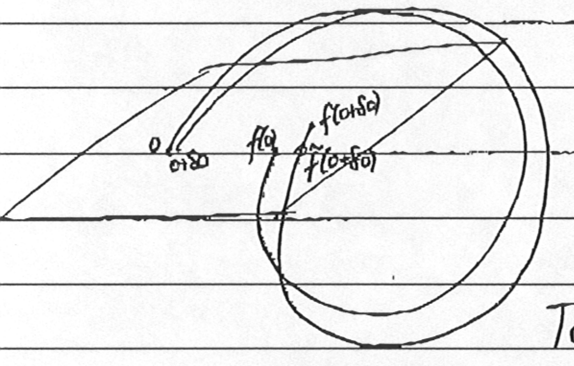
Refining periodic orbits

Fix $c = u^2 - u_x - u_{xxx}$

then indept variables are u, u_x, u_{xx} Poincaré section is $u_x = 0$ (and maybe $u_{xx} > 0$)

Start with approx periodic orbit $(u^0, u^0_{xx}), (u^1, u^1_{xx}), (u^2, u^2_{xx}), \ldots (u^{n-1}, u^{n-1}_{xx})$



$$f(0+\delta 0) = f(0) + J(0)\cdot\delta(0)$$

3×3 matrix computed for fixed "time" x_0

$$\tilde f(0+\delta 0) = f(0) + J(0).\delta(0) - v\,\delta x_0$$

flow

To enforce $\tilde f_1(0+\delta 0) = 0$, must have

$$\delta x_0 = \frac{1}{v_1}\left(J(0).\delta(0)\right)_1 = \frac{1}{v_1}\left(J_{10}\delta_0 + J_{11}\delta_1 + J_{12}\delta_2\right)$$

To reduce problem, take $\delta_1 = 0$, use $\delta_0 = (1,0,0)$, $\delta_2 = (0,0,1)$ (call this $\tilde\delta 0$)

0: $\delta\tilde f_0 = J_{00} - \frac{v_0}{v_1}J_{10}$ $\delta\tilde f_2 = J_{20} - \frac{v_2}{v_1}J_{10}$

2: $\delta\tilde f_0 = J_{02} - \frac{v_0}{v_1}J_{12}$ $\delta\tilde f_2 = J_{22} - \frac{v_2}{v_1}J_{12}$

Thus reduced matrix $\tilde J = \begin{pmatrix} J_{00} - \frac{v_0}{v_1}J_{10} & J_{02} - \frac{v_0}{v_1}J_{12} \\ J_{20} - \frac{v_2}{v_1}J_{10} & J_{22} - \frac{v_2}{v_1}J_{12}\end{pmatrix}$ Flow $v = \begin{pmatrix} u_x \\ u_{xx} \\ u^2 - u_x - c\end{pmatrix}$

$v_0 = 0$ on Poincaré section!

$$\Rightarrow \tilde J = \begin{pmatrix} J_{00} & J_{02} \\ J_{20} - \frac{v_2}{v_1}J_{10} & J_{22} - \frac{v_2}{v_1}J_{12}\end{pmatrix}$$

$\frac{v_2}{v_1} = \frac{u^2 - c}{u_{xx}}$

Problems in unlikely case $u_{xx} = 0$.

Now we want to solve

$$1 + \tilde\delta 1 = \tilde f(0 + \tilde\delta 0) = f(0) + \tilde J_0 \tilde\delta 0$$
$$2 + \tilde\delta 2 = \tilde f(1 + \tilde\delta 1) = f(1) + \tilde J_1 \tilde\delta 1$$
$$\vdots$$
$$0 + \tilde\delta 0 = \tilde f(n-1 + \tilde\delta(n-1)) = f(n-1) + \tilde J_n \tilde\delta(n-1).$$

$$\begin{pmatrix} 1 & & & -\tilde J_{n-1} \\ -\tilde J_0 & 1 & & \\ & \ddots & \ddots & \\ & & -\tilde J_{n-2} & 1 \end{pmatrix}\begin{pmatrix}\tilde\delta 0 \\ \tilde\delta 1 \\ \vdots \\ \tilde\delta(n-1)\end{pmatrix} = \begin{pmatrix} f(n-1) - 0 \\ f(0) - 1 \\ \vdots \\ f(n-2) - (n-1)\end{pmatrix}$$

Then $0' = 0 + \xi\tilde\delta 0$ etc...

damping parameter

In full

$$\begin{pmatrix} 1 & 0 & & & & & * & * \\ 0 & 1 & & & & & * & * \\ * & * & 1 & 0 & & & & \\ * & * & 0 & 1 & & & & \\ & & * & * & 1 & 0 & & \\ & & * & * & 0 & 1 & & \\ & & & & \ddots & \ddots & & \\ & & & & & \ddots & \ddots & \end{pmatrix} \begin{pmatrix} \delta u^0 \\ \delta u_x^0 \\ \delta u^1 \\ \delta u_x^1 \\ \vdots \end{pmatrix} = \begin{pmatrix} f(n-1) - 0 \\ f(0) - 1 \\ \\ \\ \end{pmatrix}$$

$$\begin{array}{cccccc|c} 1 & 0 & \cdots & \bar{J}_{00} & \bar{J}_{02} & & \delta f_{0,0} \\ 0 & 1 & \cdots & \bar{J}_{20} & \bar{J}_{22} & & \delta f_{0,2} \\ J_{00} & J_{02} & \cdots & 0 & 0 & & \delta f_{1,0} \\ J_{20} & J_{22} & \cdots & 0 & 0 & & \delta f_{1,2} \end{array}$$

$$\downarrow$$

$$\begin{array}{cccc|c} 1 & 0 & \bar{J}_{00} & \bar{J}_{02} & \delta f_{0,0} \\ 0 & 1 & \bar{J}_{20} & \bar{J}_{22} & \delta f_{0,2} \\ 0 & J_{02} & -J_{00}\bar{J}_{00} & -J_{00}\bar{J}_{02} & \delta f_{1,0} - J_{00}\,\delta f_{0,0} \\ 0 & J_{22} & -J_{20}\bar{J}_{00} & -J_{20}\bar{J}_{02} & \delta f_{1,2} - J_{20}\,\delta f_{0,0} \end{array}$$

$$\downarrow$$

$$\begin{array}{cccc|c} 1 & 0 & \bar{J}_{00} & \bar{J}_{02} & \delta f_{0,0} \\ 0 & 1 & \bar{J}_{20} & \bar{J}_{22} & \delta f_{0,2} \\ 0 & 0 & -J_{00}\bar{J}_{00} - J_{02}\bar{J}_{20} & -J_{00}\bar{J}_{02} - J_{02}\bar{J}_{22} & \delta f_{1,0} - J_{00}\,\delta f_{0,0} - J_{02}\,\delta f_{0,2} \\ 0 & 0 & -J_{20}\bar{J}_{00} - J_{22}\bar{J}_{20} & -J_{20}\bar{J}_{02} - J_{22}\bar{J}_{22} & \delta f_{1,2} - J_{20}\,\delta f_{0,0} - J_{22}\,\delta f_{0,2} \end{array}$$

ie $\begin{pmatrix} I & \cdots & \bar{J} & \delta f_0 \\ 0 & \cdots & -J\bar{J} & \delta f_1 - J\delta f_0 \end{pmatrix}$

can do elimination with 2×2 blocks.

Further elimination:

$$\begin{pmatrix} I & 0 & 0 & 0 & -\tilde{J}_{n-1} \\ 0 & I & 0 & 0 & -\tilde{J}_0\tilde{J}_{n-1} \\ 0 & 0 & I & 0 & -\tilde{J}_1\tilde{J}_0\tilde{J}_{n-1} \\ 0 & 0 & 0 & I & -\tilde{J}_2\tilde{J}_1\tilde{J}_0\tilde{J}_{n-1} \\ 0 & 0 & 0 & 0 & I - \tilde{J}_{n-2}\cdots\tilde{J}_{n-1} \end{pmatrix} \begin{pmatrix} \tilde{\delta}_0 \\ \tilde{\delta}_1 \\ \tilde{\delta}_2 \\ \vdots \\ \tilde{\delta}_{n-1} \end{pmatrix} = \begin{pmatrix} \delta f_0 \\ \delta f_1 + \tilde{J}_0\delta f_0 \\ \delta f_2 + \tilde{J}_1\delta f_1 + \tilde{J}_1\tilde{J}_0\delta f_0 \\ \vdots \\ \delta f_{n-1} + \tilde{J}_{n-2}\delta f_{n-2} + \cdots + \tilde{J}_{n-2}\tilde{J}_{n-3}\cdots\tilde{J}_0\delta f_0 \end{pmatrix}$$

Finally compute inverse of bottom right element and back-substitute