# dynamical zeta functions: what, why and what are the good for?

Predrag Cvitanović

Georgia Institute of Technology

November 2 2011

I accept chaos
I am not sure that it accepts me

—Bob Dylan, *Bringing It All Back Home*

**in physics, no problem is tractable**

requires summing up exponentially increasing # of exponentially decreasing terms

yet

**in practice**

every physical problem must be tractable

"can't do" doesn't cut it

# dynamical systems

## state space

a manifold $\mathcal{M} \in \mathbb{R}^d$ : $d$ numbers determine the state of the system

## representative point

$x(t) \in \mathcal{M}$
a state of physical system at instant in time

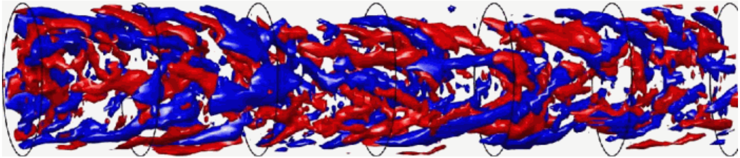
## today's experiments

### example of a representative point

$x(t) \in \mathcal{M}$, $d = \infty$

a state of turbulent pipe flow at instant in time

Stereoscopic Particle Image Velocimetry $\rightarrow$ 3-$d$ velocity field over the entire pipe[1]

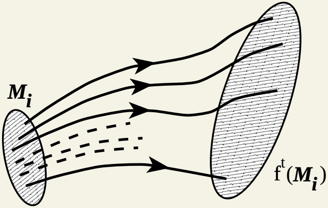


[1] Casimir W.H. van Doorne (PhD thesis, Delft 2004)

## dynamics

map $f^t(x_0)$ = representative point time $t$ later

## evolution




$f^t$ maps a region $\mathcal{M}_i$ of the state space into the region $f^t(\mathcal{M}_i)$.

# dynamics defined

## dynamical system

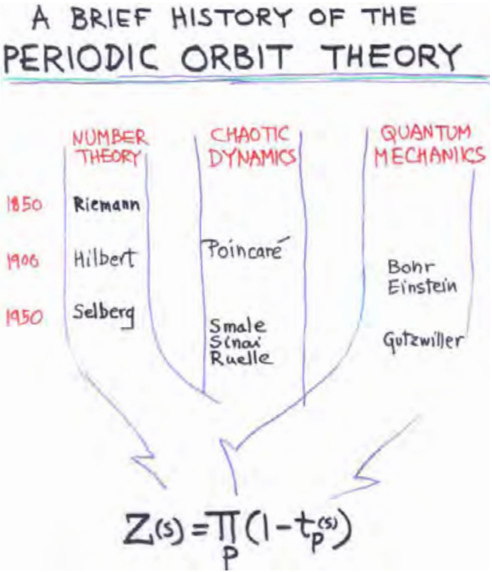
the pair $(\mathcal{M}, f)$

## the problem

enumerate, classify all solutions of $(\mathcal{M}, f)$

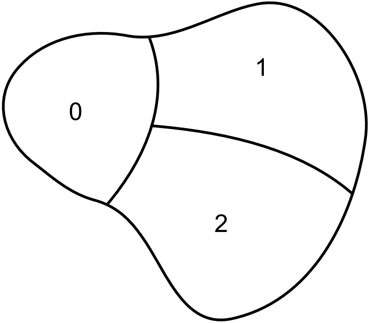
one needs to enumerate $\rightarrow$ hence zeta functions !

## dynamical systems



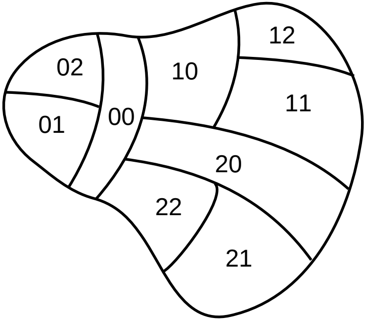
# partition into regions of similar states

**state space coarse partition**



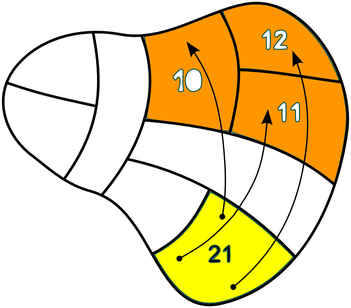

$\mathcal{M} = \mathcal{M}_0 \cup \mathcal{M}_1 \cup \mathcal{M}_2$
ternary alphabet
$\mathcal{A} = \{1, 2, 3\}$.

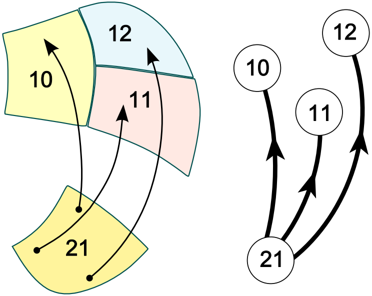
**1-step memory refinement**




$\mathcal{M}_i = \mathcal{M}_{i0} \cup \mathcal{M}_{i1} \cup \mathcal{M}_{i2}$
labeled by nine 'words'
$\{00, 01, 02, \cdots, 21, 22\}$.

# topological dynamics




## one time step

points from $\mathcal{M}_{21}$
reach $\{\mathcal{M}_{10}, \mathcal{M}_{11}, \mathcal{M}_{12}\}$
and no other regions

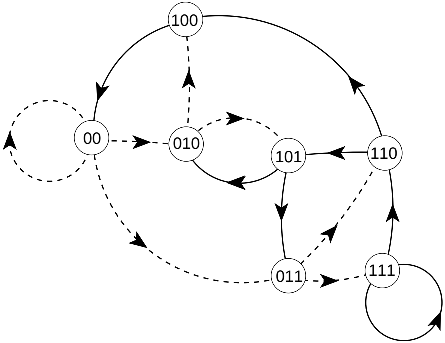
## each region = node

allowed transitions
$T_{10,21} = T_{11,21} = T_{12,21} \neq 0$
directed links

# topological dynamics

**Transition graph $T_{ba}$**

regions reached in one time step




**example: state space resolved into 7 neighborhoods**

$$\{\mathcal{M}_{00}, \mathcal{M}_{011}, \mathcal{M}_{010}, \mathcal{M}_{110}, \mathcal{M}_{111}, \mathcal{M}_{101}, \mathcal{M}_{100}\}$$

## how many ways to get there from here?

$$
\begin{aligned}
(T^n)_{ij} &= \sum_{k_1,k_2,\ldots,k_{n-1}} T_{ik_1} T_{k_1 k_2} \ldots T_{k_{n-1} j} \\
&\propto \lambda_0^n , \qquad \lambda_0 = \text{leading eigenvalue}
\end{aligned}
$$

counts topologically distinct $n$-step paths
starting in $\mathcal{M}_j$ and ending in partition $\mathcal{M}_i$.

compute eigenvalues by evaluating the determinant

**topological (or Artin-Mazur) zeta function**

$$1/\zeta_{\mathrm{top}} = \det(1 - zT)$$

## topological dynamics



$$\det(1 - zT) = \sum \text{[non-(self)-intersecting loops]}$$

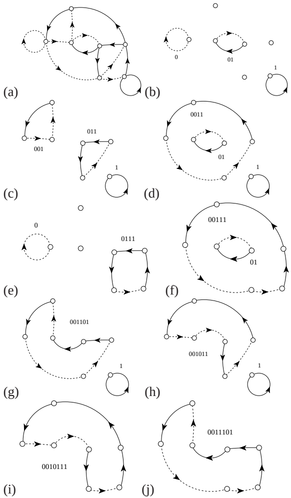


**Figure 15.1:** (a) The region labels in the nodes of transition graph figure 14.3 can be omitted, as the links alone keep track of the symbolic dynamics. (b)-(j) The fundamental cycles (15.23) for the transition graph (a), i.e., the set of its non-self-intersecting loops. Each loop represents a local trace $t_p$, as in (14.5).

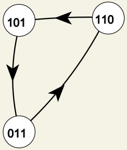
**Example 15.5** ***Nontrivial pruning:*** *The non-self-intersecting loops of the transition graph of figure 14.6 (d) are indicated in figure 14.6 (e). The determinant can be written*

# periodic orbit

## *loop, periodic orbit, cycle*

walk that ends at the starting node, for example

$$t_{011} = L_{110,011} L_{011,101} L_{101,110} =$$




.

zeta function ("partition function")

$$\det(1 - zT)$$

can be read off the graph, expanded as a polynomial in $z$, with coefficients given by products of non-intersecting loops (traces of powers of $T$)

**cycle expansion of a zeta function**

$$\begin{aligned}\det(1 - zT) = 1 - (t_0 + t_1)z - (t_{01} - t_0 t_1)\, z^2 \\ - (t_{001} + t_{011} - t_{01} t_0 - t_{01} t_1)\, z^3 \\ - (t_{0011} + t_{0111} - t_{001} t_1 - t_{011} t_0 - t_{011} t_1 + t_{01} t_0 t_1)\, z^4 \\ - (t_{00111} - t_{0111} t_0 - t_{0011} t_1 + t_{011} t_0 t_1)\, z^5 \\ - (t_{001011} + t_{001101} - t_{0011} t_{01} - t_{001} t_{011})\, z^6 \\ - (t_{0010111} + t_{0011101} - t_{001011} t_1 - t_{001101} t_1 \\ - t_{00111} t_{01} + t_{0011} t_{01} t_1 + t_{001} t_{011} t_1)\, z^7\end{aligned}$$

## if there is one idea that one should learn about chaotic dynamics

it is this

there is a fundamental local $\leftrightarrow$ global duality which says that

**eigenvalue spectrum is dual to periodic orbits spectrum**

for dynamics on the circle, this is called Fourier analysis

for dynamics on well-tiled manifolds, Selberg traces and zetas

for generic nonlinear dynamical systems the duality is embodied in the trace formulas and zeta functions

## global eigenspectrum ⇔ local periodic orbits

Twenty years of schooling
and they put you on the day shift
Look out kid, they keep it all hid

—Bob Dylan, *Subterranean Homesick Blues*

the eigenspectrum $s_0, s_1, \cdots$ of the classical evolution operator

### trace formula, infinitely fine partition

$$\sum_{\alpha=0}^{\infty} \frac{1}{s - s_\alpha} = \sum_p T_p \sum_{r=1}^{\infty} \frac{e^{r(\beta \cdot A_p - s T_p)}}{\left|\det\left(\mathbf{1} - M_p^r\right)\right|} .$$

the beauty of trace formulas lies in the fact that everything on the right-hand-side

–prime cycles $p$, their periods $T_p$ and the eigenvalues of $M_p$–

is a coordinate independent, invariant property of the flow

## deterministic chaos vs. noise

**any physical system:**

noise limits the resolution that can be attained in partitioning the state space

**noisy orbits**

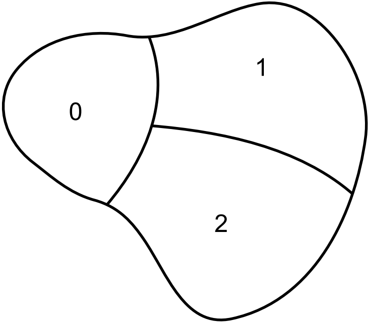
probabilistic densities smeared out by the noise:
a finite # fits into the attractor

**goal: determine**

the **finest attainable** partition

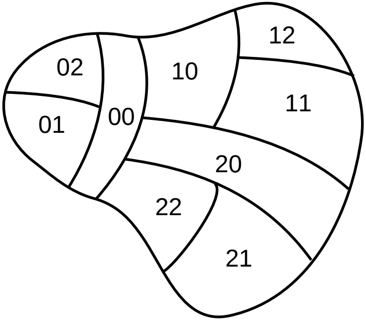
# deterministic partition

**state space coarse partition**



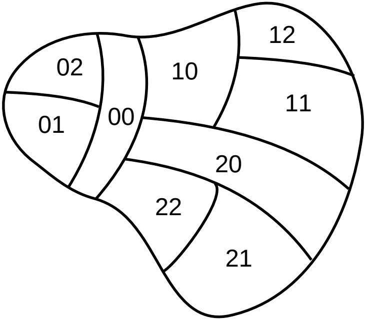

$\mathcal{M} = \mathcal{M}_0 \cup \mathcal{M}_1 \cup \mathcal{M}_2$
ternary alphabet
$\mathcal{A} = \{1, 2, 3\}$.

**1-step memory refinement**

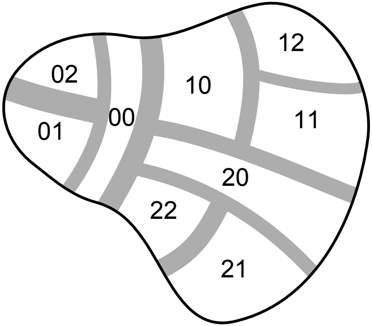



$\mathcal{M}_i = \mathcal{M}_{i0} \cup \mathcal{M}_{i1} \cup \mathcal{M}_{i2}$
labeled by nine 'words'
$\{00, 01, 02, \cdots, 21, 22\}$.

# deterministic vs. noisy partitions




deterministic partition




noise blurs the boundaries

can be refined
*ad infinitum*

when overlapping, no further refinement of partition

# periodic points instead of boundaries

- each partition contains a short periodic point smeared into a 'cigar' by noise

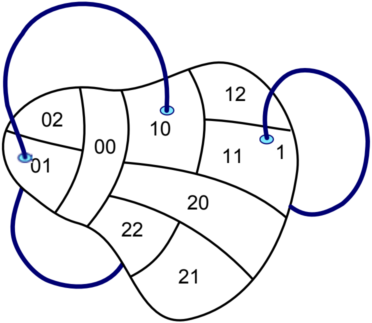
# periodic points instead of boundaries

- each partition contains a short periodic point smeared into a 'cigar' by noise

compute the size of a noisy periodic point neighborhood

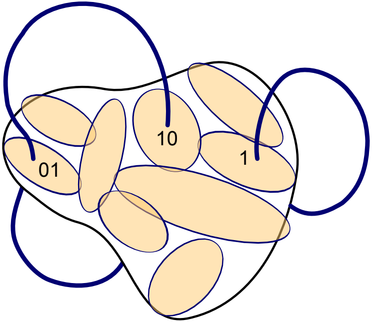
# periodic orbit partition

deterministic partition




some short periodic points:
fixed point $\overline{1} = \{x_1\}$
two-cycle $\overline{01} = \{x_{01}, x_{10}\}$

noisy partition

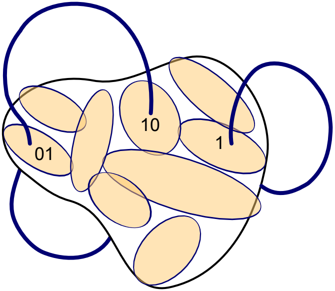



periodic points blurred by the Langevin noise into cigar-shaped densities

- successive refinements of a deterministic partition: exponentially shrinking neighborhoods
- as the periods of periodic orbits increase, the diffusion always wins:

partition stops at the finest attainable partition, beyond which the diffusive smearing exceeds the size of any deterministic subpartition.

# noisy periodic orbit partition




## optimal partition hypothesis

optimal partition:
the maximal set of resolvable periodic point neighborhoods

## why care?

if the high-dimensional flow has only a few unstable directions, the overlapping stochastic 'cigars' provide a *compact cover* of the noisy chaotic attractor, embedded in a state space of arbitrarily high dimension

## strategy

- use periodic orbits to partition state space
- compute local eigenfunctions of the Fokker-Planck operator to determine their neighborhoods
- done once neighborhoods overlap

# how big is the neighborhood blurred by the Langevin noise?

**the** (well known) **key formula**

**composition law for the covariance matrix $Q_a$**

$$Q_{a+1} = M_a Q_a M_a^T + \Delta_a$$

density covariance matrix at time $a$: $Q_a$
Langevin noise covariance matrix: $\Delta_a$
Jacobian matrix of linearized flow: $M_a$

# roll your own cigar

**evolution law for the covariance matrix $Q_a$**

$$Q_{a+1} = M_a Q_a M_a^T + \Delta_a$$

in one time step a Gaussian density distribution with covariance matrix $Q_a$ is smeared into a Gaussian 'cigar' whose widths and orientation are given by eigenvalues and eigenvectors of $Q_{a+1}$

**(1)** deterministically advected and deformed local density covariance matrix $Q \to MQM^T$

**(2)** add noise covariance matrix $\Delta$

add up as sums of squares

# noise along a trajectory

iterate $Q_{a+1} = M_a Q_a M_a^T + \Delta_a$ along the trajectory

if $M$ is contracting, over time the memory of the covariance $Q_{a-n}$ of the starting density is lost, with iteration leading to the limit distribution

$$Q_a = \Delta_a + M_{a-1} \Delta_{a-1} M_{a-1}^T + M_{a-2}^2 \Delta_{a-2} (M_{a-2}^2)^T + \cdots .$$

diffusive dynamics of a nonlinear system is fundamentally different from Brownian motion, as the flow induces a history dependent effective noise: Always!

## noise and a single attractive fixed point

if all eigenvalues of $M$ are strictly contracting, any initial compact measure converges to the unique invariant Gaussian measure $\rho_0(z)$ whose covariance matrix satisfies

**time-invariant measure condition (Lyapunov equation)**

$$Q = MQM^T + \Delta$$

[A. M. Lyapunov 1892, doctoral dissertation]

## example : Ornstein-Uhlenbeck process

width of the natural measure concentrated at the deterministic fixed point $z = 0$

$$Q = \frac{2D}{1 - |\Lambda|^2}\,, \qquad \rho_0(z) = \frac{1}{\sqrt{2\pi\, Q}}\, \exp\left(-\frac{z^2}{2\, Q}\right)\,,$$

- is balance between contraction by $\Lambda$ and diffusive smearing by $2D$ at each time step
- for strongly contracting $\Lambda$, the width is due to the noise only
- As $|\Lambda| \to 1$ the width diverges: the trajectories are no longer confined, but diffuse by Brownian motion

## things fall apart, centre cannot hold

but what if $M$ has *expanding* Floquet multipliers?

both deterministic dynamics and noise tend to smear densities away from the fixed point: no peaked Gaussian in your future

## things fall apart, centre cannot hold

but what if $M$ has *expanding* Floquet multipliers?

Fokker-Planck operator is non-selfadjoint

If right eigenvector is peaked (attracting fixed point)
the left eigenvector is flat (probability conservation)

# case of *repelling* fixed point

if $M$ has only *expanding* Floquet multipliers, both deterministic dynamics and noise tend to smear densities away from the fixed point

balance between the two is described by the *adjoint Fokker-Planck operator*, and the evolution of the covariance matrix $Q$ is now given by

$$Q_a + \Delta = M_a Q_{a+1} M_a^T \,,$$

[aside to control freaks: reachability and observability Gramians]

# optimal partition challenge

**finally in position to address our challenge:**

*determine the finest possible partition for a given noise*

## resolution of a one-dimensional chaotic repeller

As an illustration of the method, consider the chaotic repeller on the unit interval

$$x_{n+1} = \Lambda_0 \, x_n(1 - x_n)(1 - b x_n) + \xi_n \,, \quad \Lambda_0 = 8,\ b = 0.6 \,,$$

with noise strength $2D = 0.002$

## *'the best possible of all partitions'* hypothesis formulated as an algorithm

- calculate the local adjoint Fokker-Planck operator eigenfunction width $Q_a$ for every unstable periodic point $x_a$
- assign one-standard deviation neighborhood $[x_a - Q_a, x_a + Q_a]$ to every unstable periodic point $x_a$
- cover the state space with neighborhoods of orbit points of higher and higher period $n_p$
- stop refining the local resolution whenever the adjacent neighborhoods of $x_a$ and $x_b$ overlap:

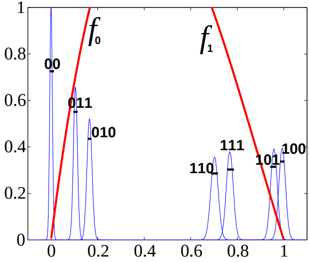
$$|x_a - x_b| < Q_a + Q_b$$

## optimal partition, 1 dimensional map

$f_0, f_1$: branches of deterministic map

local eigenfunctions $\tilde{\rho}_a$ partition state space by neighborhoods of periodic points of period 3

neighborhoods $\mathcal{M}_{000}$ and $\mathcal{M}_{001}$ overlap, so $\mathcal{M}_{00}$ cannot be resolved further



all neighborhoods $\{\mathcal{M}_{0101}, \mathcal{M}_{0100}, \cdots\}$ of period $n_p = 4$ cycle points overlap, so

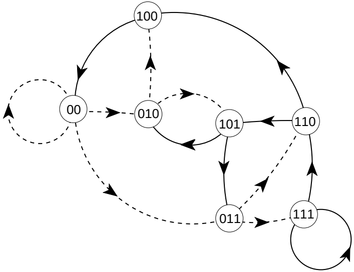
state space can be resolved into 7 neighborhoods

$$\{\mathcal{M}_{00}, \mathcal{M}_{011}, \mathcal{M}_{010}, \mathcal{M}_{110}, \mathcal{M}_{111}, \mathcal{M}_{101}, \mathcal{M}_{100}\}$$

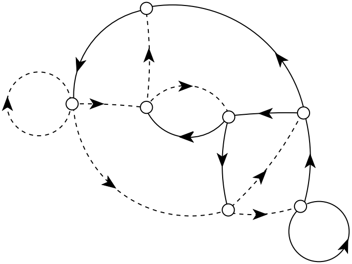
# Markov partition

evolution in time maps intervals
$\mathcal{M}_{011} \to \{\mathcal{M}_{110}, \mathcal{M}_{111}\}$
$\mathcal{M}_{00} \to \{\mathcal{M}_{00}, \mathcal{M}_{011}, \mathcal{M}_{010}\}$, etc..

summarized by the transition graph (links correspond to elements of transition matrix $T_{ba}$): the regions $b$ that can be reached from the region $a$ in one time step



# transition graph



7 nodes = 7 regions of the optimal partition

dotted links = symbol 0 (next region reached by $f_0$)

full links = symbol 1 (next region reached by $f_1$)

region labels in the nodes can be omitted, with links keeping track of the symbolic dynamics

**(1)** deterministic dynamics is full binary shift, but

**(2)** noise dynamics nontrivial and *finite*

# predictions

**escape rate and the Lyapunov exponent of the repeller**

are given by the leading eigenvalue of this $[7 \times 7]$ graph / transition matrix

tests : numerical results are consistent with the full Fokker-Planck PDE simulations

## what is novel?

- we have shown how to compute the **locally optimal partition**, for a given dynamical system and given noise, in terms of local eigenfunctions of the forward-backward actions of the Fokker-Planck operator and its adjoint

## what is novel?

- **A handsome reward:** as the optimal partition is always finite, the dynamics on this ‘best possible of all partitions’ is encoded by a finite transition graph of finite memory, and the Fokker-Planck operator can be represented by a finite matrix

## the payback

**claim:**

optimal partition hypothesis

- the best of all possible state space partitions
- optimal for the given noise

## the payback

**claim:**

optimal partition hypothesis

- optimal partition replaces stochastic PDEs by finite, low-dimensional Fokker-Planck matrices

## the payback

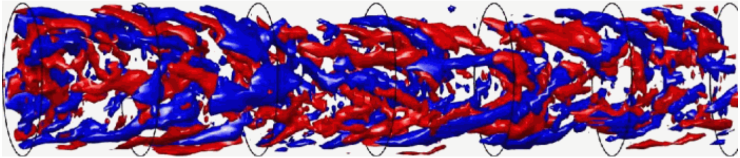
**claim:**

optimal partition hypothesis

- optimal partition replaces stochastic PDEs by finite, low-dimensional Fokker-Planck matrices
- finite matrix calculations, finite cycle expansions $\Rightarrow$ optimal estimates of long-time observables (escape rates, Lyapunov exponents, etc.)

## summary

- Computation of unstable periodic orbits in high-dimensional state spaces, such as Navier-Stokes,



is at the border of what is feasible numerically, and criteria to identify finite sets of the most important solutions are very much needed. Where are we to stop calculating orbits of a given hyperbolic flow?

## summary

- Intuitively, as we look at longer and longer periodic orbits, their neighborhoods shrink exponentially with time, while the variance of the noise-induced orbit smearing remains bounded; there has to be a *turnover time*, a time at which the noise-induced width overwhelms the exponentially shrinking deterministic dynamics, so that no better resolution is possible.

## summary

- We have described here the *optimal partition hypothesis*, a new method for partitioning the state space of a chaotic repeller in presence of weak Gaussian noise, and tested the method in a 1-dimensional setting against direct numerical Fokker-Planck operator calculation.

## references


- D. Lippolis and P. Cvitanović, *How well can one resolve the state space of a chaotic map?*, Phys. Rev. Lett. 104, 014101 (2010); arXiv.org:0902.4269
- D. Lippolis and P. Cvitanović, *Optimal resolution of the state space of a chaotic flow in presence of noise* (in preparation)