Continuous symmetry reduction for high-dimensional flows

Predrag Cvitanović[1] and Evangelos Siminos[1,2]

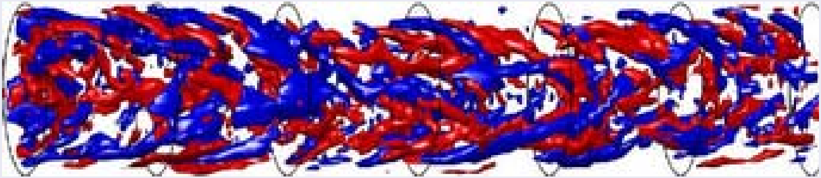
[1]Georgia Institute of Technology
[2]CEA/DAM/DIF, Paris

April 2, 2010

Outline

amazing data! amazing numerics!

3D turbulent pipe flow



solutions are

- rotationally equivariant
- translationally equivariant

Kuramoto-Sivashinsky equation

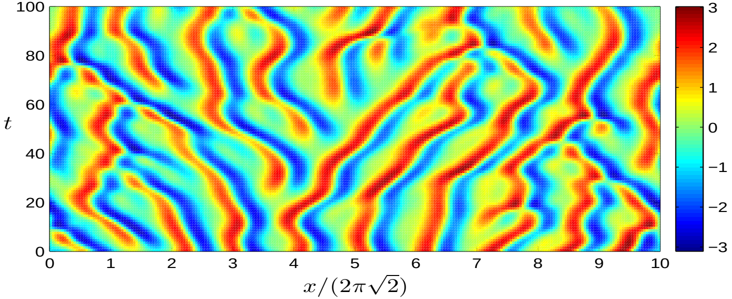
1-dimensional "Navier-Stokes"

$$u_t + u\nabla u = -\nabla^2 u - \nabla^4 u\,, \qquad x \in [-L/2, L/2]\,,$$

describes extended systems such as

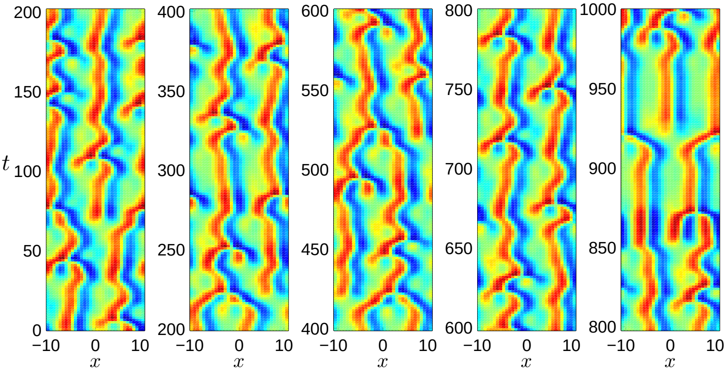
- reaction-diffusion systems
- flame fronts in combustion
- drift waves in plasmas
- thin falling films, ...

Kuramoto-Sivashinsky on a large domain



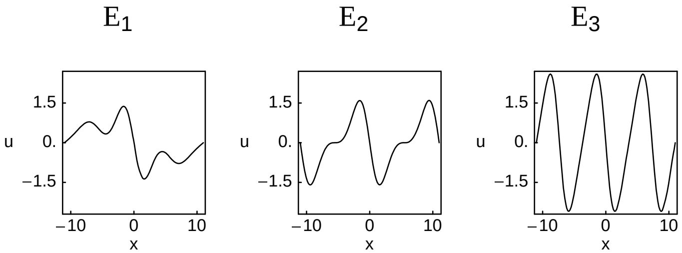
- turbulent behavior
- simpler physical, mathematical and computational setting than Navier-Stokes

evolution of Kuramoto-Sivashinsky on small $L = 22$ cell



horizontal: $x \in [-11, 11]$
vertical: time
color: magnitude of $u(x, t)$

equilibria



- E_3 invariant under $\tau_{1/3}$.
- For any E_i we have a continuous family of equilibria under rotations $\tau_{\ell/L}\, E_i$.

symmetries of Kuramoto-Sivashinsky equation

with periodic boundary condition

$$u(x,t) = u(x+L,t)$$

the symmetry group is $O(2)$:

- translations: $\tau_{\ell/L}\, u(x,t) = u(x+\ell,t)\,, \qquad \ell \in [-L/2, L/2]\ ,$
- reflections: $\kappa\, u(x) = -u(-x)\,.$

translational symmetry $\rightarrow$ traveling wave solutions

symmetries of Kuramoto-Sivashinsky equation

with periodic boundary condition

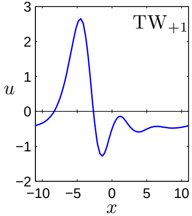
$$u(x,t) = u(x+L,t)$$

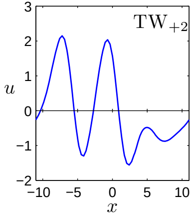
the symmetry group is $O(2)$:

- translations: $\tau_{\ell/L}\, u(x,t) = u(x+\ell,t)\,, \qquad \ell \in [-L/2, L/2]\ ,$
- reflections: $\kappa\, u(x) = -u(-x)\,.$

translational symmetry $\rightarrow$ traveling wave solutions Traveling (or relative) unstable coherent solutions are ubiquitous in turbulent hydrodynamic flows

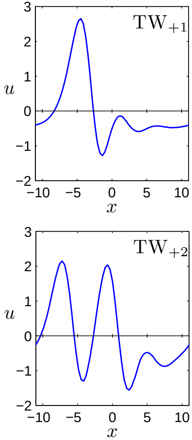
traveling waves





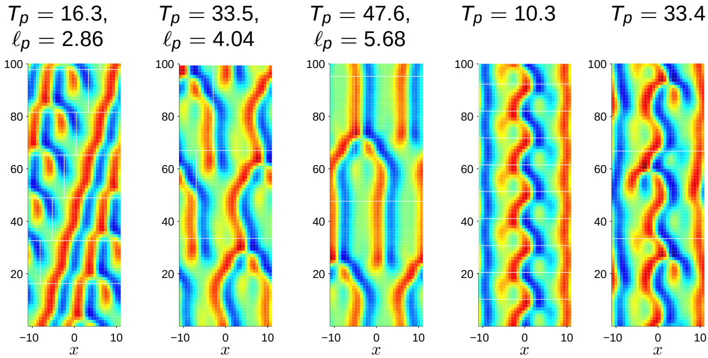
- invariant (as a set) under rotations: relative equilibria.
- They live in full space.

traveling waves



- invariant (as a set) under rotations: relative equilibria.
- They live in full space.
- Toshiba Corp and Microsoft Corp chairman Bill Gates are to work together to develop a next generation "traveling-wave reactor", which could operate for up to 100 years without refueling. [news item - Tokyo, March 23, 2010]

unstable relative periodic orbits



- have computed 40,000 unstable periodic and relative periodic orbits.
- how are they organized?

symmetries of Kuramoto-Sivashinsky equation

translational symmetry $\Rightarrow$

- traveling wave solutions
- unstable relative periodic orbits

question

what are the invariant objects that organize phase space in a spatially extended system with translational symmetry and how do they fit together to form a skeleton of the dynamics?

state space

- the space in which all possible states u's live
- ∞-dimensional:
  point $u(x)$ is a function of x on interval $x \in L$.
- in practice:
  a high but finite dimensional space (e.g. through a spectral discretization)

state space of Kuramoto-Sivashinsky on $L = 22$

intrinsic dimensionality

- dynamics are often captured by fewer variables than needed to numerically resolve the PDE.
- Lyapunov exponents:
  $(\lambda_i) = (0.048, 0, 0, -0.003, -0.189, -0.256, -0.290, -0.310, \cdots)$
- '8-dimensional' covariant Lyapunov frame? perhaps tractable?
- how do we exploit such low dimensionality to obtain dynamical systems description?

- low dimensional systems:
  equilibria, periodic orbits organize the long time dynamics.
- is this true in extended systems?

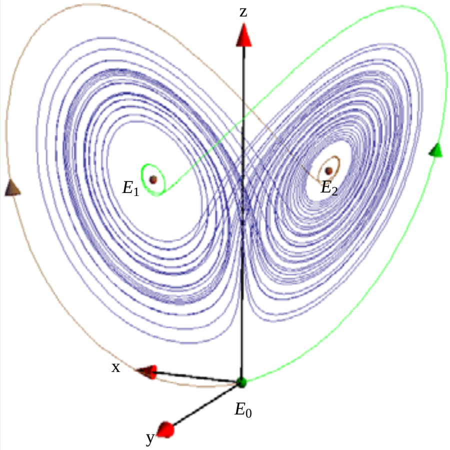
from Lorenz 3D attractor to a unimodal map

Lorenz equations

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} \sigma(y - x) \\ \rho x - y - xz \\ xy - bz \end{bmatrix}$$

with
$\sigma = 10, b = 8/3, \rho = 28.$

Lorenz attractor



from Lorenz 3D attractor to a unimodal map

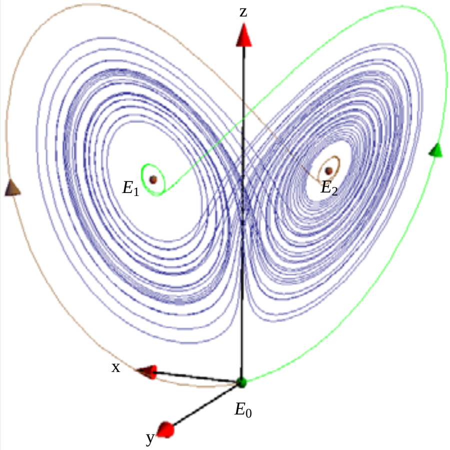
Equilibria

$$\dot{x} = v(x) = 0$$

Linear stability of equilibria

$$A_{ij} = \frac{\partial v_i}{\partial x_j}(x_{E_m})$$

Lorenz attractor



from Lorenz 3D attractor to a unimodal map

Linear stability of equilibria

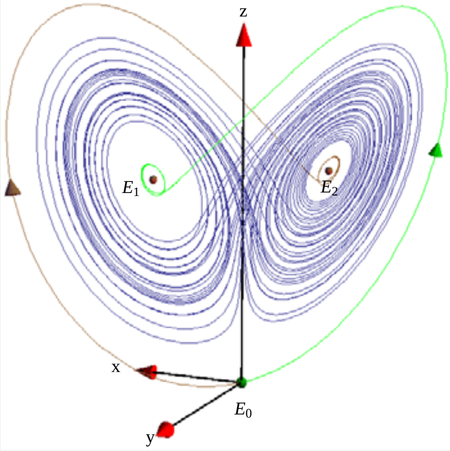
$$A_{ij} = \frac{\partial v_i}{\partial x_j}(x_{E_m})$$

Eigenvalues of A:

$$\lambda_j = \mu_j \pm i\nu_j$$

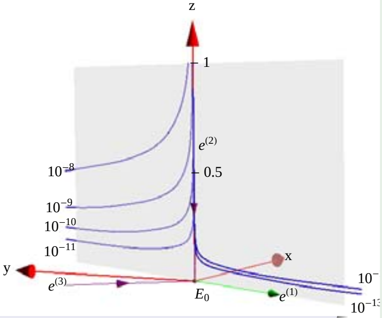
- Linearly stable if $\mu_j < 0$
- Linearly unstable if $\mu_j > 0$

Lorenz attractor

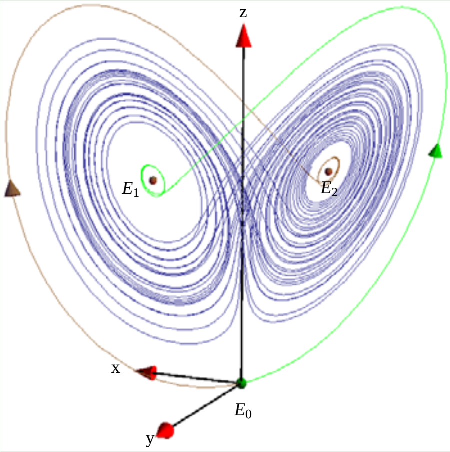


from Lorenz 3D attractor to a unimodal map

$$E_0: \quad \begin{aligned} \lambda_1 &= 11.83 \\ \lambda_2 &= -2.66 \\ \lambda_3 &= -22.83 \end{aligned}$$



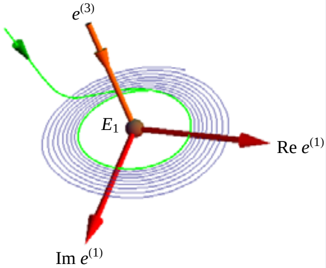
Lorenz attractor



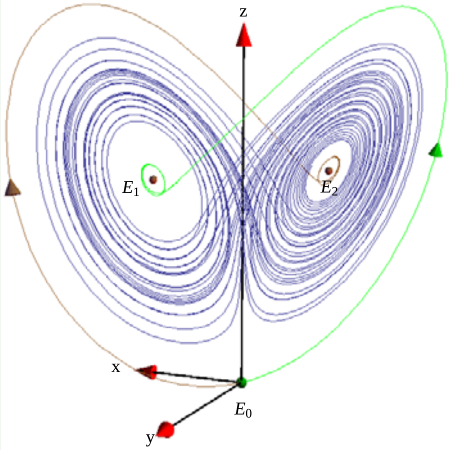
from Lorenz 3D attractor to a unimodal map

E_1

$$\lambda_{1,2} = 0.094 \pm 10.19$$
$$\lambda_3 = -13.85$$



Lorenz attractor

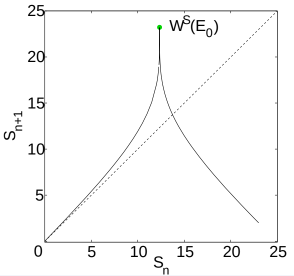


from Lorenz 3D attractor to a unimodal map

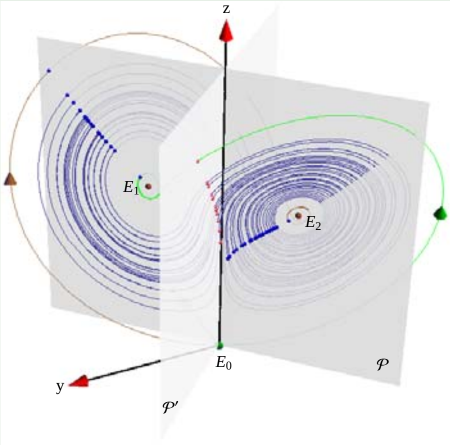
Poincaré section

$\mathcal{P}$: (N-1)-dimensional hypersurface.

Poincaré return map



Lorenz attractor

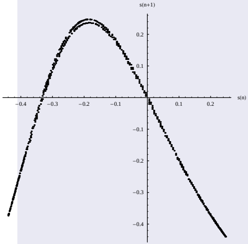


Take the hint from low dimensional systems

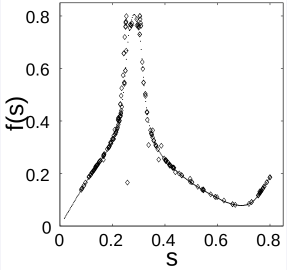
- low dimensional systems:
  equilibria, periodic orbits organize the long time dynamics.
- is this true in extended systems?

Kuramoto-Sivashinsky flow reduced to discrete maps

within the discrete $u(x) = -u(-x)$ invariant subspace



Christiansen et. al. (1996)

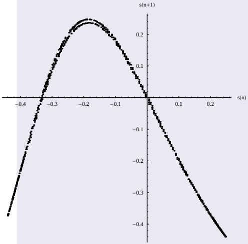


Lan and Cvitanović (2004)

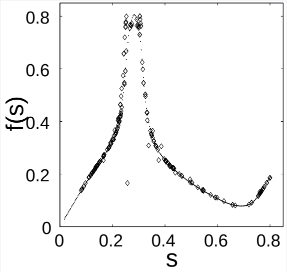
- $\infty - d$ PDE state space dynamics can be reduced to low dimensional return maps!

Kuramoto-Sivashinsky flow reduced to discrete maps

within the discrete $u(x) = -u(-x)$ invariant subspace



Christiansen et. al. (1996)



Lan and Cvitanović (2004)

- $\infty - d$ PDE state space dynamics can be reduced to low dimensional return maps!
- BUT! must reduce continuous symmetries first

from complex Lorenz flow $5D$ attractor $\to$ unimodal map

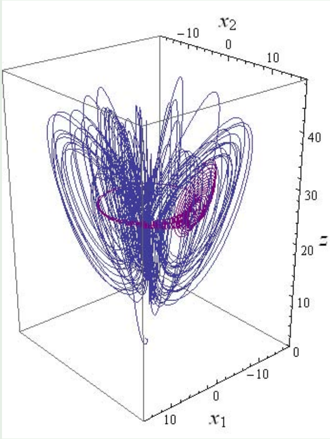
complex Lorenz equations

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{y}_1 \\ \dot{y}_2 \\ \dot{z} \end{bmatrix} = \begin{bmatrix} -\sigma x_1 + \sigma y_1 \\ -\sigma x_2 + \sigma y_2 \\ (\rho_1 - z)x_1 - \rho_2 x_2 - y_1 - e y_2 \\ \rho_2 x_1 + (\rho_1 - z)x_2 + e y_1 - y_2 \\ -bz + x_1 y_1 + x_2 y_2 \end{bmatrix}$$

$\rho_1 = 28, \rho_2 = 0, b = 8/3, \sigma = 10, e = 1/10$

A typical $\{x_1, x_2, z\}$ trajectory of the complex Lorenz flow
+ a short trajectory of whose initial point is close to the relative equilibrium Q_1 superimposed.

attractor



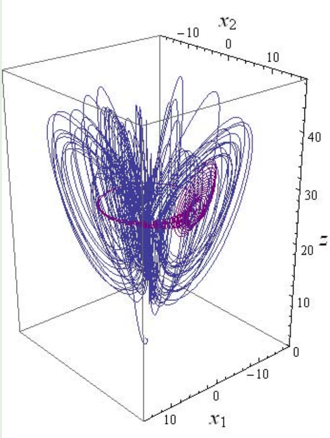
from complex Lorenz flow $5D$ attractor $\to$ unimodal map

what to do?

the goal

reduce this messy strange attractor to a 1-dimensional return map

attractor



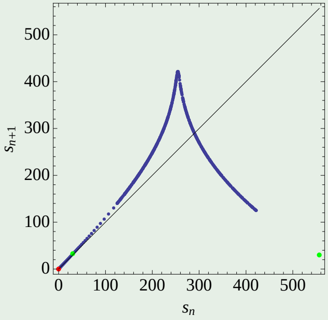
from complex Lorenz flow $5D$ attractor $\to$ unimodal map

the goal attained

but it will cost you

after symmetry reduction; must learn how to quotient the $SO(2)$ symmetry

1D return map!



Lie groups elements, Lie algebra generators

An element of a compact Lie group:

$$g(\theta) = e^{\theta \cdot \mathbf{T}}, \qquad \theta \cdot \mathbf{T} = \sum \theta_a \mathbf{T}_a, \; a = 1, 2, \cdots, N$$

$\theta \cdot \mathbf{T}$ is a *Lie algebra* element, and θ_a are the parameters of the transformation.

example: $SO(2)$ rotations for complex Lorenz equations

$SO(2)$ rotation by finite angle θ:

$$g(\theta) = \begin{pmatrix} \cos\theta & \sin\theta & 0 & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta & 0 \\ 0 & 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

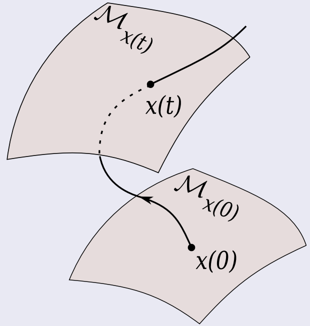
symmetries of dynamics

A flow $\dot{x} = v(x)$ is G-equivariant if

$$v(x) = g^{-1}\, v(g\, x)\,, \qquad \text{for all } g \in G\,.$$

foliation by group orbits

group orbits

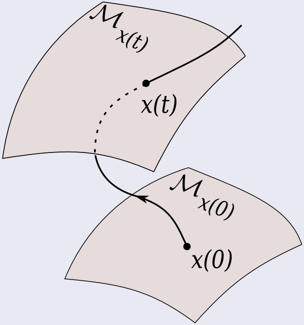


group orbit $\mathcal{M}_x$ of x is the set of all group actions

$$\mathcal{M}_x = \{g\,x \mid g \in G\}$$

foliation by group orbits

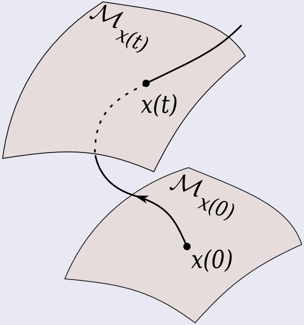
group orbits



group orbit $\mathcal{M}_{x(0)}$ of state space point $x(0)$, and the group orbit $\mathcal{M}_{x(t)}$ reached by the trajectory $x(t)$ time t later.

foliation by group orbits

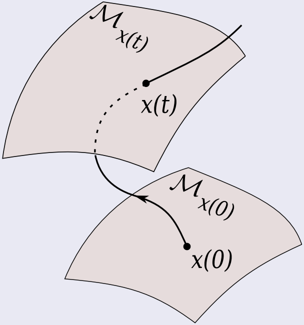
group orbits



any point on the manifold $\mathcal{M}_{x(t)}$ is equivalent to any other.

foliation by group orbits

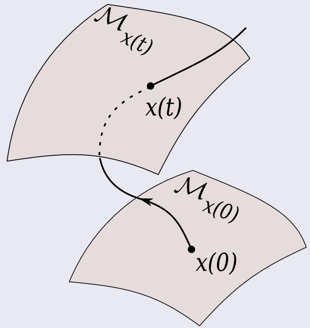
group orbits



action of a symmetry group endows the state space with the structure of a union of group orbits, each group orbit an equivalence class.

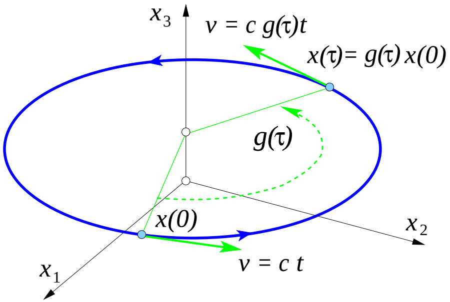
foliation by group orbits

group orbits



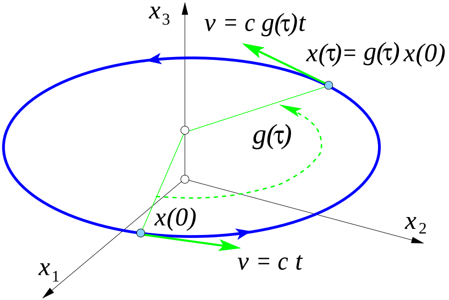
the goal:
replace each group orbit by a unique point a lower-dimensional *reduced state space* (or orbit space)

a traveling wave



relative equilibrium
(traveling wave, rotating wave)
$x_{\mathrm{TW}}(\tau) \in \mathcal{M}_{\mathrm{TW}}$: the dynamical flow field points along the group tangent field, with constant 'angular' velocity c, and the trajectory stays on the group orbit

a traveling wave

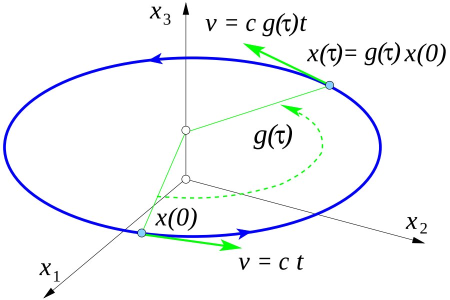


relative equilibrium

$$v(x) = c \cdot t(x)\,, \qquad x \in \mathcal{M}_{\mathrm{TW}}$$

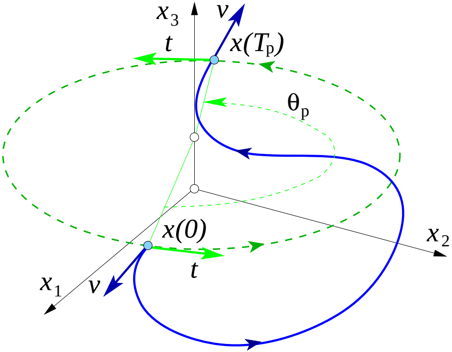
$$x(\tau) = g(-\tau\, c)\, x(0) = e^{-\tau\, c \cdot \mathbf{T}} x($$

a traveling wave



group orbit $g(\tau)\,x(0)$ coincides with the dynamical orbit $x(\tau) \in \mathcal{M}_{\mathrm{TW}}$ and is thus flow invariant

a relative periodic orbit

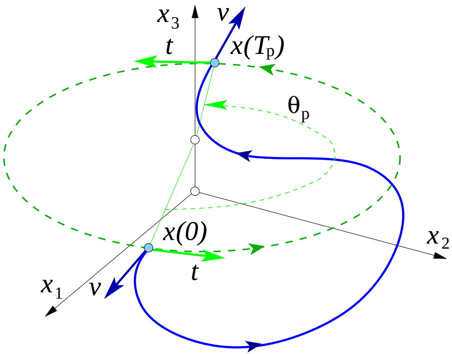


relative periodic orbit

$$x_p(0) = g_p x_p(T_p)$$

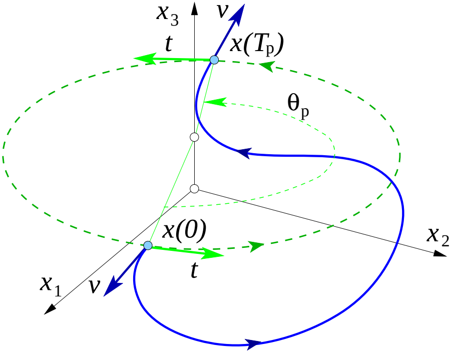
exactly recurs at a fixed *relative period* T_p, but shifted by a fixed group action g_p

a relative periodic orbit



relative periodic orbit starts out at $x(0)$, returns to the group orbit of $x(0)$ after time T_p, a rotation of the initial point by g_p

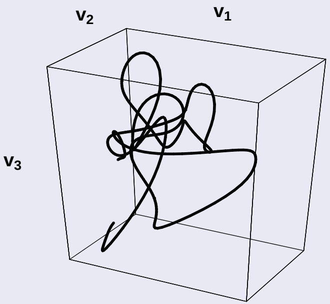
a relative periodic orbit



The group action parameters $\theta = (\theta_1, \theta_2, \cdots \theta_N)$ are irrational: trajectory sweeps out ergodically the group orbit without ever closing into a periodic orbit.

relativity for pedestrians

try a co-moving coordinate frame?

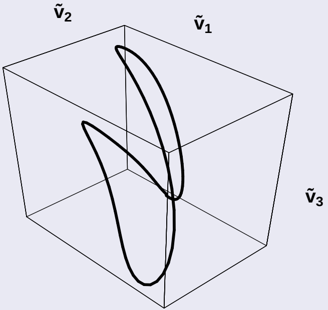


(a)

A relative periodic orbit of the Kuramoto-Sivashinsky flow, traced for four periods T_p, projected on
(a) a stationary state space coordinate frame $\{v_1, v_2, v_3\}$;

relativity for pedestrians

try a co-moving coordinate frame?

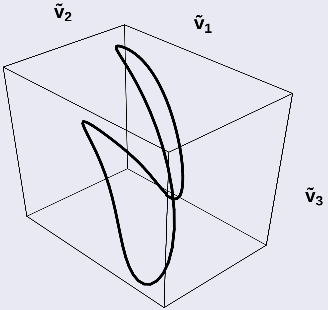


(b)

A relative periodic orbit of the Kuramoto-Sivashinsky flow, traced for four periods T_p, projected on
(b) a co-moving $\{\tilde{v}_1, \tilde{v}_2, \tilde{v}_3\}$ frame

relativity for pedestrians

no good global co-moving frame!



(*b*)

this is no symmetry reduction at all; all other relative periodic orbits require their own frames,moving at different velocities.

symmetry reduction

- all points related by a symmetry operation are mapped to the same point.
- relative equilibria become equilibria and relative periodic orbits become periodic orbits in reduced space.
- families of solutions are mapped to a single solution

reduction methods

1. **Hilbert polynomial basis**: rewrite equivariant dynamics in invariant coordinates
2. **moving frames, or slices**: cut group orbits by a hypersurface (kind of Poincareé section), each group orbit of symmetry-equivalent points represented by the single point

reduction methods

1. **Hilbert polynomial basis**: rewrite equivariant dynamics in invariant coordinates: global
2. **moving frames, or slices**: cut group orbits by a hypersurface (kind of Poincareé section), each group orbit of symmetry-equivalent points represented by the single point: local

invariant polynomials

- rewrite the equations in variables invariant under the symmetry transformation

invariant polynomials

- rewrite the equations in variables invariant under the symmetry transformation
- or compute solutions in original space and map them to invariant variables

invariant polynomials basis

Hilbert basis for complex Lorenz equations

$$
\begin{aligned}
u_1 &= x_1^2 + x_2^2 \,, & u_2 &= y_1^2 + y_2^2 \\
u_3 &= x_1 y_2 - x_2 y_1 \,, & u_4 &= x_1 y_1 + x_2 y_2 \\
u_5 &= z
\end{aligned}
$$

invariant under $SO(2)$ action on a 5-dimensional state space

polynomials related through syzygies:

$$u_1 u_2 - u_3^2 - u_4^2 = 0$$

invariant polynomials basis

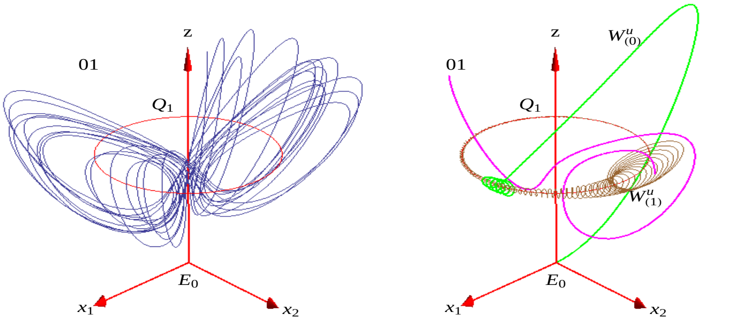
complex Lorenz equations in invariant polynomial basis

$$
\begin{aligned}
\dot{u}_1 &= 2\,\sigma\,(u_3 - u_1) \\
\dot{u}_2 &= -2\,u_2 - 2\,u_3\,(u_5 - \rho_1) \\
\dot{u}_3 &= \sigma\,u_2 - (\sigma - 1)\,u_3 - e\,u_4 + u_1\,(\rho_1 - u_5) \\
\dot{u}_4 &= e\,u_3 - (\sigma + 1)\,u_4 \\
\dot{u}_5 &= u_3 - b\,u_5
\end{aligned}
$$

A 4-dimensional $\mathcal{M}/SO(2)$ reduced state space, a symmetry-invariant representation of the 5-dimensional $SO(2)$ equivariant dynamics

state space portrait of complex Lorenz flow

drift induced by continuous symmetry



A generic chaotic trajectory (blue), the E_0 equilibrium, a representative of its unstable manifold (green), the Q_1 relative equilibrium (red), its unstable manifold (brown), and one repeat of the $\overline{01}$ relative periodic orbit (purple).

invariant polynomials basis

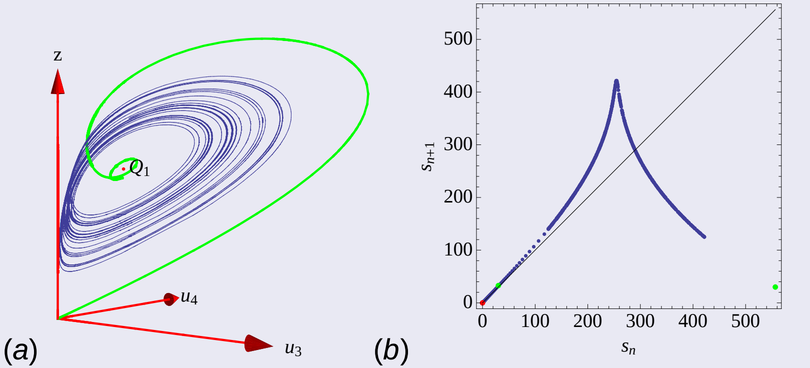
complex Lorenz equations in invariant polynomial basis

$$
\begin{aligned}
\dot{u}_1 &= 2\,\sigma\,(u_3 - u_1) \\
\dot{u}_2 &= -2\,u_2 - 2\,u_3\,(u_5 - \rho_1) \\
\dot{u}_3 &= \sigma\,u_2 - (\sigma\; - 1)\,u_3 - e\,u_4 + u_1\,(\rho_1 - u_5) \\
\dot{u}_4 &= e\,u_3 - (\sigma\; + 1)\,u_4 \\
\dot{u}_5 &= u_3 - b\,u_5
\end{aligned}
$$

the image of the full state space relative equilibrium Q_1 group orbit is an equilibrium point, while the image of a relative periodic orbit, such as $\overline{01}$, is a periodic orbit

Hilbert invariant coordinates

projected onto invariant polynomials basis



(a) The unstable manifold connection from the equilibrium E_0 at the origin to the strange attractor controlled by the rotation around the reduced state space image of relative equilibrium Q_1;

higher-dimensional invariant bases? an example

first 11 invariants for the standard action of $SO(2)$

$$u_1 = r_1 = \sqrt{b_1^2 + c_1^2}$$
$$u_3 = \frac{b_2\left(b_1^2 - c_1^2\right) + 2b_1 c_1 c_2}{r_1^2}$$
$$u_4 = \frac{-2b_1 b_2 c_1 + \left(b_1^2 - c_1^2\right) c_2}{r_1^2}$$
$$u_5 = \frac{b_1 b_3\left(b_1^2 - 3c_1^2\right) - c_1\left(-3b_1^2 + c_1^2\right) c_3}{r_1^3}$$
$$u_6 = \frac{-3b_1^2 b_3 c_1 + b_3 c_1^3 + b_1^3 c_3 - 3b_1 c_1^2 c_3}{r_1^3}$$

higher-dimensional invariant bases? an example

first 11 invariants for the standard action of $SO(2)$

$$u_7 = \frac{b_4\left(b_1^4-6b_1^2c_1^2+c_1^4\right)+4b_1c_1\left(b_1^2-c_1^2\right)c_4}{r_1^4}$$

$$u_8 = \frac{4b_1b_4c_1\left(-b_1^2+c_1^2\right)+\left(b_1^4-6b_1^2c_1^2+c_1^4\right)c_4}{r_1^4}$$

$$u_9 = \frac{b_1b_5\left(b_1^4-10b_1^2c_1^2+5c_1^4\right)+c_1\left(5b_1^4-10b_1^2c_1^2+c_1^4\right)c_5}{r_1^5}$$

$$u_{10} = \frac{-b_5c_1\left(5b_1^4-10b_1^2c_1^2+c_1^4\right)+b_1\left(b_1^4-10b_1^2c_1^2+5c_1^4\right)c_5}{r_1^5}$$

$$u_{11} = \frac{b_6\left(b_1^6-15b_1^4c_1^2+15b_1^2c_1^4-c_1^6\right)+2b_1c_1\left(3b_1^4-10b_1^2c_1^2+3c_1^4\right)c_6}{r_1^6}$$

$$u_{12} = \frac{-2b_1b_6c_1\left(3b_1^4-10b_1^2c_1^2+3c_1^4\right)+\left(b_1^6-15b_1^4c_1^2+15b_1^2c_1^4-c_1^6\right)c_6}{r_1^6}$$

invariant polynomials - how to find them?

- invariant polynomials (Hilbert basis)

invariant polynomials - how to find them?

- invariant polynomials (Hilbert basis): computationally prohibitive for high-dimensional flows

invariant polynomials - how to find them?

- invariant polynomials (Hilbert basis)
- Cartan moving frame method / method of slices

invariant polynomials - how to find them?

- invariant polynomials (Hilbert basis)
- Cartan moving frame method / method of slices

invariant polynomials - how to find them?

- invariant polynomials (Hilbert basis)
- Cartan moving frame method / method of slices: singularities

Lie algebra generators

$\mathbf{T}_a$ generate infinitesimal transformations: a set of N linearly independent $[d \times d]$ anti-hermitian matrices, $(\mathbf{T}_a)^\dagger = -\mathbf{T}_a$, acting linearly on the d-dimensional state space $\mathcal{M}$

example: $SO(2)$ rotations for complex Lorenz equations

$$\mathbf{T} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The action of $SO(2)$ on the complex Lorenz equations state space decomposes into $m = 0$ G-invariant subspace (z-axis) and $m = 1$ subspace with multiplicity 2.

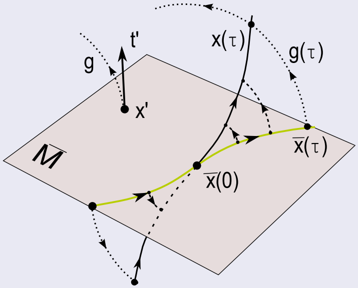
group tangent fields

flow field at the state space point x induced by the action of the group is given by the set of N *tangent fields*

$$t_a(x)_i = (\mathbf{T}_a)_{ij} x_j$$

slice & dice

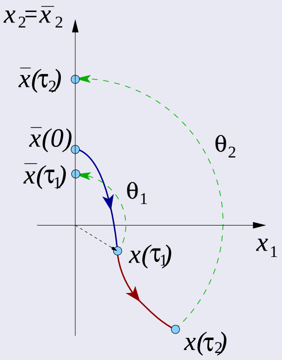
flow reduced to a slice



Slice $\bar{\mathcal{M}}$ through the slice-fixing point x', normal to the group tangent t' at x', intersects group orbits (dotted lines). The full state space trajectory $x(\tau)$ and the reduced state space trajectory $\overline{x}(\tau)$ are equivalent up to a group rotation $g(\tau)$.

method of moving frames for $SO(2)$-equivariant flow

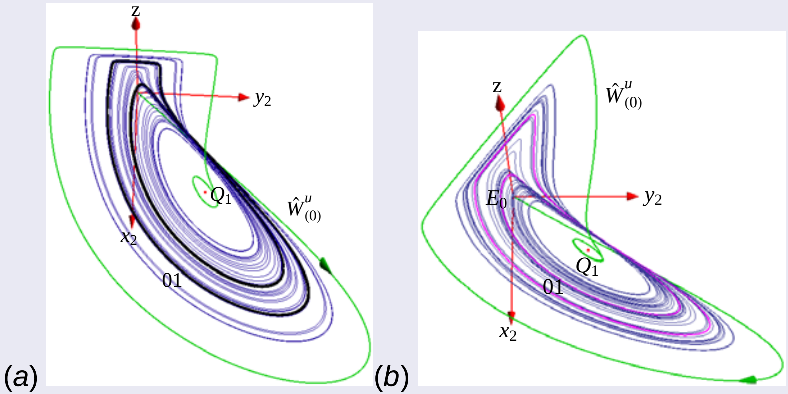
flow reduced to a slice



slice through $x' = (0, 1, 0, 0, 0)$
group tangent $t' = (-1, 0, 0, 0, 0)$
Start on the slice at $\bar{x}(0)$, evolve.
Compute angle θ_1 to the slice
rotate $x(\tau_1)$ by θ_1 to
$\bar{x}(\tau_1) = g(\theta_1)\, x(\tau_1)$ back into the slice,
$\bar{x}_1(\tau_1) = 0$. Repeat for points $x(\tau_i)$
along the trajectory.

slice trouble 1

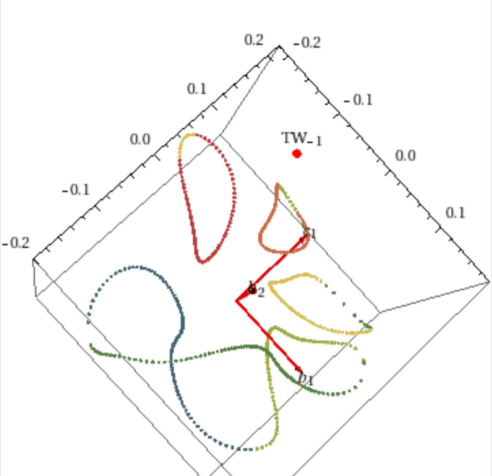
portrait of complex Lorenz flow in reduced state space



all choices of the slice fixing point x' exhibit flow discontinuities / jumps

slice trouble 2

slice cuts an relative periodic orbit multiple times



Relative periodic orbit intersects a hyperplane slice in 3 closed-loop images of the relative periodic orbit and 3 images that appear to connect to a closed loop.

summary

conclusion

- Symmetry reduction: efficient implementation allows exploration of high-dimensional flows with continuous symmetry.
- stretching and folding of unstable manifolds in reduced state space organizes the flow

to be done

- construct Poincaré sections and return maps
- find all (relative) periodic orbits up to a given period.
- use the information quantitatively (periodic orbit theory).