turbulence in spacetime

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Working Across Scales

April 13, 2023 - Emory

overview

1. what this talk is about
2. turbulence in large domains
3. space is time
4. bye bye, dynamics

do clouds solve PDEs?

do clouds integrate Navier-Stokes equations?

NO!

$\Longrightarrow$ other swirls $\Longrightarrow$

do clouds satisfy Navier-Stokes equations?

yes!

they satisfy them locally, everywhere and at all times

does cat's visual cortex solve a PDE?

do brains integrate Hodgkin-Huxley equations?

NO!

$\Longrightarrow$ other swirls $\Longrightarrow$

do neurons satisfy FitzHugh-Nagumo equations?

yes!

they satisfy them locally, everywhere and at all times

1. turbulence in large domains
2. space is time
3. spacetime
4. bye bye, dynamics

goal : enumerate the building blocks of turbulence

Navier-Stokes equations

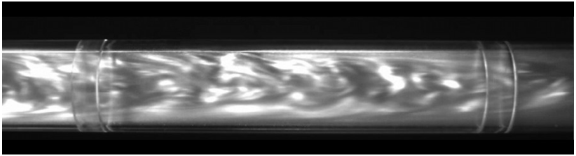
$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = \frac{1}{R}\nabla^2\mathbf{v} - \nabla p + \mathbf{f}\,, \qquad \nabla \cdot \mathbf{v} = 0,$$

velocity field $\mathbf{v} \in \mathbb{R}^3$; pressure field p ; driving force $\mathbf{f}$

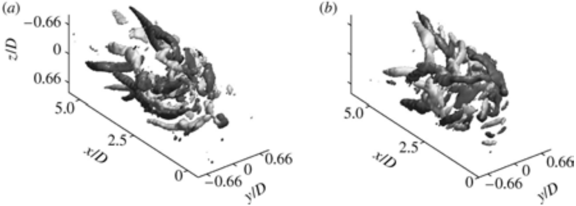
describe turbulence

starting from the equations (no statistical assumptions)

challenge : experiments are amazing



T. Mullin lab



B. Hof lab

pipe theory and numerics

amazing experiments!
amazing numerics!
beautiful visualizations !

“Exact Coherent Structures” :
numerical Navier-Stokes

isosurfaces and cross sections
of the streamwise velocity

red (blue) streaks
are faster (slower)
than the base flow

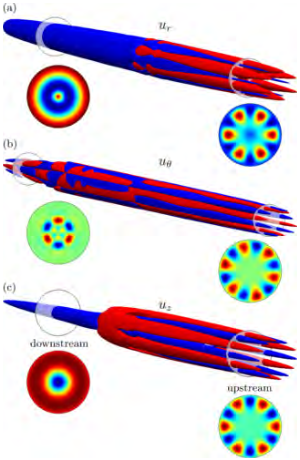


figure from[1]

[1] P. Ritter et al., Phys. Rev. Fluids **3**, 013901 (2018).

geometry of turbulence in wall-bounded shear flows :

a stroll through 61,506 dimensions

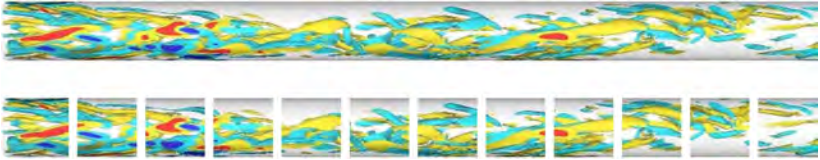
- online self-study tutorial ChaosBook.org/tutorials
- here you you get : personalized YouTube guided tour through turbulent statespace

the tutorial[2]

[2] J. F. Gibson and P. Cvitanović, *Movies of plane Couette*, tech. rep. (Georgia Inst. of Technology, 2015).

building blocks of turbulence ?

pipe flow close to onset of turbulence [3]



we have a detailed theory of small turbulent fluid cells

can we can we construct the infinite pipe by coupling small turbulent cells ?

what would that theory look like ?

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[3] M. Avila and B. Hof, Phys. Rev. **E 87** (2013)

goal : we can do 3D turbulence, but for today

Navier-Stokes equations

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = \frac{1}{R}\nabla^2 \mathbf{v} + (\cdots)$$

velocity field $\mathbf{v}(\mathbf{x}; t) \in \mathbb{R}^3$

not helpful for developing intuition

we cannot visualize 3D velocity field at every 3D spatial point

look instead at 1D ‘flame fronts’

(3+1) spacetime dimensional “Navier-Stokes”

Navier-Stokes equations **(1822)**

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = \frac{1}{R}\nabla^2 \mathbf{v} + (\cdots)$$

▼
▼
▼

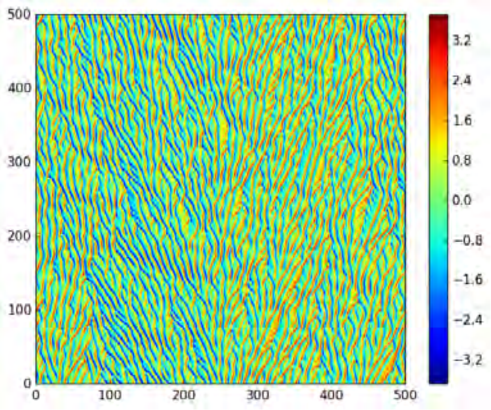
Kuramoto-Sivashinsky (1+1)-dimensional PDE **(1975)**

$$u_t + u\nabla u = -\nabla^2 u - \nabla^4 u\,, \qquad x \in \mathbb{R}\,,$$

describes spatially extended systems such as

- flame fronts in combustion
- reaction-diffusion systems
- …

an example : Kuramoto-Sivashinsky on a large domain



[horizontal] space $x \in [0, L]$ [up] time evolution

- turbulent behavior
- simpler physical, mathematical and computational setting than Navier-Stokes

Q: what is the physical dimension of a turbulent flow ?

question

does an attractor of a dissipative flow have a “dimension” ?

Foias *et al*[4]

mathematician’s answer

dimension of ‘inertial manifold’ is finite

[4] C. Foias et al., C. R. Acad. Sci. Paris, Ser. I **301**, 285–288 (1985).

Q: what is the physical dimension of a turbulent flow ?

question

does an attractor of a dissipative flow have a “dimension” ?

Ginelli, Chaté, Radons, *et al*[4,5,6,7]

physicist’s answer

‘Lyapunov covariant vectors’ split into

(a) finite number of ‘physical,’ entangled directions, in the tangent space of the attractor

(b) infinitely many hyperbolically decaying directions that are isolated and do not mix

[4] A. Politi et al., Physica D **224**, 90 (2006).

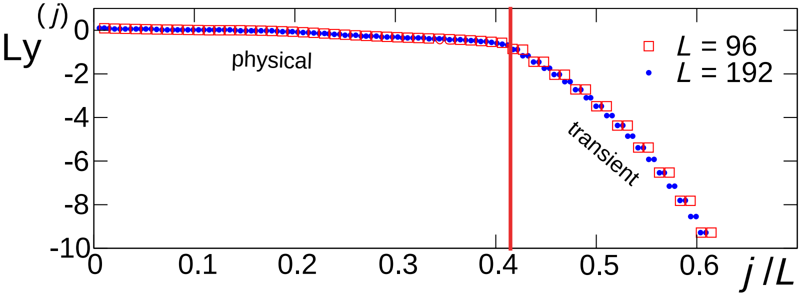
[5] F. Ginelli et al., Phys. Rev. Lett. **99**, 130601 (2007).

[6] H. L. Yang et al., Phys. Rev. Lett. **102**, 074102 (2009).

[7] K. A. Takeuchi et al., Phys. Rev. Lett. **103**, 154103 (2009).

the killer plot : physical dimension grows linearly with the domain size!

Kuramoto-Sivashinsky Lyapunov spectrum
cells $L = 22, 96, 192$: it scales!



Now double # computational elements, fixed L :
all new ones go to the transient spectrum[8,9] !

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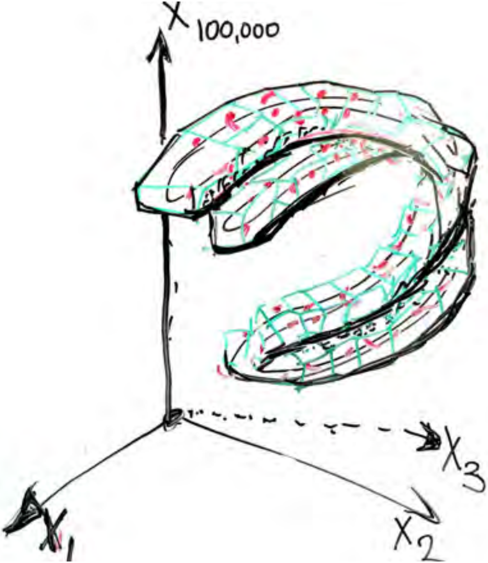
[8] H. L. Yang et al., Phys. Rev. Lett. **102**, 074102 (2009).

[9] X. Ding et al., Phys. Rev. Lett. **117**, 024101 (2016).

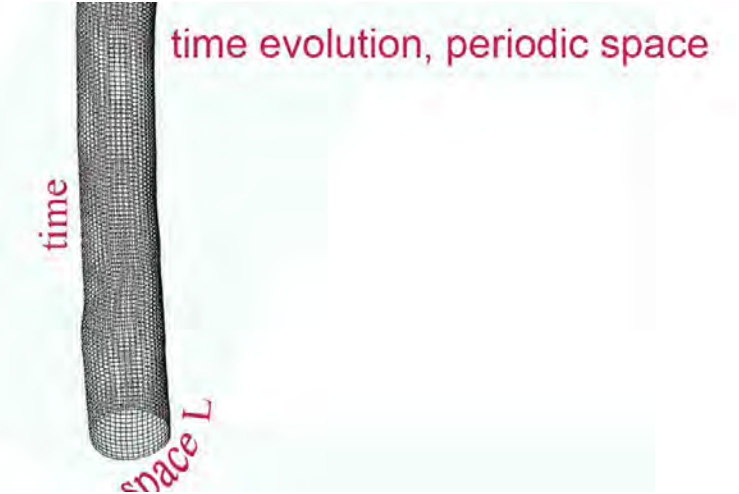
a finite physical manifold embedded in ∞ dimensions



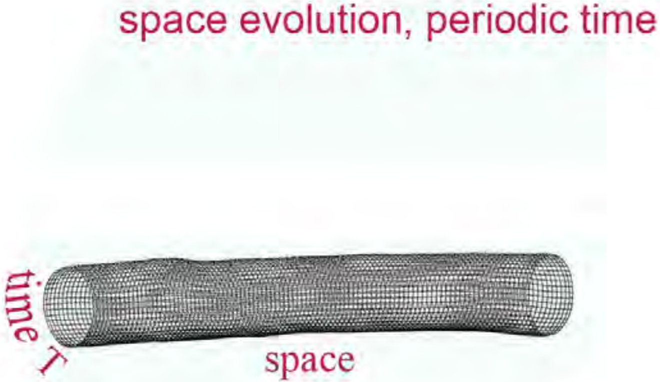
inertial manifold

the attractor is stuffed into a finite-dimensional body bag

conventional computations : compact space, infinite time cylinder



so far : Navier-Stokes on compact spatial domains, all times can do also : compact time, infinite space cylinder



but integrations are uncontrollably unstable

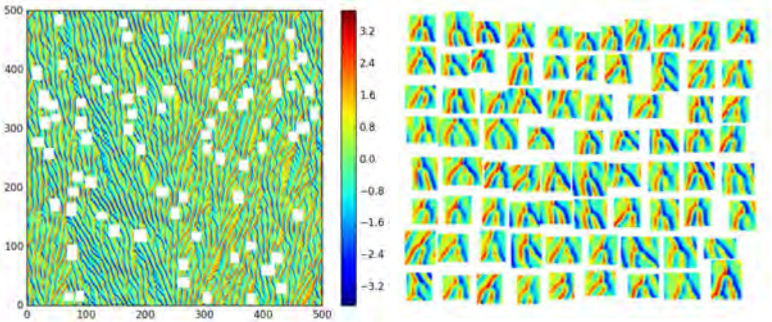
neither time nor space integration works
for large domains

rethink the formulation!

part 3

Kuramoto-Sivashinsky on a large spacetime domain

the same small tile recurs often in a turbulent pattern



goal : define, enumerate nearly recurrent tiles

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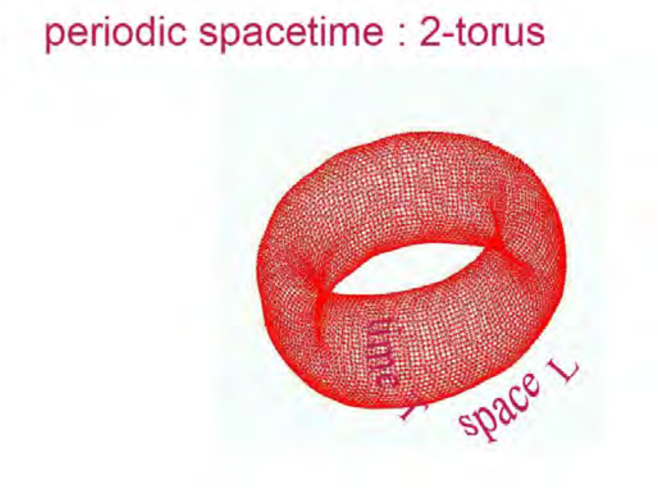
use spatiotemporally compact solutions as spacetime ‘tiles’



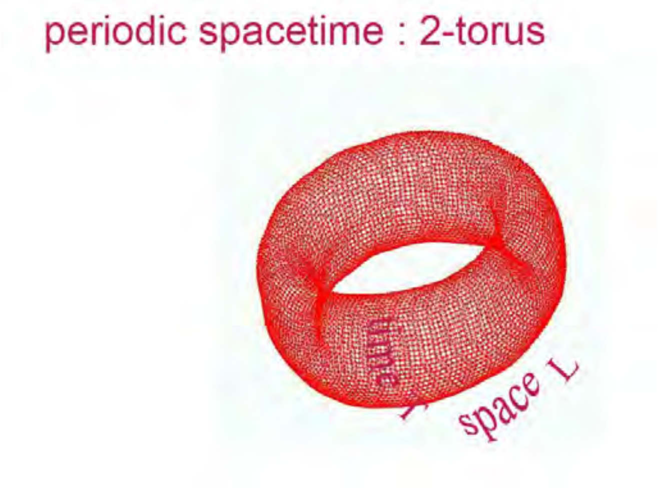
shadows a small patch of spacetime

spatiotemporal shadowing

plane Couette doubly periodic shadow: click here

every calculation is a spatiotemporal lattice calculation

field is discretized as $\tilde{u}_{k\ell}$ values
over NM points of a periodic lattice



there is no more time evolution

solution to Kuramoto-Sivashinsky is now given as

condition that

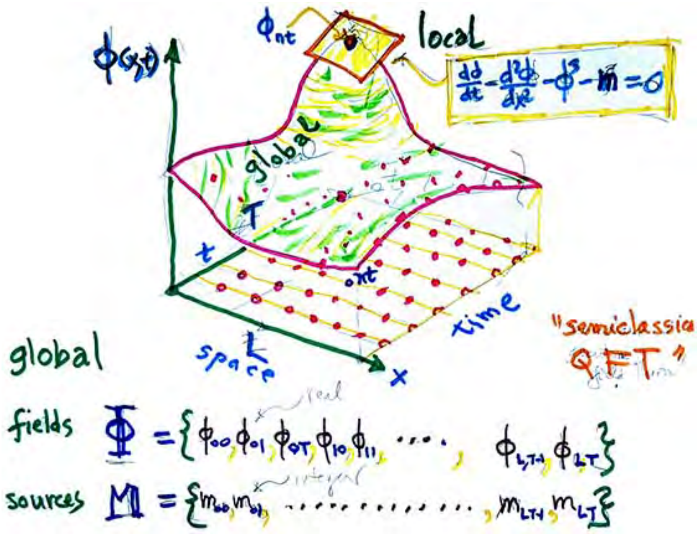
at each lattice point $k\ell$
the tangent field at $\tilde{u}_{k\ell}$

satisfies the equations of motion

$$\left[-i\omega_\ell - (q_k^2 - q_k^4)\right]\tilde{u}_{k\ell} + i\frac{q_k}{2}\sum_{k'=0}^{N-1}\sum_{m'=0}^{M-1}\tilde{u}_{k'm'}\tilde{u}_{k-k',m-m'} = 0$$

this is a local tangent field constraint on a global solution

think globally, act locally



for each symbol array M, a periodic lattice state X_M

unexpected gift from nature

robust : no exponential instabilities
as there are no finite time / space integrations

no need for $\sim 10^{-11}$ accuracies,

SO

accuracy to a few % suffices,
you only need to get the shape of a solution right

part 4

1. turbulence in large domains
2. space is time
3. spacetime
4. **spacetime computations**
5. bye bye, dynamics

how do clouds solve PDEs?

clouds do not **NOT** integrate Navier-Stokes equations

$\Longrightarrow$ other swirls $\Longrightarrow$

do clouds satisfy Navier-Stokes equations?

yes!

they satisfy them **locally**, everywhere and at all times

the equations imposed as local constraints

Kuramoto-Sivashinsky equation

$$F(u) = u_t + u_{xx} + u_{xxxx} + u u_x = 0$$

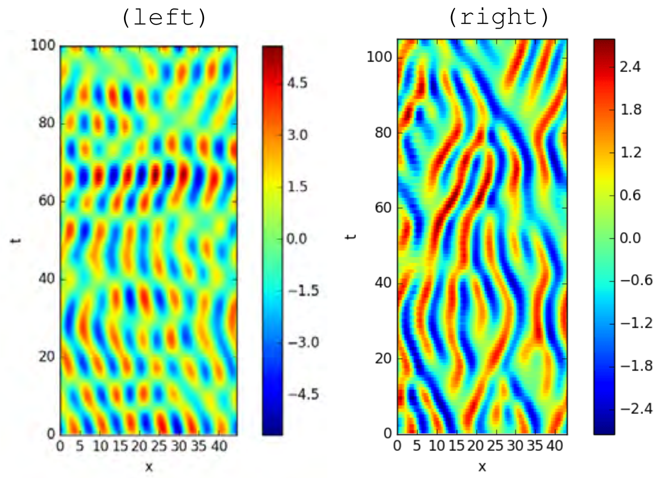
for example, minimize

cost function

$$G \equiv \frac{1}{2} |F(u)|^2_{L \times T}$$

we can do this !

KS invariant 2-torus found variationally



(left) initial spatiatiotemporal guess
(right) converged invariant 2-torus

Gudorf 2018

1. turbulence in large domains
2. space is time
3. spacetime
4. **fundamental tiles**
5. bye bye, dynamics

building blocks of turbulence

how do we recognize a cloud?

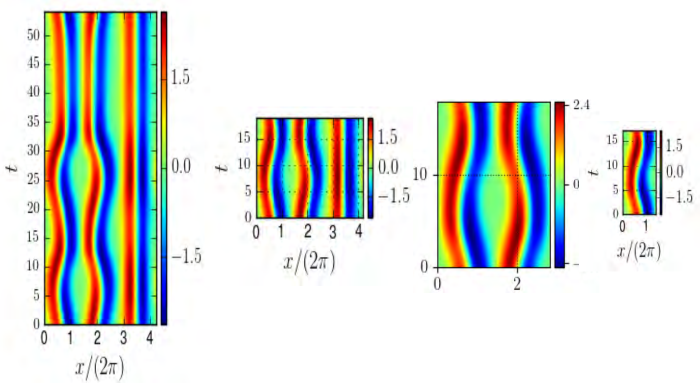
WATCH

 $\Longrightarrow$ other swirls $\Longrightarrow$ 

by recurrent shapes!

so, construct an alphabet of possible shapes

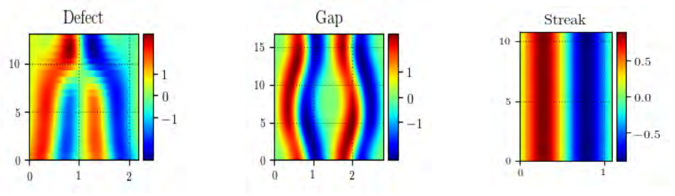
extracting a fundamental tile



1) invariant 2-torus
2) invariant 2-torus computed from initial guess cut out from 1)
3) “gap" invariant 2-torus, initally cut out from 2)
4) the “gap" prime invariant 2-torus fundamental domain

a trial set of prime (rubber) tiles

an alphabet of Kuramoto-Sivashinsky fundamental tiles



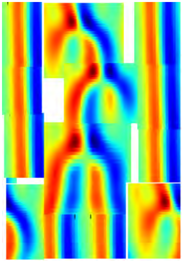
utilize also discrete symmetries :
spatial reflection, spatiotemporal shift-reflect, $\cdots$

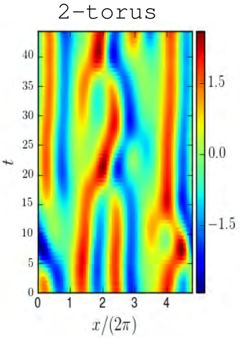
part 5

1. turbulence in large domains
2. space is time
3. spacetime
4. fundamental tiles
5. **gluing tiles**
6. bye bye, dynamics

a qualitative tiling guess

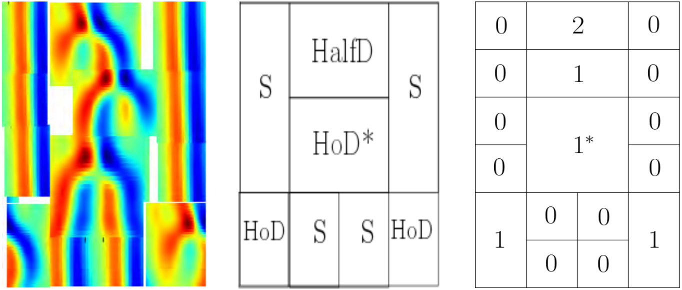
a tiling and the resulting solution





turbulence.zip : each solution has a unique symbolic name

symbolic dynamics is 2-dimensional!



- each symbol indicates a corresponding spatiotemporal tile

part 5

1. turbulence in large domains
2. space is time
3. bye bye, dynamics

do this ?

compute locally, adjust globally

Navier-Stokes and orbit hunting codes

- T. M. Schneider : developing a matrix-free variational Navier-Stokes code, machine learning initial guesses
- M. N. Gudorf : Orbithunter - framework for nonlinear dynamics and chaos
- D. Lasagna and A. Sharma : developing variational adjoint solvers to find periodic orbits with long periods
- Q. Wang : parallelizing spatiotemporal computation is FLOPs intensive, but more robust than integration forward in time

it's rocket science[10,11,12,13]

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[10] J. P. Parker and T. M. Schneider, J. Fluid. Mech. **941**, A17 (2022).

[11] M. N. Gudorf, *Orbithunter: Framework for Nonlinear Dynamics and Chaos*, tech. rep. (School of Physics, Georgia Tech, 2021).

[12] M. V. Lakshmi et al., Physica D **427**, 133009 (2021).

[13] Q. Wang et al., Phys. Fluids **25**, 110818 (2013).

does a bird flock solve a PDE?

does motor cortex integrate Hodgkin-Huxley equations?

NO!

$\Longrightarrow$ near recurrence $\Longrightarrow$

do dragonflies satisfy Navier-Stokes equations?

yes!

they satisfy them locally, everywhere and at all times

summary

1. study turbulence in infinite spatiatemporal domains
2. theory : classify all spatiotemporal tilings
3. numerics : future is spatiotemporal

there is no more time

there is only enumeration of spacetime solutions