillouin zone

traditional periodic orbit theory[7,8,9]

alles falsch :(

is not smart :
finite periodic states Hill determinants are only approximations

[7] M. C. Gutzwiller, J. Math. Phys. **8**, 1979–2000 (1967).

[8] D. Ruelle, Bull. Amer. Math. Soc **82**, 153–157 (1976).

[9] P. Cvitanović et al., *Chaos: Classical and Quantum*, (Niels Bohr Inst., Copenhagen, 2023).

finite primitive cell Hill determinant

free massive compact boson

square-lattice explicit formula for a finite primitive cell $\mathbb{A}$ orbit Jacobian matrix, in terms of lattice momentum $p(k)$

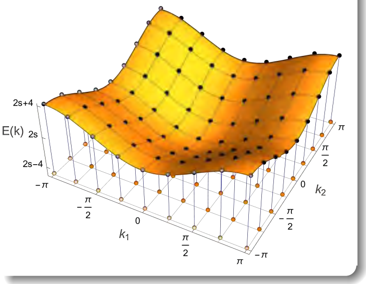
$$\begin{aligned} \mathrm{Det}\,\mathcal{J}_{\mathbb{A}} &= \mathrm{Det}\,(p^2+\mu^2) \\ &= \prod_{m_1=0}^{L-1}\prod_{m_2=0}^{T-1}\left[p(k_1)^2+p(k_2-k_1 S/T)^2+\mu^2\right] \\ p(k) &= 2\sin(k/2)\,,\quad k_1=\frac{2\pi}{L}\,m_1\,,\quad k_2=\frac{2\pi}{T}\,m_2\,. \end{aligned}$$

however,
finite periodic states' Hill determinants are only approximations

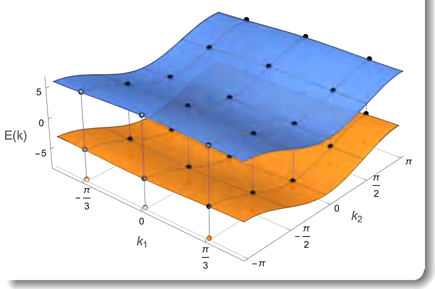
spatiotemporal lattice orbit Jacobian operator spectra (k_1, k_2)

smooth surfaces : Brillouin zone bands

massive compact boson



ϕ^4 theory in 2D



black dots : orbit Jacobian matrix eigenvalues,
finite volume primitive cells

[left] primitive cell periodicity $[8\times 8]_0$
[right] primitive cell tiled by repeats of $[2\times 1]_0$ periodic state

wisdom of solid state physicists

in 2D spacetime, free massive boson theory, the stability exponent $\ln \mathrm{Det}\, \mathcal{J}_c$ is given by the band integral over the Brillouin zone

exact stability exponent of a periodic state c

$$\lambda_c = \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} dk_1 dk_2 \, \ln\left(p(k_1)^2 + p(k_2)^2 + \mu^2\right) \,,$$

1. what this is about
2. semiclassical field theory
3. deterministic field theory
4. **chaotic field theories**
5. bye bye, dynamics

chaotic field theory

deterministic field theory partition function

has support only on lattice $\mathcal{L}$ field values that are solutions to the Euler–Lagrange equations. The partition function is the

sum over geometries

$$Z_{\mathcal{L}}[0] = \sum_{c} \frac{1}{|\mathrm{Det}\, \mathcal{J}_c|}$$

(or, sum over all Bravais lattices)

where the $[N_{\mathcal{L}} \times N_{\mathcal{L}}]$ matrix

$$(\mathcal{J}_c)_{z'z} = \frac{\delta F_{z'}[\Phi_c]}{\delta \phi_z}$$

is the orbit Jacobian matrix, and $\mathrm{Det}\, \mathcal{J}_c$ is its Hill determinant

field theorist's chaos

definition : chaos is

expanding	Hill determinants	$\mathrm{Det}\,\mathcal{J}_c$
exponential ♯	periodic states	$N_{\mathcal{L}}$

the precise sense in which
a (discretized) field theory is deterministically chaotic

note : there is no 'time' in this definition use shadowing

new !

spatiotemporal zeta function

wisdom of dynamicists

a periodic state Φ_c is either prime, or a repeat of a prime periodic state Φ_p

periodic state weight

$$t_p = e^{-N_p(\lambda_p - s)}\,, \qquad N_p = N_{\mathbb{A}} = L_{\mathbb{A}} T_{\mathbb{A}}$$

λ_p : stability exponent of periodic state p
N_p : lattice volume (number of sites of primitive cell $\mathbb{A}$)
s : a parameter, to be used soon

wisdom of Euler, ..., Weierstrass

2D spatiotemporal

prime orbit zeta function

$$1/\zeta_p(s) = \tau_p^{-\frac{1}{24}} \eta(\tau_p)$$

with the imaginary phase parameter

$$\tau_p = i\frac{N_p}{2\pi}(-\lambda_p + s)\,,$$

$\eta(\tau)$: Dedekind eta function

is a modular function[10,11,12]

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[10] J. L. Cardy, Nucl. Phys. B **270**, 186–204 (1986).

[11] E. V. Ivashkevich et al., J. Phys. A **35**, 5543–5561 (2002).

[12] A. Maloney and E. Witten, J. High Energy Phys. **2010**, 029 (2010).

2D spatiotemporal zeta function - a product over all prime orbit zeta functions

$$\zeta(s) = \prod_p \zeta_p(s)$$

related to the deterministic partition function by the logarithmic derivative relation between the partition sum and the zeta function

$$Z(s) = \frac{d}{ds} \ln \zeta(s) \,,$$

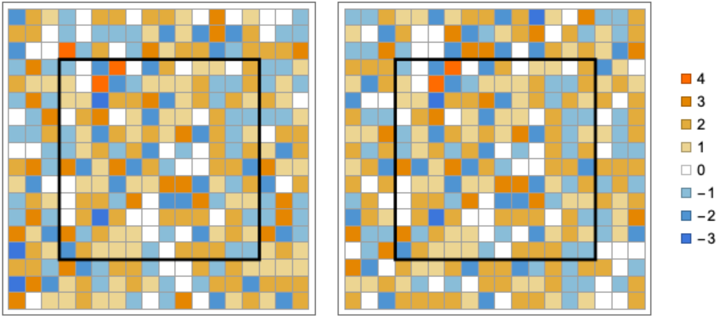
see, for example, ChaosBook eq. (22.12)

why does it work ?

shadowing !

short-periods periodic states dominate

2 periodic states, shared sub-mosaic



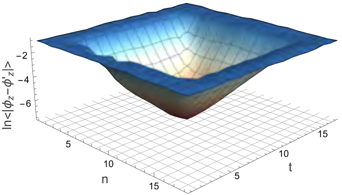
Color coded : 8-letter alphabet

Mosaics of two $[18\times18]_0$ periodic states which share the the black square $[12\times12]$ sub-mosaic

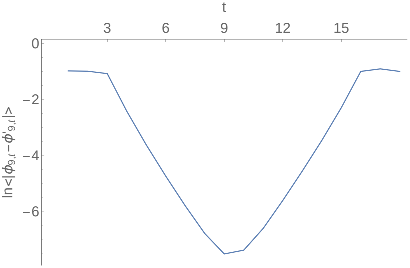
shadowing

log of the mean site-wise field value distances $|\phi_z - \phi'_z|$

across the primitive cell



along $z = (9, t)$ line



the slope = approx. Klein-Gordon mass μ
(as in the massive boson Green's function)

what is all this good for ?

use partition function to predict

expectation of an observable '*a*'

$$\begin{aligned} \langle a \rangle &= \sum_c \frac{1}{|\mathrm{Det}\,\mathcal{J}_c|} \langle a \rangle_c \\ \Delta &= \langle (a - \langle a \rangle)^2 \rangle \\ &\cdots \\ \text{higher cumulants} &= \ldots \end{aligned}$$

where

evaluate observable on each prime orbit

$$\langle a \rangle = \sum_p^{prime} w(\Phi_p)\, \langle a \rangle_p$$

$\cdots$ details $\cdots\cdots$

read Das Buch,
take the `ChaosBook.org/course1`

1. what this is about
2. semiclassical field theory
3. deterministic field theory
4. chaotic field theories
5. bye bye, dynamics

bye bye, dynamics

1. goal : describe states of turbulence in infinite spatiatemporal domains
2. theory : classify, enuremate all spatiotemporal tilings

there is no more time

there is only enumeration of
admissible spacetime field configurations

insight 1 : how is turbulence described?

not by the evolution of an initial state

exponentially unstable system have finite (Lyapunov) time and space prediction horizons

but

by enumeration of admissible field configurations

and their stability weights

insight 2 : description of turbulence by d-tori

1 time, 0 space dimensions

a state space point is *periodic* if its orbit returns to itself after a finite time T; such orbit tiles the time axis by infinitely many repeats

1 time, d-1 space dimensions

a state space point is *spatiotemporally periodic* if it belongs to an invariant d-torus,
i.e., a block $\mathsf{M}_{\mathcal{L}}$ that tiles the periodic state M,
with period ℓ_j in jth lattice direction

insight 3 : can hierarchically compute ‘all’ solutions

orbitHunter

rough initial guesses converge

no exponential instabilities

stability spectrum : compute on reciprocal lattice

gitHub code[13]

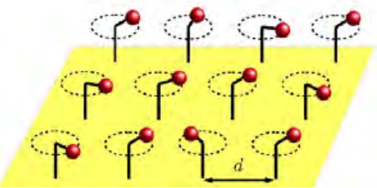
[13] M. N. Gudorf, *Orbithunter: Framework for Nonlinear Dynamics and Chaos*, tech. rep. (School of Physics, Georgia Inst. of Technology, 2021).

take-home :

traditional field theory

Helmholtz

chaotic field theory



damped Poisson, Yukawa

if anyone asks : extra slides

ODEs, PDEs linear operators wisdom

Hill's 1886 formula[14]

Gel'fand-Yaglom 1960 theorem[15]

orbit Jacobian matrix $\mathcal{J}$ is fundamental

temporal evolution Jacobian matrix J is merely
one of the methods to compute it

all of dynamical systems theory is subsumed in spatiotemporal field-theoretic formulation

[14] G. W. Hill, Acta Math. **8**, 1–36 (1886).

[15] I. M. Gel'fand and A. M. Yaglom, J. Math. Phys. **1**, 48–69 (1960).

orbit stability vs. temporal stability

orbit Jacobian matrix

$\mathcal{J}_{z'z} = \frac{\delta F[\Phi]_{z'}}{\delta \phi_z}$ stability under global perturbation of the whole orbit

for n large, a huge $[dn \times dn]$ matrix

temporal Jacobian matrix

J propagates initial perturbation n time steps

small $[d \times d]$ matrix

J and $\mathcal{J}$ are related by[16]

Hill's 1886 remarkable formula

$$|\mathrm{Det}\,\mathcal{J}_\mathrm{M}| = |\det(\mathbf{1} - J_\mathrm{M})|$$

$\mathcal{J}$ is huge, even ∞-dimensional matrix
J is tiny, few degrees of freedom matrix

[16] G. W. Hill, Acta Math. **8**, 1–36 (1886).

how is deterministic field theory different from other theories?

- we always work in the ‘broken-symmetry’ regime, as almost every ‘turbulent’ (spatiotemporally chaotic) solution breaks all symmetries
- we work ‘beyond perturbation theory’, in the anti-integrable, strong coupling regime. This are not weak coupling expansions around a ground state
- our ‘far from equilibrium’ field theory is not driven by external noise. All chaoticity is due to the intrinsic deterministic instabilities
- our formulas are exact, not merely saddle points approximations to the exact theory

for a deep dive

chaotic field theory talks, papers $\Rightarrow$

ChaosBook.org/overheads/spatiotemporal