ChaosBook.org chapter

average

June 26, 2014 version 14.5.6, you can't always get what you want

Outline

1. dynamical averaging
2. evolution operators

average

time average of an observable

is its integral along a trajectory:
split into a sum of integrals over trajectory segments;
exponentiated,
get a *multiplicative* weight for a trajectory segments;

evolution operators

formulas for averages are thus recast in a multiplicative form that leads to evolution operators:
any *dynamical* average measurable in a chaotic system can be extracted from the spectrum of an appropriately constructed evolution operator

time average

integrated observable

A^t = time integral of observable $a(x)$ along the trajectory

$$A^t(x_0) = \int_0^t d\tau \, a(x(\tau)) \,, \qquad x(t) = f^t(x_0)$$

time average

$$\overline{a(x_0)} = \lim_{t\to\infty} \frac{1}{t} A^t(x_0)$$

average

time average is a property of the orbit, independent of the initial point on that orbit

periodic orbit

integrated observable $A^t(x_0)$ and the time average $\overline{a(x_0)}$ evaluated on a periodic orbit

$$A_p = a_p T_p = \int_0^{T_p} d\tau\, a(f^\tau(x_0)) \qquad x_0 \in \mathcal{M}_p$$

p is a prime cycle, T_p its period

space average

of a evaluated over all state space trajectories $x(t)$ at time t is given by the d-dimensional integral over all initial points x_0 at time $t = 0$:

$$\langle a \rangle(t) = \frac{1}{|\mathcal{M}|}\int_{\mathcal{M}} dx_0\, a(x(t))\,, \qquad x(t) = f^t(x_0)$$

$$|\mathcal{M}| = \int_{\mathcal{M}} dx = \text{volume of } \mathcal{M}$$

expectation value

of an observable a as the asymptotic time and space average over the state space $\mathcal{M}$

$$\langle a \rangle = \frac{1}{|\mathcal{M}|}\int_{\mathcal{M}} dx\, \overline{a(x)} = \lim_{t\to\infty} \frac{1}{|\mathcal{M}|}\int_{\mathcal{M}} dx_0\, \frac{1}{t}\int_0^t d\tau\, a(x(t))$$

intractable in practice ...

a simple idea

the basis of all that follows:

$$\left\langle e^{\beta \cdot A^t} \right\rangle = \frac{1}{|\mathcal{M}|} \int_{\mathcal{M}} dx \; e^{\beta \cdot A^t(x)}$$

β is an auxiliary variable of no physical significance

recover the space average

by differentiation

$$\langle A^t \rangle = \frac{\partial}{\partial \beta} \left\langle e^{\beta \cdot A^t} \right\rangle \Big|_{\beta=0}$$

rate of growth

given by

$$s(\beta) = \lim_{t\to\infty} \frac{1}{t} \ln \left\langle e^{\beta \cdot A^t} \right\rangle .$$

computing $\langle \exp(\beta \cdot A^t) \rangle$ smarter than $\langle a \rangle$

expectation value of the observable and its variance

$$\left.\frac{\partial s}{\partial \beta}\right|_{\beta=0} = \lim_{t\to\infty} \frac{1}{t} \left\langle A^t \right\rangle = \langle a \rangle ,$$

$$\left.\frac{\partial^2 s}{\partial \beta^2}\right|_{\beta=0} = \lim_{t\to\infty} \frac{1}{t} \left\langle (A^t - t\langle a \rangle)^2 \right\rangle ,$$

evolution operators

average

insert the identity

$$1 = \int_{\mathcal{M}} dy \, \delta\big(y - f^t(x)\big) \ ,$$

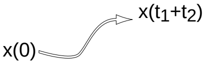
to make dependence on the flow explicit

$$\left\langle e^{\beta \cdot A^t} \right\rangle = \frac{1}{|\mathcal{M}|} \int_{\mathcal{M}} dx \int_{\mathcal{M}} dy \, \delta\big(y - f^t(x)\big) \, e^{\beta \cdot A^t(x)} \, .$$

evolution operator

$$\mathcal{L}^t(y, x) \ = \ \delta\big(y - f^t(x)\big) \ e^{\beta \cdot A^t(x)} \, .$$

integral over observable a is additive along the trajectory



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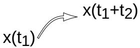

$$\begin{aligned} A^{t_1+t_2}(x_0) &= \int_0^{t_1} d\tau\, a[f^\tau(x)] + \int_{t_1}^{t_1+t_2} d\tau\, a[f^\tau(x)] \\ &= A^{t_1}(x_0) + A^{t_2}(f^{t_1}(x_0))\,. \end{aligned}$$

$A^t(x)$ is additive along the trajectory, so

evolution operator generates a semigroup

$$\mathcal{L}^{t_1+t_2}(y,x) = \int_{\mathcal{M}} dz\, \mathcal{L}^{t_2}(y,z)\mathcal{L}^{t_1}(z,x)$$

linear operator has a spectrum of

eigenvalues s_α and eigenfunctions $\varphi_\alpha(x)$

$$\left[\mathcal{L}^t \varphi_\alpha\right](x) = e^{s_\alpha t} \varphi_\alpha(x)\,, \qquad \alpha = 0, 1, 2, \ldots$$

space average

$$\left\langle e^{\beta \cdot A^t} \right\rangle = \frac{1}{|\mathcal{M}|} \int_{\mathcal{M}} dx \int_{\mathcal{M}} dy \, \phi(y) \mathcal{L}^t(y, x) \phi(x)$$

dominated by the leading eigenvalue in $t \to \infty$ limit

evolution operator

expectation value $\langle a \rangle$

of observable $a(x)$ integrated, $A^t(x) = \int_0^t d\tau\, a(x(\tau))$, and time averaged, A^t/t, over the trajectory $x \to x(t)$ is given by the derivative

$$\langle a \rangle = \left. \frac{\partial s}{\partial \beta} \right|_{\beta=0}$$

of the leading eigenvalue $e^{ts(\beta)}$ of the evolution operator $\mathcal{L}^t$.

the next question:

how do we evaluate the eigenvalues of $\mathcal{L}$?