A GUIDE TO THE READER

As our intended audience spans many disjoint specialties, from fluid dynamics to quantum field theory, the exposition entails much pedagogical detail.

To aid the reader, here we lay out the flow of the argument in the reverse,[1] by starting with

Part I: Derivation

of the main result of the paper, the deterministic zeta function, Sec. XI *Chaotic field theory*,

$$\zeta = \prod_p \zeta_p \,, \qquad 1/\zeta_p = \phi(t_p)\,. \tag{3}$$

It follows from our deterministic partition sum, Eq. (122),

$$Z[\beta,z] \,=\, \sum_c t_c\,, \quad t_c \,=\, \left(e^{\beta\cdot a_c - \lambda_c}\, z\right)^{N_c}, \tag{122}$$

where λ_c is the stability exponent of deterministic solution Φ_c, Sec. X *Bravais lattice stability*, and z is a generating function variable. Each Φ_c is multi-periodic, Sec. VII A *Prime orbits over two-dimensional lattices* (see also Appendix B *Bravais sublattices*), and defines its Bravais lattice $\mathcal{L}_c$, Sec. VI *Bravais lattices*, with N_c the Bravais lattice volume. The prime orbits Φ_p are deterministic zeta function's Eq. (3) building blocks.

What is new is that the partition sum is over probabilities of deterministic field configurations, the exact solutions Φ_c of defining equations of the system, Sec. III *Deterministic field theory*, so the probability density is a porcupine of Dirac deltas, Fig. 2 (b),

$$\begin{aligned} Z_{\mathbb{A}}[\beta] &= \sum_c \int_{\mathcal{M}_c} d\Phi_{\mathbb{A}}\, \delta(F[\Phi])\, e^{\beta\cdot A_{\mathbb{A}}[\Phi]} \\ &= \sum_c \frac{1}{|\mathrm{Det}\mathcal{J}_{\mathbb{A},c}|}\, e^{\beta\cdot A_c}\,, \end{aligned} \tag{42}$$

The weight of the solution Φ_c is given by its orbit Jacobian $\mathrm{Det}\mathcal{J}_c$, Sec. V *Spatiotemporal stability of a periodic state*, the key innovation of our field theory of chaos and turbulence.

In Gutzwiller-Ruelle[6–8] temporal periodic orbit formulation of chaotic dynamics, orbit Jacobian is a determinant of a matrix, Sec. IX *Primitive cell stability*, evaluated over a finite number of lattice sites. Such weight is *not* multiplicative for orbit repeats, for example

$$\mathrm{Det}\,\mathcal{J}_{r*01} \neq (\mathrm{Det}\,\mathcal{J}_{01})^r\,. \tag{109}$$

Our chaotic field theory, however, is formulated on the totality of infinite Bravais lattices, with infinite-dimensional orbit Jacobian operators $\mathcal{J}_p$, Fig. 9 (c). The orbit Jacobian is now a functional determinant, with stability exponent $\lambda_c = \ln \mathrm{Det}\mathcal{J}_p$ per lattice site evaluated by integrating Bloch bands over the Sec. VIII *Reciprocal lattice* first Brillouin zone,

$$\lambda \,=\, \frac{1}{(2\pi)^d}\int_{\mathbb{B}} dk^d\, \ln\left[\mathsf{p}(\mathsf{k})^2 + \mu^2\right]\,. \tag{112}$$

In contrast to Eq. (109), Bravais lattice weight is multiplicative, see Eq. (120), the essential property that underpins our derivation of deterministic zeta function Eq. (3).

So, the spatiotemporal deterministic zeta function is beautiful enough to grace a T-shirt. But no child is born understanding a zeta function.

Part II: Applications

To make it tangible, we define its essential ingredients in Sec. I *Lattice field theory*, and introduce in Sec. IV *Examples of spatiotemporal lattice field theories*, in particular the simplest of chaotic field theories that captures the essence of spatiotemporal chaos, the piecewise linear Sec. IV A *Spatiotemporal cat*, a discretization of the Klein-Gordon equation,

$$(-\Box + \mu^2)\,\Phi - \mathsf{M} = 0\,. \tag{4}$$

Its history is reviewed and credits given in Appendix A 1 *Spatiotemporal cat*, and results of our calculations are presented in Appendix C *Computation of spatiotemporal cat periodic states*.

Dynamical zeta functions convergence is in part due to periodic orbits shadowing, ChaosBook Sec. 23.1 *Pseudocycles and shadowing*. In Sec. XII *Shadowing* we show that spatiotemporal periodic states also shadow each other.

The piecewise linear spatiotemporal cat Eq. (4) is too simple to illustrate the band structure of orbit Jacobians. In Appendix D *Spectra of orbit Jacobian operators for nonlinear field theories*, we give a glimpse of calculations undertaken in the companion paper III.[4]

In summary, every section of this paper is necessary to derive, or illustrate the main result, our deterministic zeta function Eq. (3). We see no way of splitting the derivation into several self-contained short papers or references to literature. Hence, a paper that is long. Even this overview is too long.

Sincerely
Predrag Cvitanović and Han Liang

[1] Purple text, such as Eq. (3), is a live hyperlink to the Eq. (3) of the permanent ver. 1 of the article, arXiv.org/pdf/2503.22972v1. Concepts and results that we believe are new, are marked blue: for example, deterministic field theory.