high (dimensional) life

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ChaosBook.org/overheads/dimension

Mathematical Methods in Computational Neuroscience
Fred Kavli Science Center – Eresfjord

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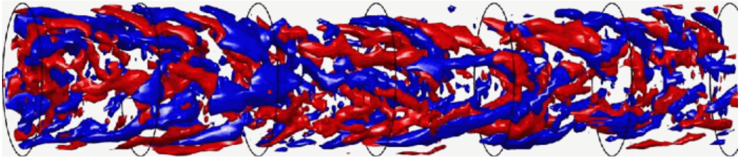
overview

1. what is "dimension"
2. neural manifold
3. state space
4. dimension of the neural manifold

example : pipe experiment data point

a state of turbulent pipe flow at instant in time

Stereoscopic Particle Image Velocimetry $\rightarrow$ 3-*d* velocity field over the entire pipe[1]



[1] B. Hof et al., Science **305**, 1594–1598 (2004).

Navier-Stokes equation

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \frac{1}{\mathrm{Re}} \nabla^2 \mathbf{u}$$

requires at least $\approx 100,000$-dimensional DNS (direct numerical simulation)

the ‘manifold’ dimension is still unknown: at least 100?

where are we?

you have a brain

neuron i activity = axis x_i : 86 billion neurons

experiment ⇒ neural manifolds data

embedded in state space : many low-dimensional manifolds

data ⇒ neural manifold models

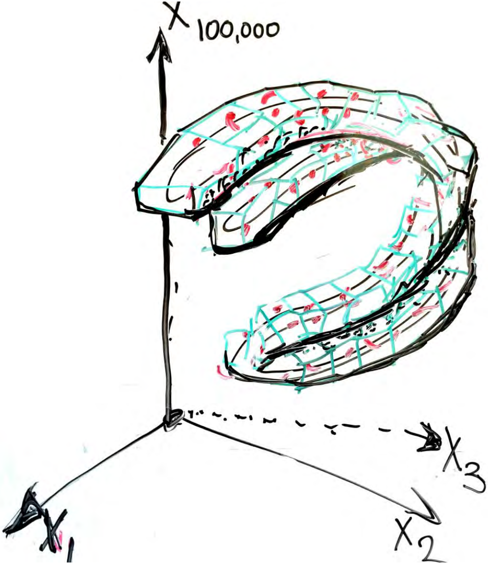
each can be modeled as a dynamical system

[2,3]

[2] J. A. Gallego et al., Nat. Commun. **9**, 4233 (2018).

[3] J. Ladenbauer et al., Nat. Commun. **10**, 4933 (2019).

neural manifold



strange attractor stuffed into a finite-dimensional body bag

Q: what is the dimension of a neural manifold ?

question

does an attractor of a dissipative flow have a “dimension” ?

Q: what is the dimension of a neural manifold ?

question

does an attractor of a dissipative flow have a "dimension" ?

Ginelli, Chaté, Radons, *et al*[4,5,6,7]

covariant vectors answer

'covariant vectors' split into

(a) finite number of directions, in the tangent space of the neural manifold

(b) infinitely many hyperbolically decaying directions that are isolated and do not mix and

[4] A. Politi et al., Physica D **224**, 90–101 (2006).

[5] F. Ginelli et al., Phys. Rev. Lett. **99**, 130601 (2007).

[6] H.-l. Yang et al., Phys. Rev. Lett. **102**, 074102 (2009).

[7] K. A. Takeuchi et al., Phys. Rev. Lett. **103**, 154103 (2009).

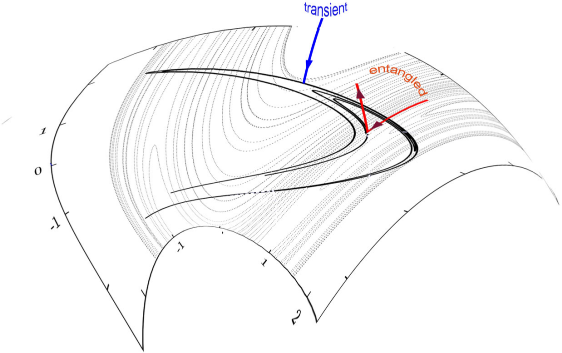
part 2

1. neural manifold
2. state space
3. dimension of the neural manifold

what is "dimension"

the attracting set of a dissipative flow

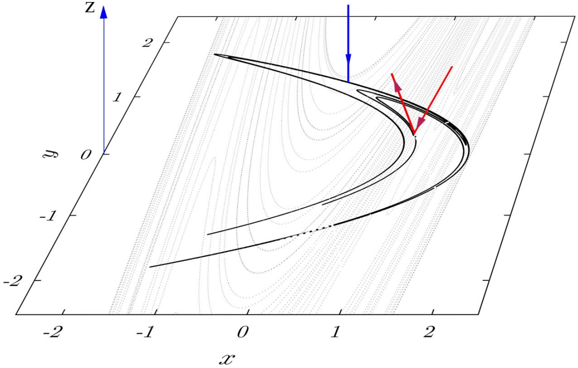
is the curvilinear neural manifold
embedded into ∞-dimensional state space



but try to draw THAT :)

what is "dimension":

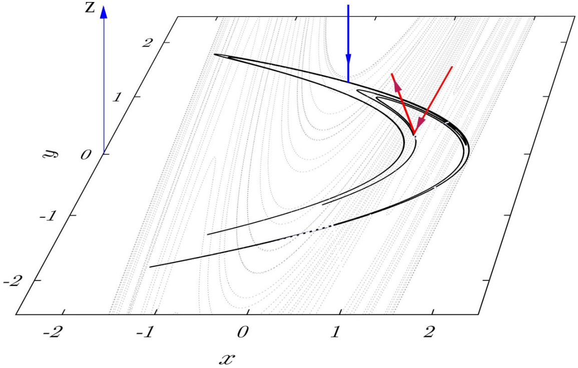
it is believed that the attracting set of a dissipative flow



- is confined to :
  a finite-dimensional smooth *neural manifold*
- "z" directions :
  the remaining ∞ of *transient dimensions*

what is "dimension":

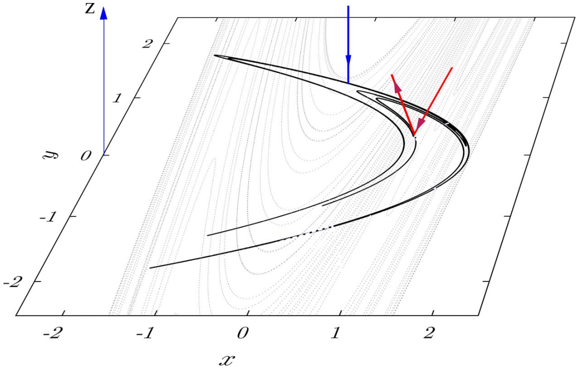
state space of dissipative flow is split into



- neural manifold : spanned locally by entangled covariant vectors, tangent to unstable / stable manifolds
- the rest : spanned by the remaining ∞ of the contracting, decoupled, transient covariant vectors

what is "dimension":

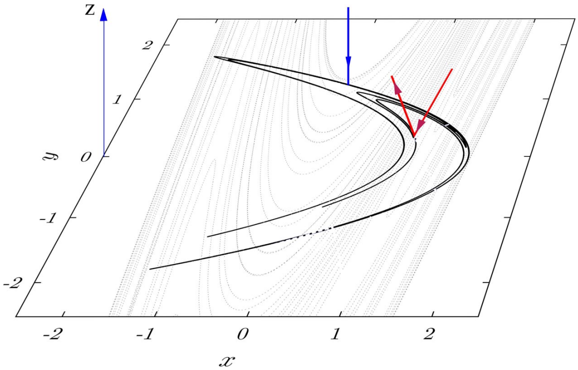
neural manifold



- dynamics of the vectors that span the neural manifold is entangled, with small angles and frequent tangencies
- any transient covariant vector : isolated, nearly orthogonal to all other covariant vectors

what is "dimension":

goal : construct neural manifold



- tile it with a finite collection of bricks centered on recurrent states, each brick $\approx 10 - 100$ dimensions
- span of ∞ of transient covariant vectors : no intersection with the entangled modes

part 3

1. neural manifold
2. **state space**
3. dimension of the neural manifold

if all this works out, it is kinda amazing

simulatation of a neuronal network

requires at least
$\rightarrow$ integration of 10^4-10^6 coupled computational modes

neuronal manifold, tiled

50(?) linear tiles cover the (nonlinear, curved) attracting manifold

each tile low-dimensional

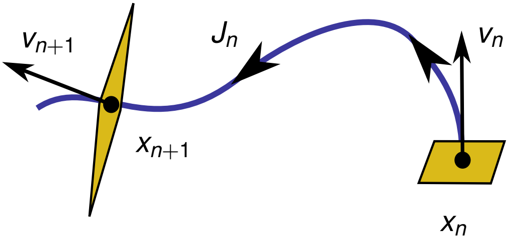
1. neural manifold
2. state space
3. dimension of the neural manifold

what is the dimension of a neural manifold?

determine it in many independent ways

- Lyapunov exponents (diagnostic only)
- covariant vectors (sharp)
- periodic orbits (sharp)

linearized deterministic flow



$$x_{n+1} + z_{n+1} = f(x_n) + J_n\, z_n\,, \quad J_{ij} = \partial f_i / \partial x_j$$

in one time step a linearized neighborhood of x_n is

(1) advected by the flow

(2) transported by the Jacobian matrix J_n into a neighborhood given by the J eigenvalues and eigenvectors[8]

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[8] (F. Ginelli et al., Phys. Rev. Lett. **99**, 130601 [2007]) call these "covariant Lyapunov vectors"

algorithmic advance

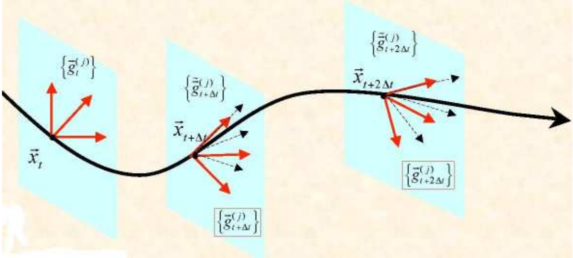
F. Ginelli, H. Chaté, G. Radons, A. Politi, P. Poggi, A. Turchi, R. M. Samelson, C. L. Wolfe:

computation of covariant “Lyapunov” vectors

Phys. Rev. Lett. 99, 130601 (2007); Tellus A 59, 355 (2007);

J. Phys. A 46, 254005 (2013)

covariant vectors are non-normal



(references are hyperlinked)

beautiful insight of

F. Ginelli, H. Chaté, G. Radons, A. Politi, P. Poggi, A. Turchi, H.-l. Yang, K. A. Takeuchi

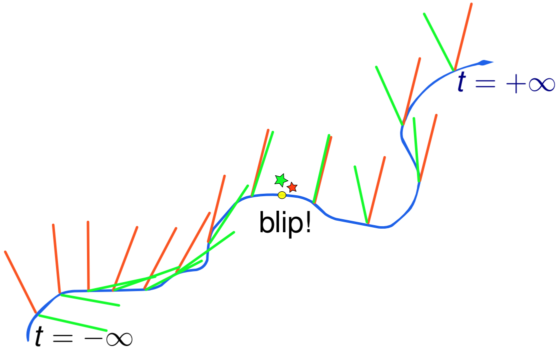
neural manifold dynamics is hyperbolically separated from the infinity of transient modes :

dimension of a neural manifold

Phys. Rev. Lett. 102, 074102 (2009); Phys. Rev. E 84, 046214 (2011);
Phys. Rev. Lett. 117, 024101 (2016)

(references are hyperlinked)

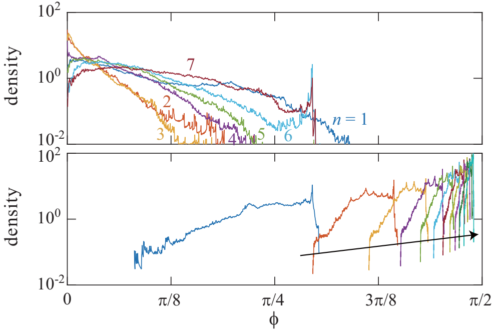
eigenvectors span the neural manifold



a pair of “entangled” eigenvectors
distinct Lyapunov exponents
dance along t from $-\infty$ to $-\infty$ orbit

at the instant “blip!” they are (nearly) collinear

distribution of angles between eigenvectors



histogram of angles between nth leading covariant vector and the next, accumulated over many long orbits :

- (top) For $n = 1 \cdots 7$ (eigenvector within the entangled manifold) the angles can be arbitrarily small
- (bottom) For the remaining, transient eigenvectors, $n = 8, 11, 12, \cdots$: angles are bounded away from zero

part 5

1. neural manifold
2. state space
3. dimension of the neural manifold
4. **cartography of the neural manifold**

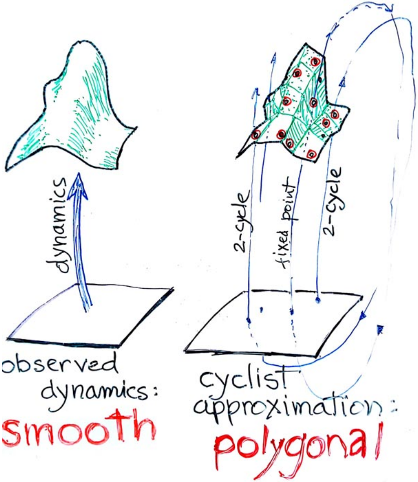
cartography for neuro dynamicists

cover the neural manifold with a set of flat charts

we can do this with

finite-dimensional bricks embedded in $10^{100\,000}$ dimensions!

tile the neural manifold by **recurrent flows**

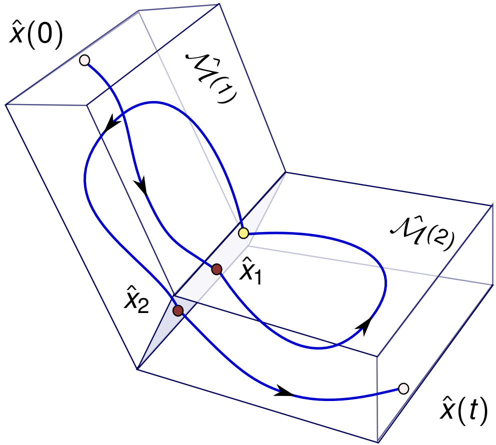


a fixed point

a 2-cycle, etc.

smooth dynamics (left frame)
tesselated by the skeleton of recurrent flows,
together with (right frame) their linearized neighborhoods

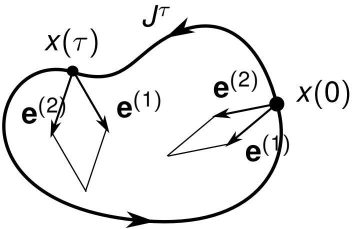
charting the neural manifold



two tangent “entangled” tiles = finite-dimensional bricks

shaded plane :
when integrating your equations, switch the local chart

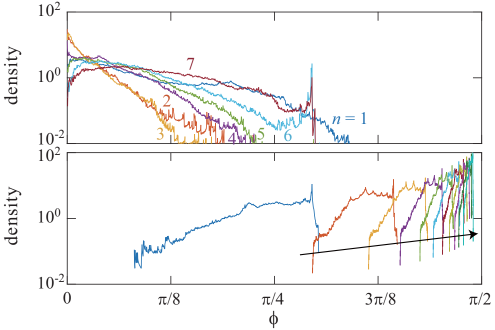
(2) covariant vectors



a parallelepiped spanned by a pair of covariant eigenvectors (‘covariant vectors’) transported along the orbit

- Jacobian matrix not self-adjoint : the eigenvectors are not orthogonal, the eigenframe is ‘non-normal’
- Measure the angle between eigenvectors $\mathbf{e}^{(i)}(x(t))$ and $\mathbf{e}^{(j)}(x(t))$

(2) distribution of principal angles between covariant subspaces



histogram of angles between S_n (n leading covariant vectors) and $\bar{S}_n$ (the rest), accumulated over the 400 orbits :

- (top) For $n = 1 \cdots 7$ (S_n within the entangled manifold) the angles can be arbitrarily small
- (bottom) For the $\bar{S}_n$ spanned by transient modes, $n = 8, 10, 12, \cdots, 28$: angles bounded away from unity

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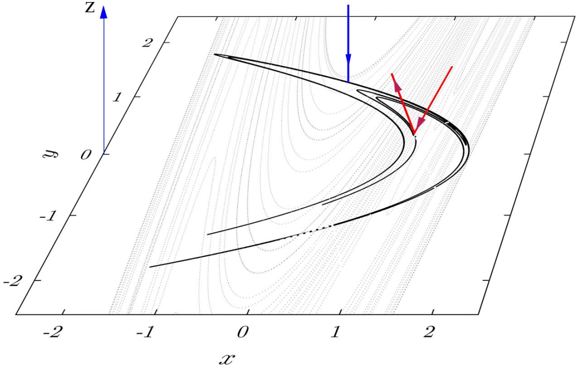
summary for the impatient

state space of dissipative flow is split into

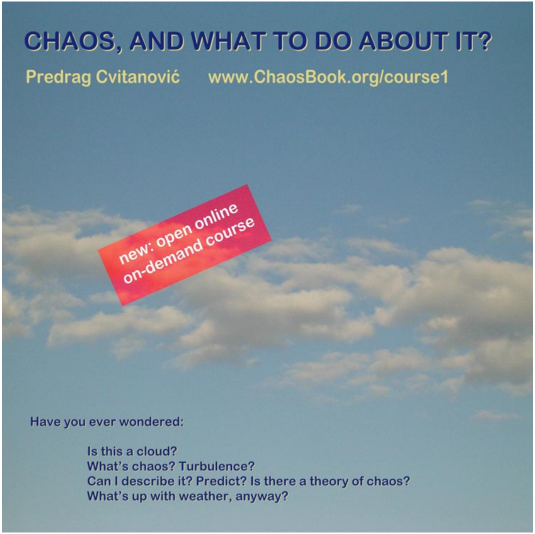


- neural manifold : spanned locally by entangled covariant vectors, tangent to unstable / stable manifolds
- the rest : spanned by the remaining ∞ of the contracting, decoupled, transient covariant vectors

the leading covariant vectors are tangent to the attractor. Perturbations that are on the attractor can be found in the subspace of the leading covariant vectors

the approximate orthogonality of the ‘isolated’ ones provides a clear threshold between the neural manifold and the rest

what next? take the course!



student raves :
...10^6 times harder than any other online course...