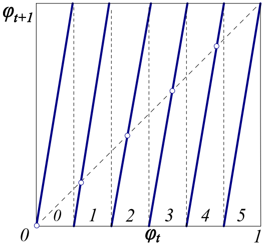
a fair dice throw

slope 6 Bernoulli map



$$\phi_{t+1} = 6\phi_t - m_{t+1}\,,\ \ \phi_t \in \mathcal{M}_{m_t}$$

6-letter alphabet
$m_t \in \mathcal{A} = \{0, 1, 2, \cdots, 5\}$

6 subintervals $\{\mathcal{M}_0, \mathcal{M}_1, \cdots, \mathcal{M}_5\}$

what is (mod 1) ?

map with integer-valued 'stretching' parameter $s > 1$:

$$x_{t+1} = s\,x_t$$

(mod 1) : subtract the integer part $m_{t+1} = \lfloor s x_t \rfloor$
so fractional part ϕ_{t+1} stays in the unit interval $[0, 1)$

$$\phi_{t+1} = s\phi_t - m_{t+1}\,, \qquad \phi_t \in \mathcal{M}_{m_t}$$

m_t takes values in the s-letter alphabet

$$m \in \mathcal{A} = \{0, 1, 2, \cdots, s-1\}$$

lattice Bernoulli

recast the time-evolution Bernoulli map

$$\phi_{t+1} = s\phi_t - m_{t+1}$$

as 1-step difference equation on the temporal lattice

$$\phi_t - s\phi_{t-1} = -m_t \,, \qquad \phi_t \in [0, 1)$$

field ϕ_t, source m_t
on each site t of a 1-dimensional lattice $t \in \mathbb{Z}$

write an n-sites lattice segment as
the lattice state and the symbol block

$$\mathsf{X} = (\phi_{t+1}, \cdots, \phi_{t+n}) \,, \quad \mathsf{M} = (m_{t+1}, \cdots, m_{t+n})$$

'M' for 'marching orders' : come here, then go there, $\cdots$

think globally, act locally

Bernoulli condition at every lattice site t, local in time

$$\phi_t - s\phi_{t-1} = -m_t$$

is enforced by the global equation

$$\mathcal{J}\mathsf{X} + \mathsf{M} = 0\,,$$

where $\mathcal{J}$ is $[n \times n]$ Hill matrix (orbit Jacobian matrix)

$$\mathcal{J} = \begin{pmatrix} 1 & 0 & & & -s \\ -s & 1 & 0 & & \\ & -s & 1 & \ddots & \\ & & -s & 1 & 0 \\ 0 & & & -s & 1 \end{pmatrix}$$

think globally, act locally

solving the lattice Bernoulli system

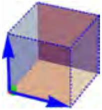
$$\mathcal{J}\mathsf{X} + \mathsf{M} = 0\,,$$

is a search for zeros of the function

$$F[\mathsf{X}] = \mathcal{J}\mathsf{X} + \mathsf{M} = 0$$

the entire global lattice state X_M is now

a single fixed point $(\phi_1, \phi_2, \cdots, \phi_n)$



in the *n*-dimensional unit hyper-cube $\mathsf{X} \in [0,1)^n$

what does this global orbit Jacobian matrix do?

$[n \times n]$ orbit Jacobian matrix

$$\mathcal{J}_{ij} = \frac{\delta F[\mathsf{X}]_i}{\delta \phi_j}$$

- global stability of lattice state X, perturbed everywhere

next : we derive Hill's formula

orbit Jacobian matrix

$\mathcal{J}_{ij} = \frac{\delta F[X]_i}{\delta \phi_j}$ stability under global perturbation of the whole orbit

for n large, a huge $[dn \times dn]$ matrix

temporal Jacobian matrix

J propagates initial perturbation n time steps

small $[d \times d]$ matrix

J and $\mathcal{J}$ are related by[1]

Hill's 1886 remarkable formula

$$|\mathrm{Det}\, \mathcal{J}_\mathrm{M}| = |\det(\mathbf{1} - J_\mathrm{M})|$$

$\mathcal{J}$ is huge, even ∞-dimensional matrix
J is tiny, few degrees of freedom matrix

[1] G. W. Hill, Acta Math. **8**, 1–36 (1886).

temporal stability

any discrete time dynamical system : an n-periodic lattice state X_p satisfies the first-order difference equation

$$\phi_t - f(\phi_{t-1}) = 0\,, \quad t = 1, 2, \cdots, n\,.$$

A deviation $\Delta\mathsf{X}$ from X_p satisfies the linear equation

$$\Delta\phi_t - \mathbb{J}_{t-1}\,\Delta\phi_{t-1} = 0\,, \qquad (\mathbb{J}_t)_{ij} = \left.\frac{\partial f(\phi)_i}{\partial \phi_j}\right|_{\phi_j=\phi_{t,j}}\,,$$

where $\mathbb{J}_t$ is the 1-time step $[d \times d]$ Jacobian matrix.

temporal period $n = 3$ example

in terms of the $[3d \times 3d]$ shift matrix σ, the orbit Jacobian matrix takes block matrix form

$$\mathcal{J}_p = \mathbb{1} - \sigma^{-1}\mathbb{J}\,, \quad \sigma^{-1} = \begin{bmatrix} 0 & 0 & \mathbb{1}_d \\ \mathbb{1}_d & 0 & 0 \\ 0 & \mathbb{1}_d & 0 \end{bmatrix}, \quad \mathbb{J} = \begin{bmatrix} \mathbb{J}_1 & 0 & 0 \\ 0 & \mathbb{J}_2 & 0 \\ 0 & 0 & \mathbb{J}_3 \end{bmatrix},$$

where $\mathbb{1}_d$ is the d–dimensional identity matrix

the third repeat of $\sigma^{-1}\mathbb{J}$ is block-diagonal

$$(\sigma^{-1}\mathbb{J})^2 = \sigma^{-2} \begin{bmatrix} \mathbb{J}_2\mathbb{J}_1 & 0 & 0 \\ 0 & \mathbb{J}_3\mathbb{J}_2 & 0 \\ 0 & 0 & \mathbb{J}_1\mathbb{J}_3 \end{bmatrix}$$

$$(\sigma^{-1}\mathbb{J})^3 = \begin{bmatrix} \mathbb{J}_2\mathbb{J}_1\mathbb{J}_3 & 0 & 0 \\ 0 & \mathbb{J}_3\mathbb{J}_2\mathbb{J}_1 & 0 \\ 0 & 0 & \mathbb{J}_1\mathbb{J}_3\mathbb{J}_2 \end{bmatrix} \quad \text{as } \sigma^3 = \mathbb{1}$$

period n temporal stability

as $\sigma^n = \mathbb{1}$, the trace of the $[nd \times nd]$ matrix for a period n lattice state

$$\mathrm{tr}\,(\sigma^{-1}\mathbb{J})^k = \delta_{k,rn}\, n\, \mathrm{tr}\,\mathbb{J}_p^r\,, \quad \mathbb{J}_p = \mathbb{J}_n \mathbb{J}_{n-1} \cdots \mathbb{J}_2 \mathbb{J}_1$$

non-vanishing only if k is a multiple of n, where $\mathbb{J}_p$ is the forward-in-time $[d \times d]$ Jacobian (or Floquet) matrix of the periodic orbit p.

orbit stability vs. temporal stability

evaluate the Hill determinant $\mathrm{Det}\,(\mathcal{J}_p)$ by expanding

$$\begin{aligned}
\ln \mathrm{Det}\,(\mathcal{J}_p) &= \mathrm{tr}\,\ln(\,\mathbb{1} - \sigma^{-1}\mathbb{J}) \,=\, -\sum_{k=1}^{\infty}\frac{1}{k}\,\mathrm{tr}\,(\sigma^{-1}\mathbb{J})^k \\
&= -\mathrm{tr}\,\sum_{r=1}^{\infty}\frac{1}{r}\mathbb{J}_p^r = \ln\det(\,\mathbb{1}_d - \mathbb{J}_p)\,.
\end{aligned}$$

orbit stability vs. temporal stability

The orbit Jacobian matrix $\mathcal{J}_p$ evaluated on a lattice state X_p satisfying the temporal lattice first-order difference equation

and the dynamical, forward in time Jacobian matrix $\mathbb{J}_p$ are thus related by Hill's formula

$$\mathrm{Det}\, \mathcal{J}_p = \det\left(\mathbb{1}_d - \mathbb{J}_p \right) ,$$

which relates the global orbit stability to the Floquet, forward in time evolution stability

for any dynamical system, dissipative as well as Hamiltonian

linear force : a cat map evolving in time

force $F(x) = Kx$ linear in the displacement x , $K \in \mathbb{Z}$

$$\begin{aligned} x_{t+1} &= x_t + p_{t+1} && \bmod 1 \\ p_{t+1} &= p_t + Kx_t && \bmod 1 \end{aligned}$$

Continuous Automorphism of the Torus, or

Hamiltonian cat map

temporal stability of the nth iterate given bythe area preserving map

$$J^n = \begin{bmatrix} 0 & 1 \\ -1 & s \end{bmatrix}^n$$

for 'stretching' $s = \mathrm{tr}\, J > 2$ the map is hyperbolic

nonlinear force : a Hamiltonian Hénon map evolving in time

force $F(x)$ nonlinear in the displacement x ,

$$\begin{aligned} x_{t+1} &= a - x_t^2 - p_t \\ p_{t+1} &= x_t \end{aligned}$$

Hamiltonian Hénon map

temporal stability of the nth iterate given by a nonlinear, area preserving map

$$J^n(x_0, x_1) = \prod_{m=0}^{n-1} \begin{bmatrix} -2x_m & -1 \\ 1 & 0 \end{bmatrix}$$

for 'stretching' $a > 5.69931\ldots$ the map is hyperbolic

cat map in Lagrangian form

replace momentum by velocity

$$p_{t+1} = (\phi_{t+1} - \phi_t)/\Delta t$$

result[2] : discrete time lattice field ϕ equations

2-step difference equation

$$\phi_{t+1} - s\,\phi_t + \phi_{t-1} = -m_t$$

integer m_t ensures that

ϕ_t lands in the unit interval

$$m_t \in \mathcal{A}\,, \quad \mathcal{A} = \{\text{finite alphabet}\}$$

[2] I. Percival and F. Vivaldi, Physica D **27**, 373–386 (1987).

Hamiltonian Hénon map in Lagrangian form

replace momentum by velocity

$$p_{t+1} = (\phi_{t+1} - \phi_t)/\Delta t$$

nonlinear 2-step difference equation

$$\phi_{t+1} + 2\,\phi_t^2 + \phi_{t-1} = -a$$

spatiotemporally infinite ‘spatiotemporal cat’

spatiotemporal cat

consider a 1 spatial dimension lattice, with field ϕ_{nt}
(the angle of a kicked rotor "particle" at instant t, at site n)

require

- each site couples to its nearest neighbors $\phi_{n\pm1,t}$
- invariance under spatial translations
- invariance under spatial reflections
- invariance under the space-time exchange

Gutkin & Osipov[3] :

2-dimensional coupled cat map lattice

$$\phi_{n,t+1} + \phi_{n,t-1} - 2s\,\phi_{nt} + \phi_{n+1,t} + \phi_{n-1,t} = -m_{nt}$$

[3] B. Gutkin and V. Osipov, Nonlinearity **29**, 325–356 (2016).

temporal cat orbit Jacobian matrix

$$\mathcal{J}\,\mathsf{X} + \mathsf{M} = 0$$

with

$$\mathsf{X} = (\phi_{t+1}, \cdots, \phi_{t+n})\,, \quad \mathsf{M} = (m_{t+1}, \cdots, m_{t+n})$$

are a lattice state, and a symbol block

and $[n\times n]$ orbit Jacobian matrix $\mathcal{J}$ is

$$\sigma - s\,\mathbb{1} + \sigma^{-1} = \begin{pmatrix} -s & 1 & & & 1 \\ 1 & -s & 1 & & \\ & 1 & & \ddots & \\ & & & -s & 1 \\ 1 & & & & -s \end{pmatrix}$$

temporal Hénon orbit Jacobian matrix

$[n \times n]$ orbit Jacobian matrix is

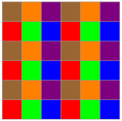
$$\mathcal{J} = \begin{pmatrix} 2\phi_1 & 1 & & & 1 \\ 1 & 2\phi_2 & 1 & & \\ & 1 & & \ddots & \\ & & & 2\phi_{n-1} & 1 \\ 1 & & & 1 & 2\phi_n \end{pmatrix}$$

spatiotemporal cat periodic $[3\times 2]_0$ lattice state

$$F[\mathsf{X}] = \mathcal{J}\mathsf{X} + \mathsf{M} = 0$$

6 field values, on 6 lattice sites $z=(n,t)$, $[3\times 2]_0$ tile :

$$\mathsf{X}_{[3\times2]_0} = \left[\begin{array}{ccc} \phi_{01} & \phi_{11} & \phi_{21} \\ \phi_{00} & \phi_{10} & \phi_{20} \end{array}\right], \qquad 6\,\mathsf{M}_{[3\times2]_0} =$$



where the region of symbol plane shown is tiled by 6 repeats of the $\mathsf{M}_{[3\times2]_0}$ block, and tile color = value of symbol m_z

'stack up' vectors and matrices, vectors as 1-dimensional arrays,

$$\mathsf{X}_{[3\times2]_0} = \left(\begin{array}{c} \phi_{01} \\ \phi_{00} \\ \hline \phi_{11} \\ \phi_{10} \\ \hline \phi_{21} \\ \phi_{20} \end{array}\right), \qquad \mathsf{M}_{[3\times2]_0} = \left(\begin{array}{c} m_{01} \\ m_{00} \\ \hline m_{11} \\ m_{10} \\ \hline m_{21} \\ m_{20} \end{array}\right)$$

with the $[6\times 6]$ orbit Jacobian matrix in block-matrix form

$$
\mathcal{J}_{[3\times 2]_0} = \left(\begin{array}{cc|cc|cc}
-2s & 2 & 1 & 0 & 1 & 0 \\
2 & -2s & 0 & 1 & 0 & 1 \\
\hline
1 & 0 & -2s & 2 & 1 & 0 \\
0 & 1 & 2 & -2s & 0 & 1 \\
\hline
1 & 0 & 1 & 0 & -2s & 2 \\
0 & 1 & 0 & 1 & 2 & -2s
\end{array}\right)
$$

summary : orbit stability vs. temporal stability

orbit Jacobian matrix

$\mathcal{J}_{ij} = \frac{\delta F[X]_i}{\delta \phi_j}$ stability under global perturbation of the whole orbit

for n large, a huge $[dn \times dn]$ matrix

temporal Jacobian matrix

J propagates initial perturbation n time steps

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orbit stability vs. temporal stability

J and $\mathcal{J}$ are always related by[4]

Hill's formula

$$|\mathrm{Det}\,\mathcal{J}_\mathsf{M}| = |\det\,(\mathbf{1} - J_\mathsf{M})|$$

$\mathcal{J}$ is huge, even ∞-dimensional matrix
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