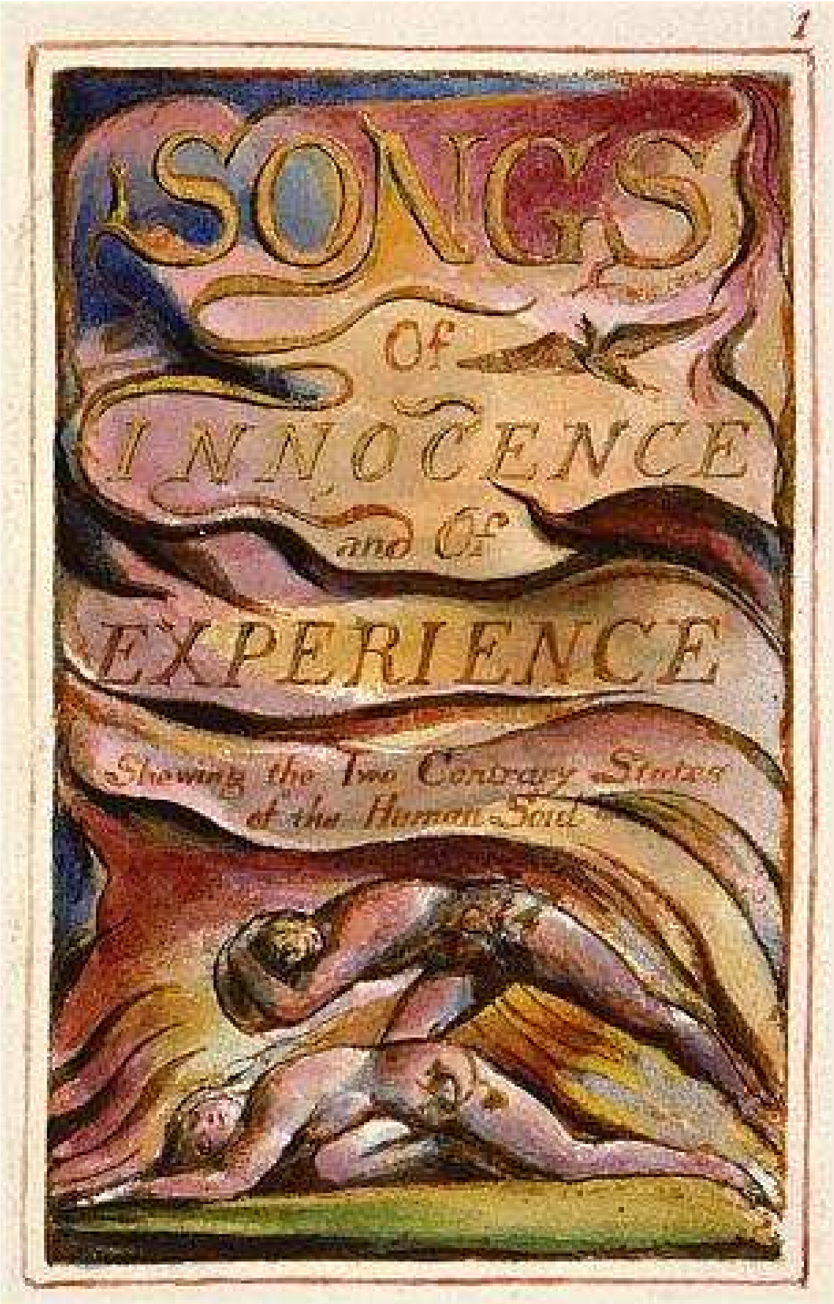


AIM Seminar

U of Michigan Sep 7 2007

Turbulence?

a stroll through 61,506 dimensions

Predrag Cvitanović
John F Gibson

School of Physics

Georgia Tech, Atlanta GA, USA

Turbulence: A walk through a repertoire of unstable recurrent patterns?

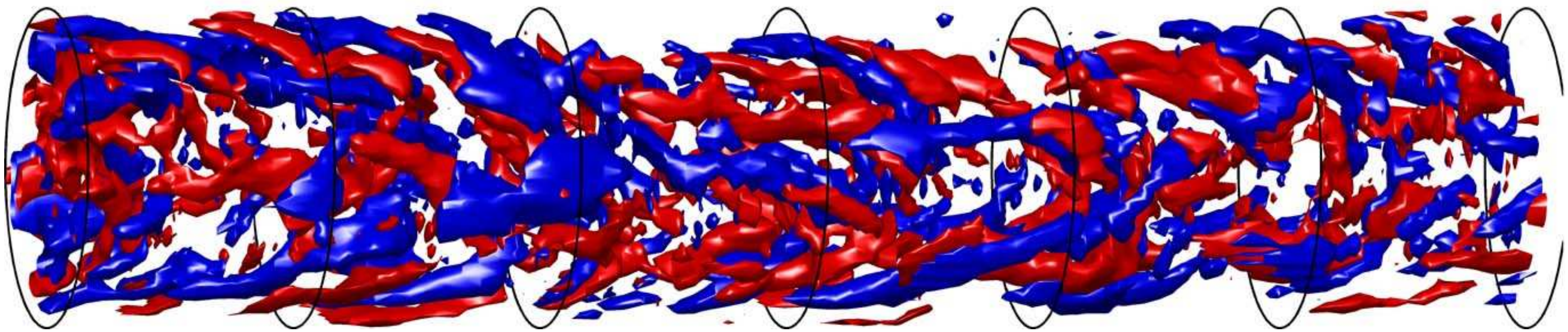
As a turbulent flow evolves, every so often we catch a glimpse of a familiar pattern:

⟹ other swirls ⟹

For any finite spatial resolution, the system follows approximately for a finite time a pattern belonging to a finite alphabet of admissible patterns. The long term dynamics = a walk through the space of such unstable patterns.

New experiments: Unstable Coherent Structures

Stereoscopic Particle Image Velocimetry → 3-d velocity field over the entire pipe[1]

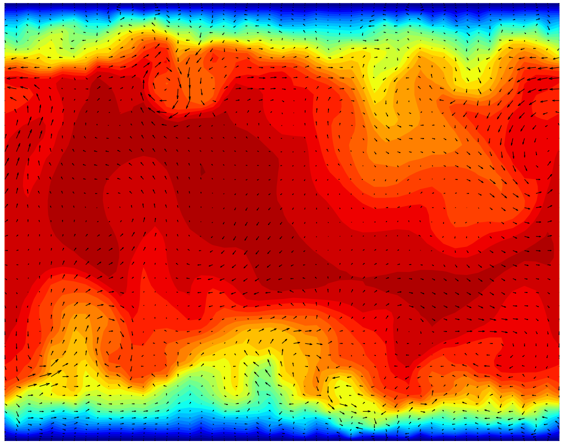


Observed structures resemble numerically computed traveling waves

What lies beyond?

[1] Casimir W.H. van Doorne (PhD thesis, Delft 2004); Hof et al., Science (Sep 10, 2004)

Wall-bounded turbulence in channel flow



Pressure driven turbulent channel flow, Re = 3700. Walls top/bottom, periodic BCs front/back and sides. Red/blue is fast/slow into screen.[2]

* Near-wall rolls generate turbulence.
* Mix high and low speed fluids near wall, generating drag.

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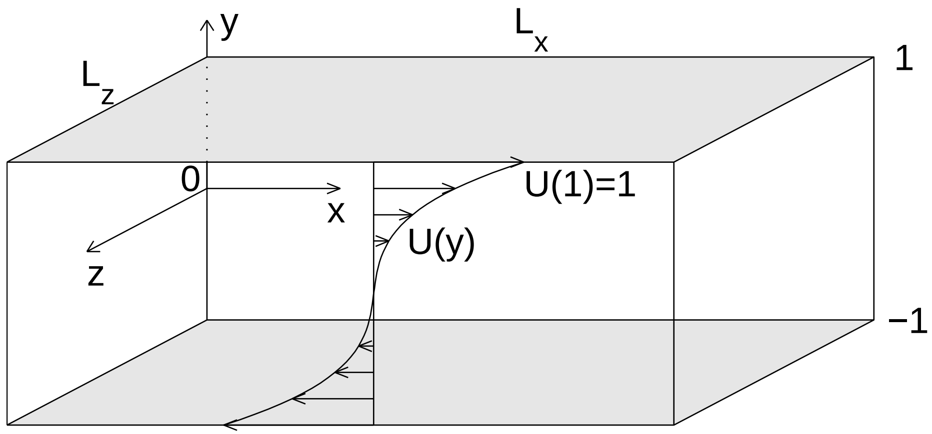
[2] J.F. Gibson: www.channelflow.org

Plane Couette flow

Navier-Stokes for fluid velocity $\mathbf{u}(\mathbf{x},t)$ and pressure $p(\mathbf{x},t)$

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} = -\nabla p + \nu\nabla^2\mathbf{u}\,, \qquad \nabla\cdot\mathbf{u} = 0$$



periodic, wall-bounded domain $\Omega = [0,\mathcal{L}_x] \times [-1,1] \times [0,\mathcal{L}_z]$ with BCs

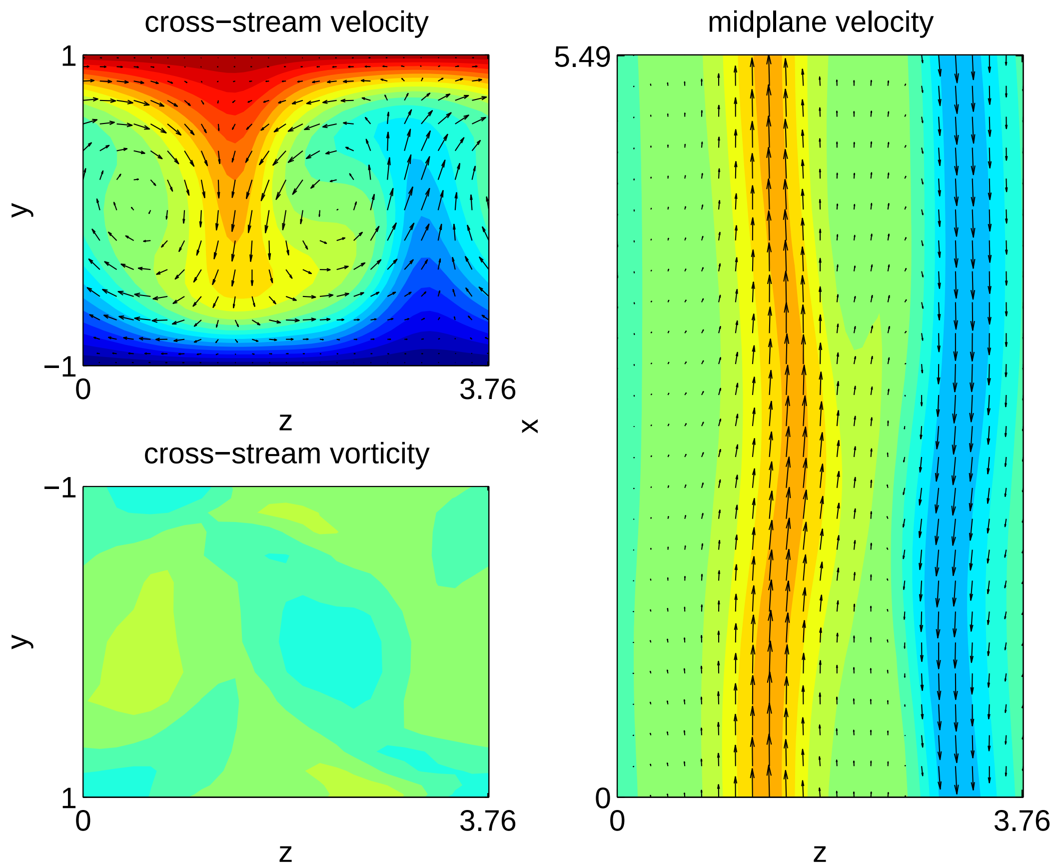
$$\mathbf{u}(x,\pm 1,z,t) = \pm 1$$

$$\mathbf{u}(x+\mathcal{L}_x,y,z) = \mathbf{u}(x,y,z,t)\,, \qquad \mathbf{u}(x,y,z+\mathcal{L}_z) = \mathbf{u}(x,y,z,t).$$

Turbulent plane Couette Flow

simplest possible turbulent flow:

Re = 400



3,4

[3] Numerical study: Hamilton, Kim, Waleffe, JFM 287 (1995)

[4] Self-sustaining process: Waleffe, Phys. Fluids 9 (1997)

THE POINT OF THIS TALK

!!! THE POINT OF THIS TALK !!!

UNLEARN: 3-d VISUALIZATION

instant in turbulent evolution:

a 3-d video frame,

each pixel a 3-d velocity field

THINK: ∞-d PHASE SPACE

instant in turbulent evolution:

a unique point

theory of turbulence =
geometry of the state space

[E. Hopf 1948]

THINK IN STATE SPACE!

.

Q: How do you treat Navier-Stokes as a dynamical system?

A: dual ODE/CDF representations

ODE in $\mathbb{R}^d$ CFD algorithm

$$
\begin{array}{ccc}
a(t) & \xrightarrow{\mathbf{u}(\mathbf{x},t)=a_n(t)\mathbf{\Phi}_n(x)} & \mathbf{u}(\mathbf{x},t) \\
\Big\downarrow \dot{a}=F(a) & & \Big\downarrow \mathrm{CFD} \\
a(t+T) & \xleftarrow[a_n=(\mathbf{u},\mathbf{\Phi}_n)_\Omega]{} & \mathbf{u}(\mathbf{x},t+T)
\end{array}
$$

State space portraits = projections on well-chosen states $\hat{\mathbf{u}}_n$:

$\hat{a}_n(t) = (\mathbf{u}(t),\ \hat{\mathbf{u}}_n)_\Omega$ (integral over the box)

ODE vs. CFD reps. of Navier–Stokes

ODE formulation

- Closed-form, unconstrained, real-valued dynamical system
- Orthonormal $\mathbf{\Phi}_n$: $||a||^2 = ||\mathbf{u}||^2_\Omega$
- Impossible to integrate: F quadratic in $\mathbb{R}^d$, $d \approx 10^5$

CFD algorithm

- Efficient time integration of Navier–Stokes
- Constraints: pressure, BCs, complex symmetries
- No 1-order ODE formulation, no clear set of independent variables

ODE: Orthonormal, divergence-free basis

inner product:

$$(\mathbf{f}, \mathbf{g})_\Omega = \frac{1}{V} \int_0^{\mathcal{L}_x} \int_{-1}^{1} \int_0^{\mathcal{L}_z} \mathbf{f} \cdot \mathbf{g} \, dx \, dy \, dz$$

construct basis $\{ \mathbf{\Phi}_n(x) \mid n = 1, \ldots, \infty \}$ with properties

real, vector-valued: $\mathbf{\Phi}_n = \Phi_n^u \mathbf{e}_x + \Phi_n^v \mathbf{e}_y + \Phi_n^w \mathbf{e}_w$

orthonormal: $(\mathbf{\Phi}_n, \mathbf{\Phi}_m)_\Omega = \delta_{mn}$

divergence-free: $\nabla \cdot \mathbf{\Phi}_n = 0$

Dirichlet at walls: $\mathbf{\Phi}_n(x, \pm 1, z) = 0$

periodic in x, z: $\mathbf{\Phi}_n(x, y, z) = \mathbf{\Phi}_n(x + \mathcal{L}_x, y, z) = \mathbf{\Phi}_n(x, y, z + \mathcal{L}_z)$

ODE: Galerkin projection of Navier-Stokes

expand $\mathbf{u}$ (deviation of velocity from laminar)

$$\mathbf{u}(\mathbf{x}, t) = a_n(t)\mathbf{\Phi}_n(\mathbf{x}), \quad n, = 1, \ldots, d$$

Galerkin projection of NS onto $\mathbf{\Phi}_m$ produces ODE in $\mathbb{R}^d$

$$\dot{a}_m = F(a)_m = L_{mn}\, a_n + N_{mnp}\, a_n a_p, \quad m, n, p = 1, \ldots, d\,.$$

where

- $L_{mn} = (\nu \nabla^2 \mathbf{\Phi}_n - \partial \mathbf{\Phi}_n / \partial x, -\Phi_n^v \mathbf{e}_x, \mathbf{\Phi}_m)_\Omega$ and $N_{mnp} = -(\mathbf{\Phi}_n \cdot \nabla \mathbf{\Phi}_p, \mathbf{\Phi}_m)_\Omega$
- Indices range from 1 to $d \approx 10^5$ (2×32^3 to 2×48^3)
- ODE system too big to integrate

CFD/ODE: State space portraits

Visualize state space by projecting ODE a(t) or CFD $\mathbf{u}(t)$ onto a few well-chosen $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ representative velocity fields

(e.g., a few equilibria and their unstable eigenvectors).

Construct $\{\hat{\mathbf{u}}_1, \hat{\mathbf{u}}_2, \hat{\mathbf{u}}_3\}$ by Gram-Schmidt orthogonalization and inner product

$$(\mathbf{u}_1, \mathbf{u}_2)_\Omega = \frac{1}{V} \int_0^{\mathcal{L}_x} \int_{-1}^{1} \int_0^{\mathcal{L}_z} \mathbf{u}_1 \cdot \mathbf{u}_2 \, dx \, dy \, dz$$

State space portraits = projections

$$\hat{a}_n(t) = (\mathbf{u}(t), \hat{\mathbf{u}}_n)_\Omega$$

.

The devil is in the details

Turbulent flows cannot be modeled by a few modes

Attractor is “low dimensional,” but has to be tracked in the full 10^3 to 10^5 dimensions

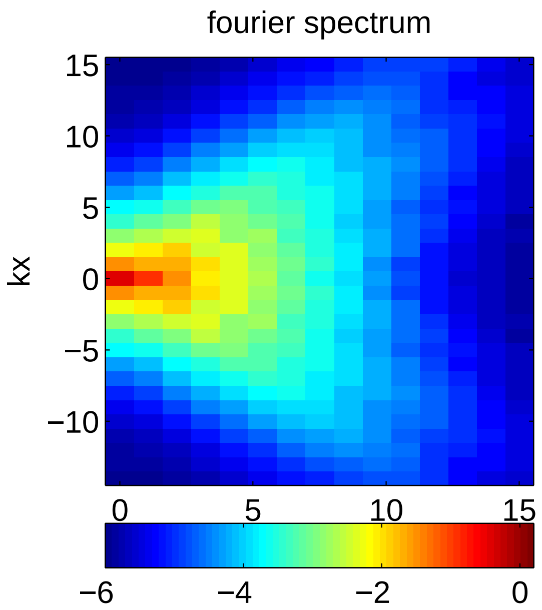
www.channelflow.org CFD software

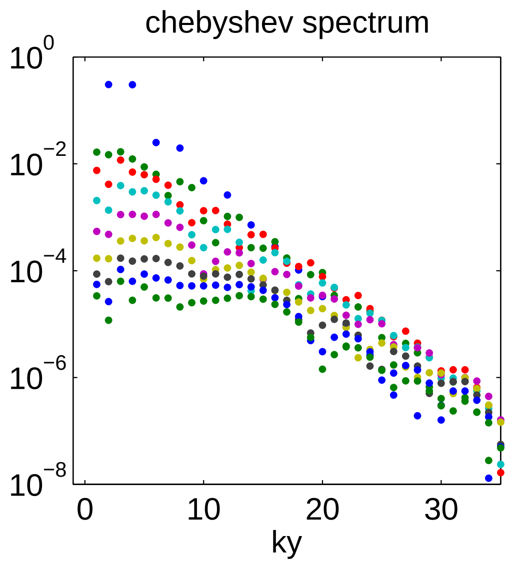
John F. Gibson:

- High-level representation of CFD objects: fields, DNS algorithms, differential operators, etc
- Compact, readable programs
- C++ library of spectral CFD building blocks
- Automated test suite, verification against known solutions

CFD: Geometry and spectral convergence

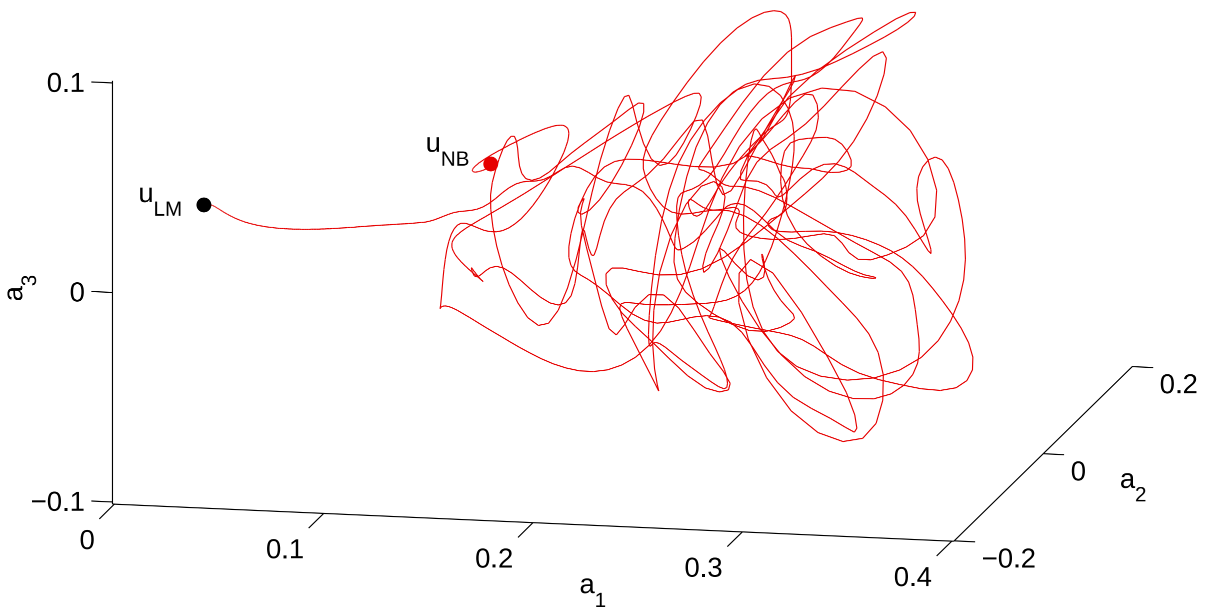
	Re	$\mathcal{L}_x$	$\mathcal{L}_y$	$\mathcal{L}_z$	
"Minimal" PCF	~ 400	$\sim 2\pi$	2	$\sim \pi$	
Hamilton, Kim, Waleffe (HWK)	400	$7\pi/4$	2	$6\pi/5$	sustained turbulence





Adequate resolution: $32 \times 33 \times 32$ to $48 \times 49 \times 48$ grids

A "turbulent Plane Couette" trajectory Re = 400



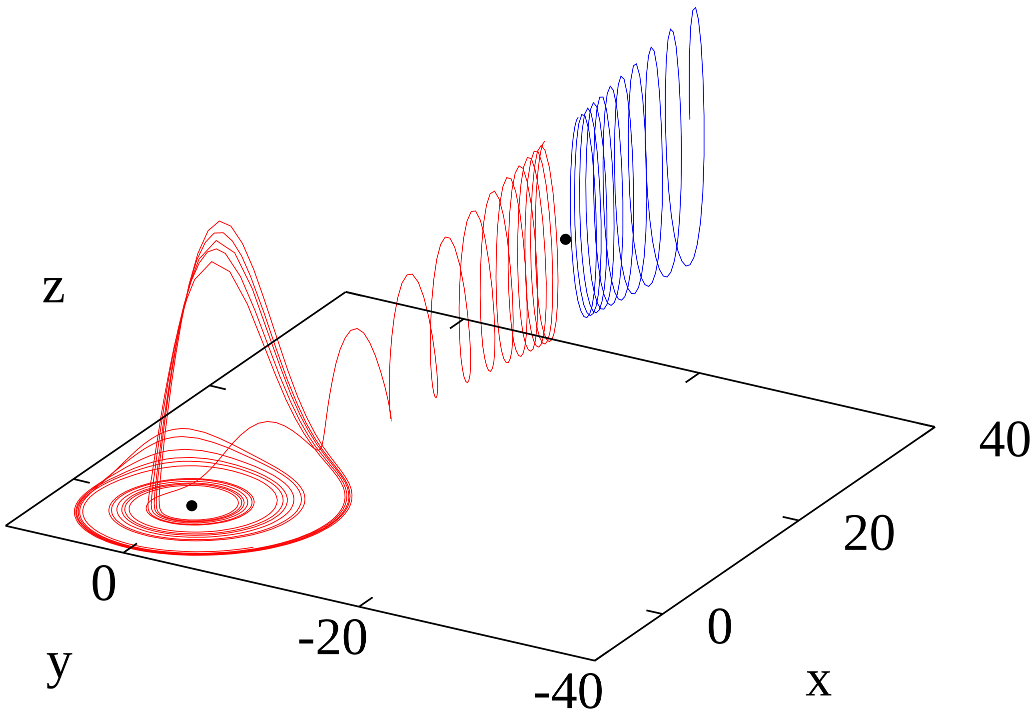
a long transient to the laminar state

60K modes 3D Navier-Stokes DNS, a projection from

Fourier × Fourier × Chebyshev → well-chosen statespace 3d frame

Equilibria / Traveling waves

Role of Rössler flow equilibria

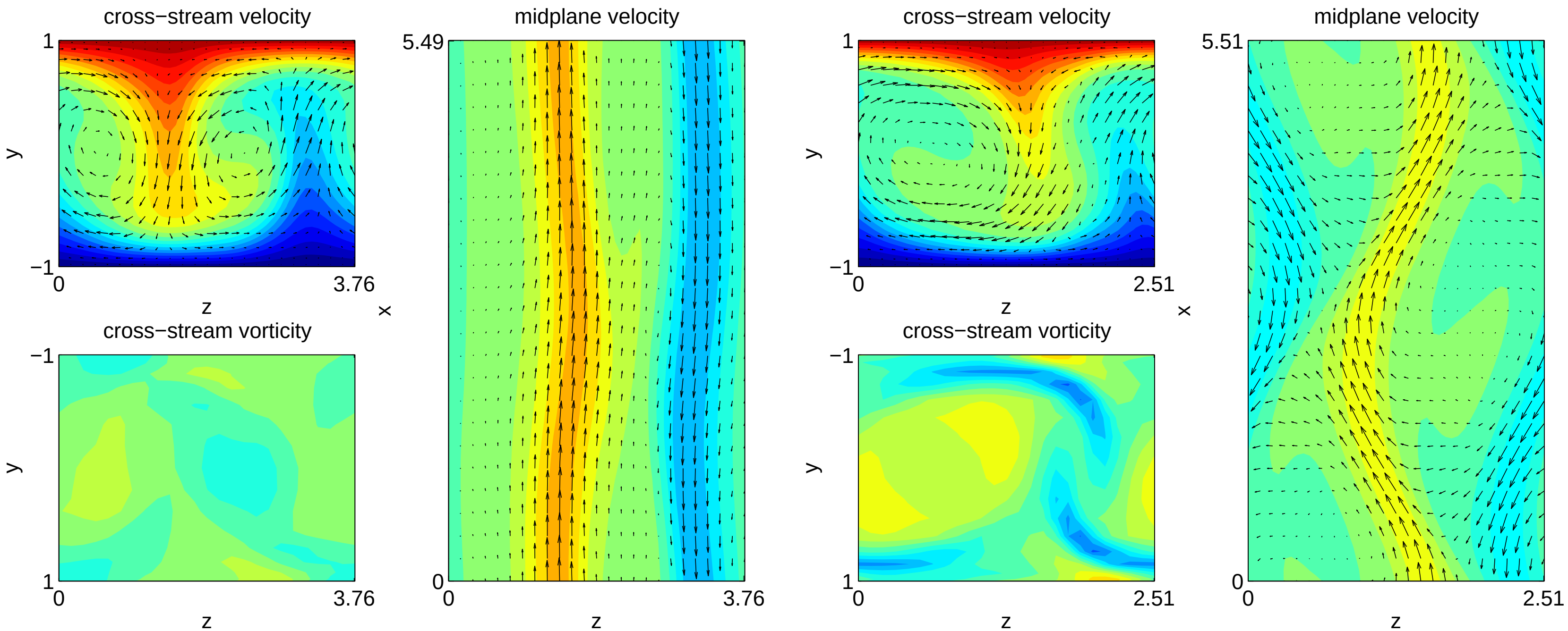


``+'' equilibrium point stable manifold = basin boundary

right of the ``+'' trajectories escape

left of the ``+'' fall into chaotic attractor circling the ``−'' equilibrium point

Turbulence vs. upper-branch equilibrium

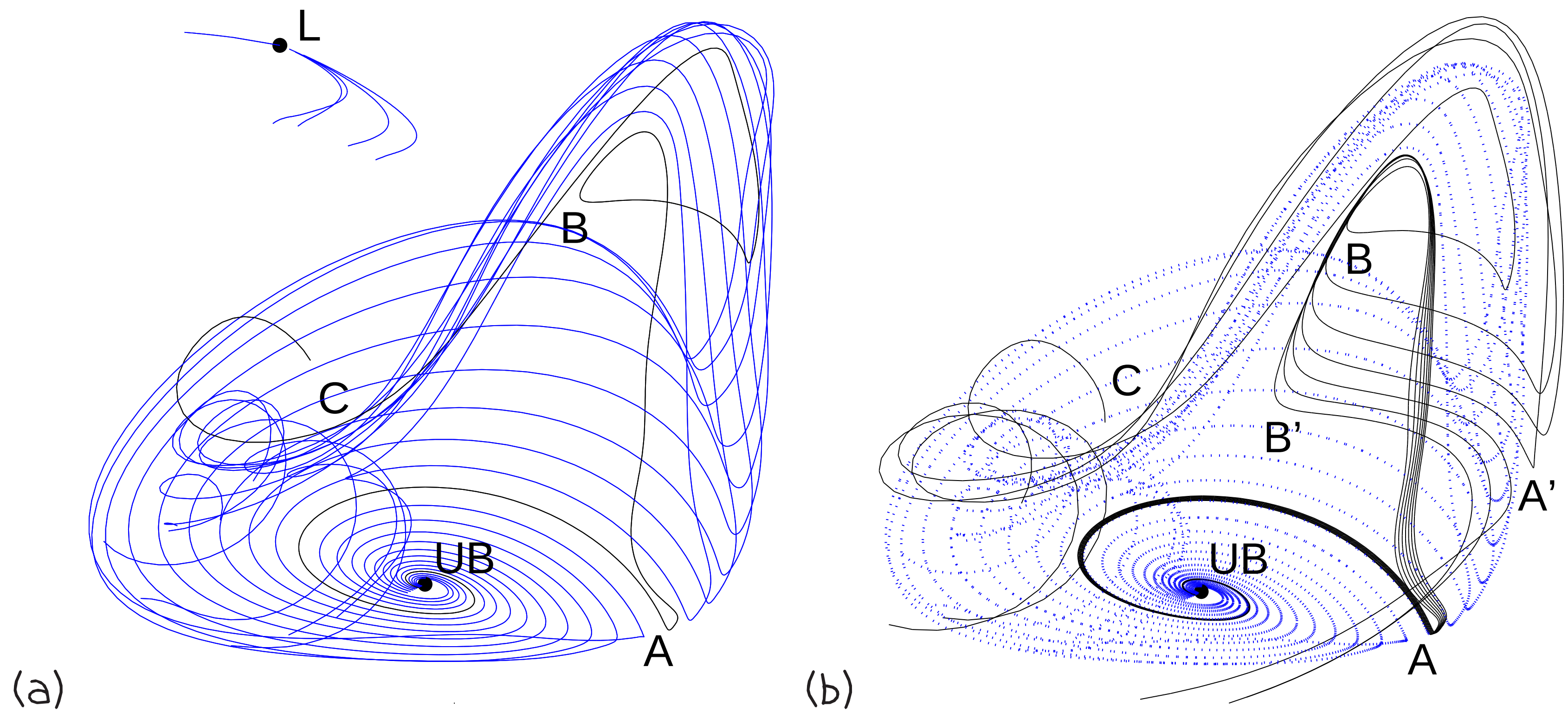


Typical turbulent field

Upper-branch equilibrium[5]

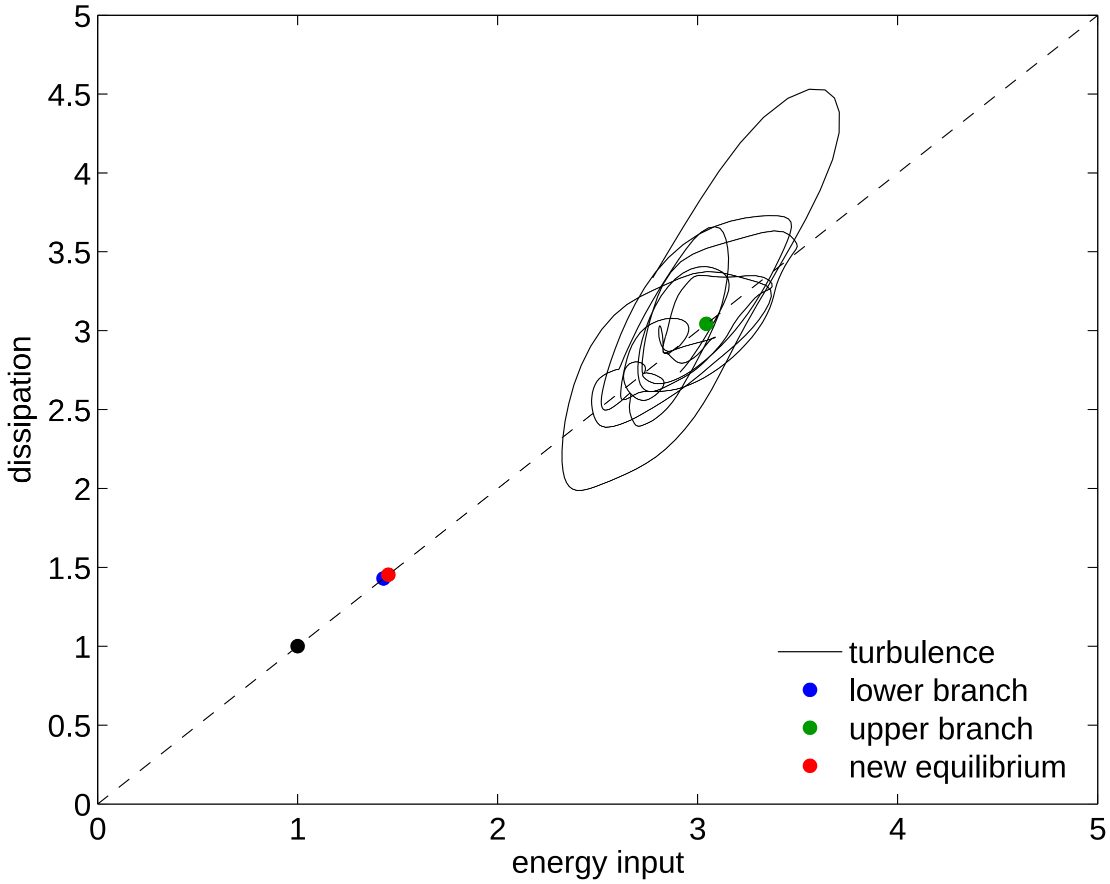
[5] Waleffe, Phys. Fluids 15 (2003)

UB unstable manifold, symmetric subspace

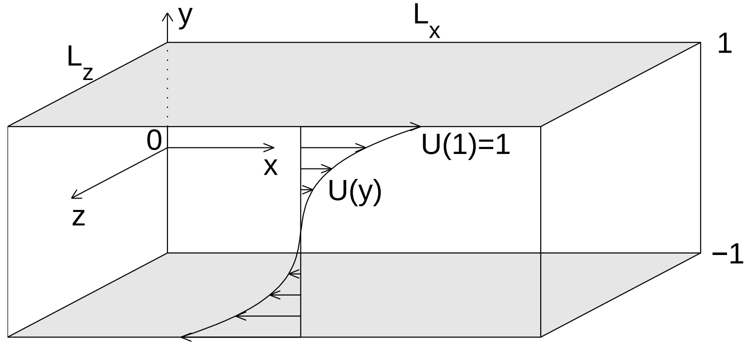


Shift-reflect, shift-rotate unstable manifold of upper branch.

Dissipation versus energy input



Symmetries of plane Couette

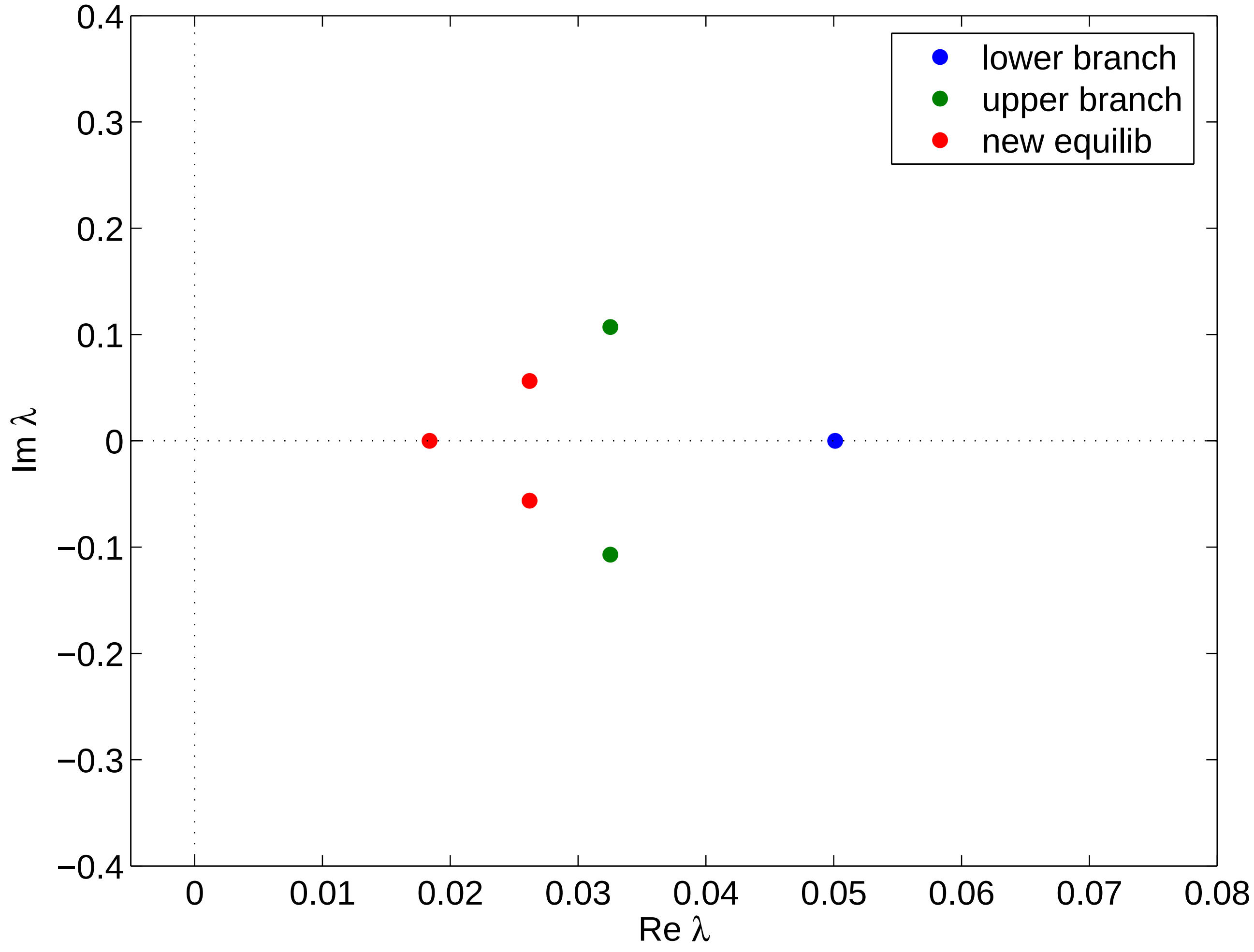


$$\begin{pmatrix}u\\v\\w\end{pmatrix}(x,y,z) \to \begin{pmatrix}u\\v\\-w\end{pmatrix}\left(\frac{\mathcal{L}_x}{2}+x,y,-z\right)$$ shift-reflect

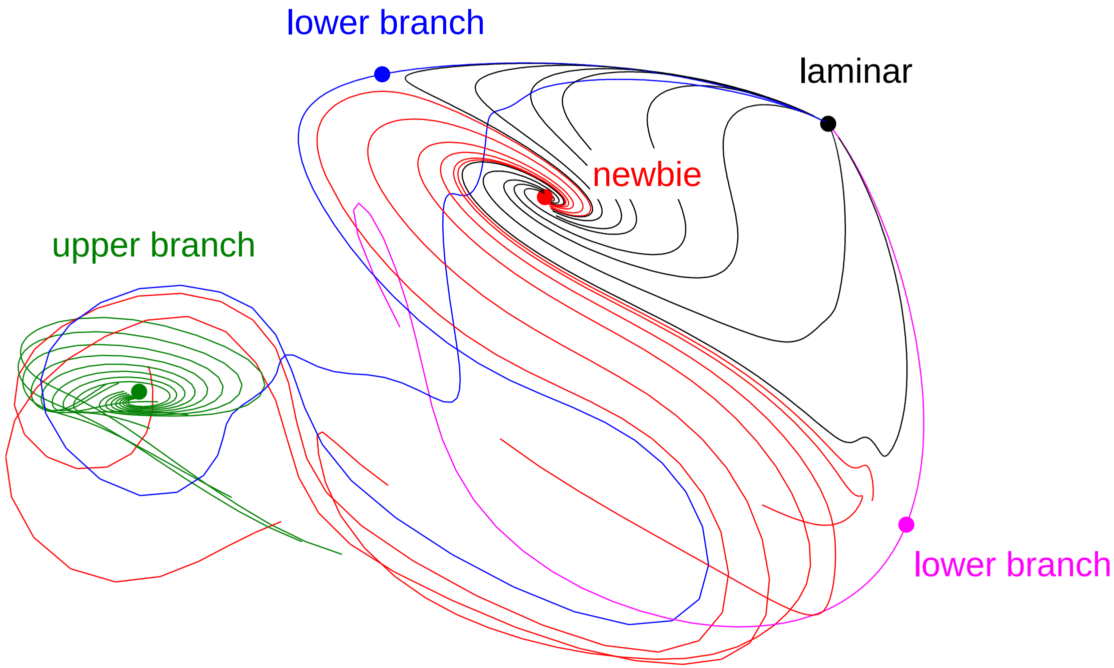
$$\to \begin{pmatrix}-u\\-v\\w\end{pmatrix}\left(\frac{\mathcal{L}_x}{2}-x,-y,\frac{\mathcal{L}_z}{2}+z\right)$$ shift-rotate

$$\to \begin{pmatrix}u\\v\\w\end{pmatrix}(x+\tau_x,-y,z+\tau_z)$$ translate

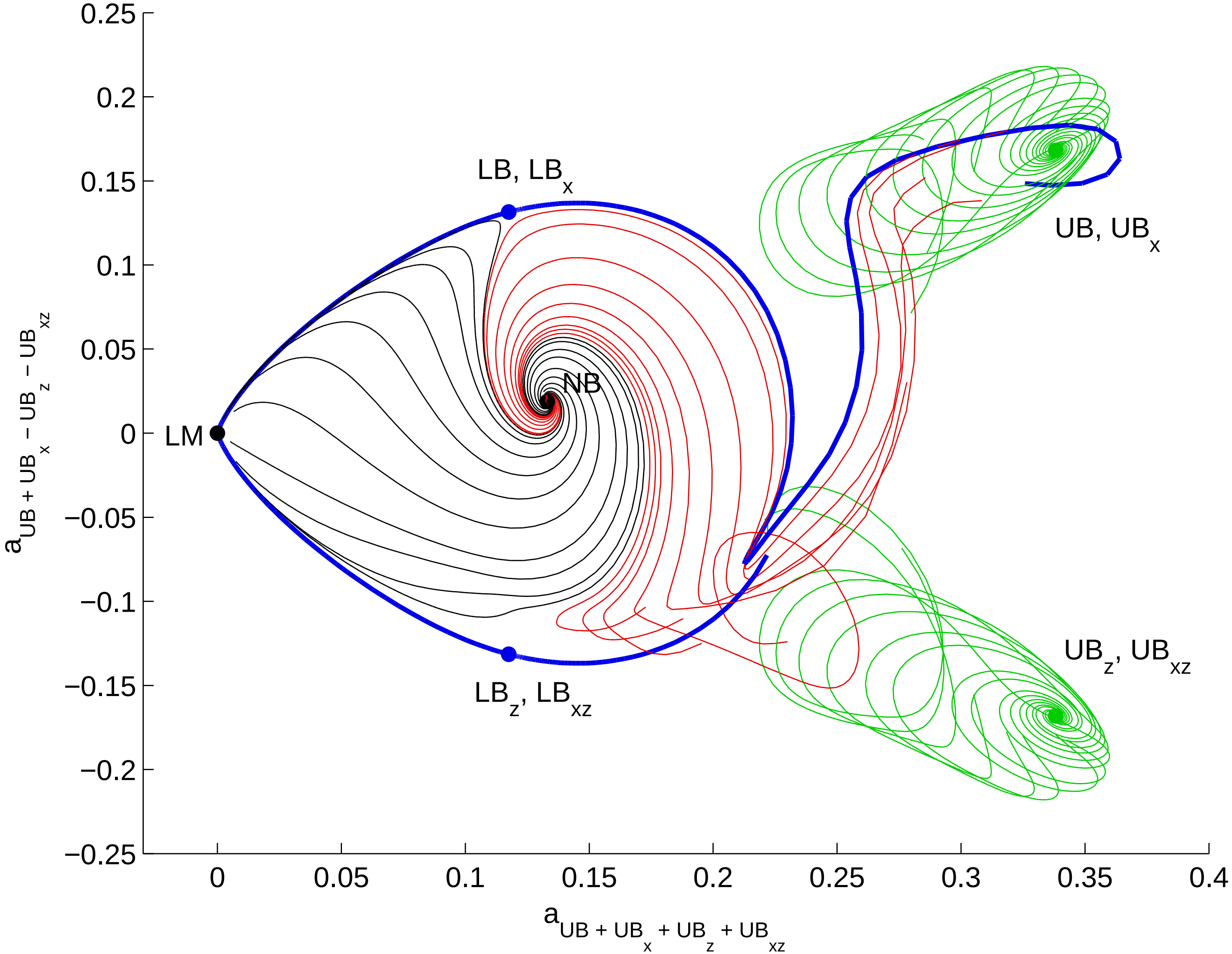
Unstable symmetric LB, UB, NB eigenvalues



UB, LB and NB symmetric state-space portrait

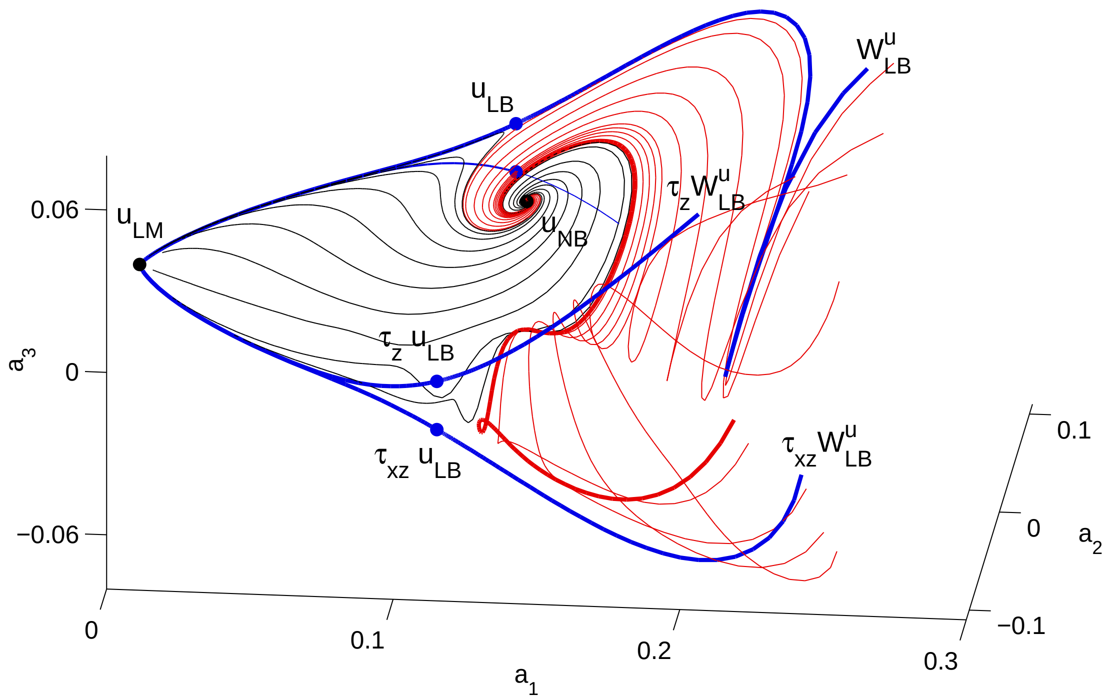


Coordinates of phase portrait are orthogonalized LB, NB, UB.

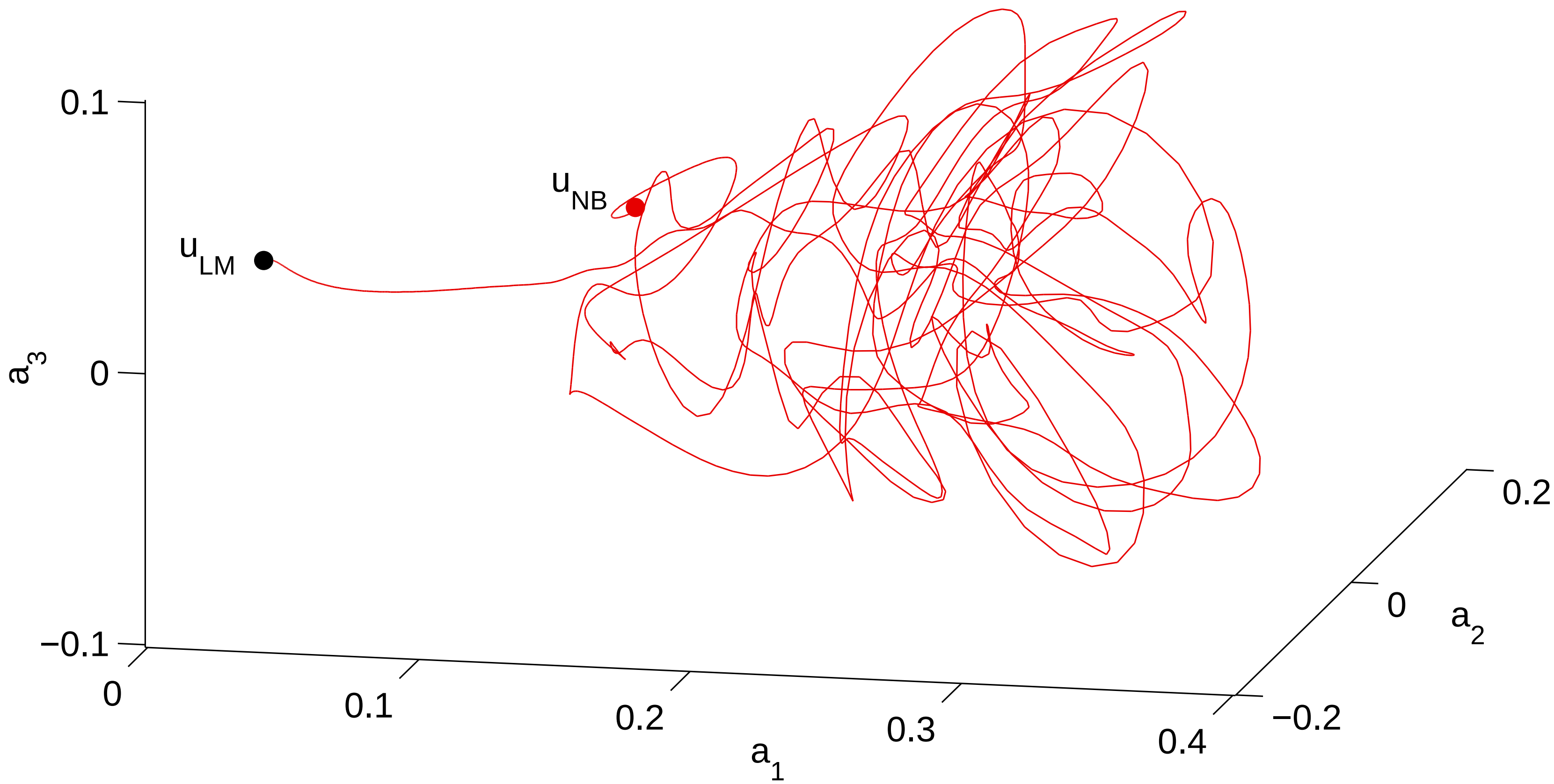


LB, LB_x
LB_z, LB_xz
LM
NB
UB, UB_x
UB_z, UB_xz
a_UB + UB_x − UB_z − UB_xz
a_UB + UB_x + UB_z + UB_xz

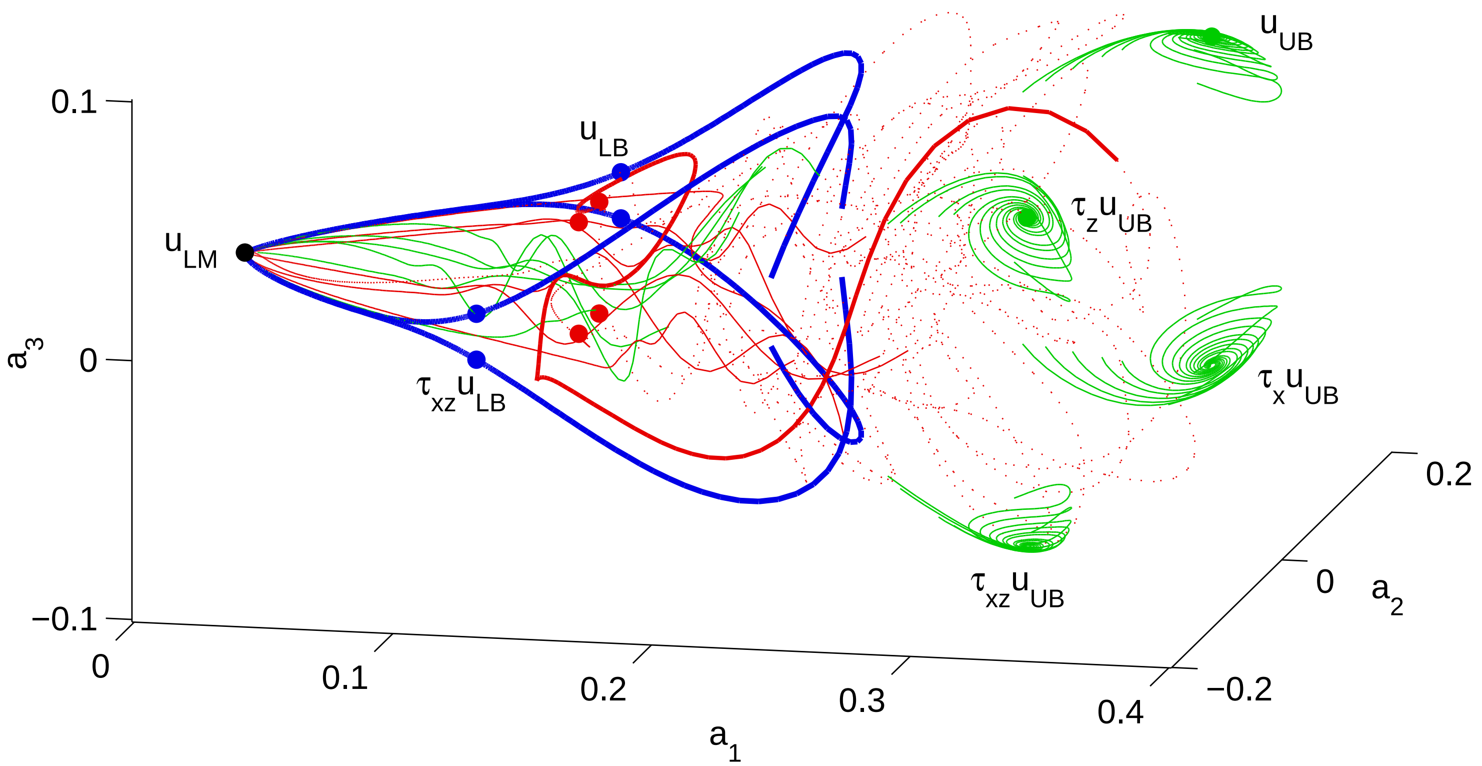
A stroll in 61,506 dimensions



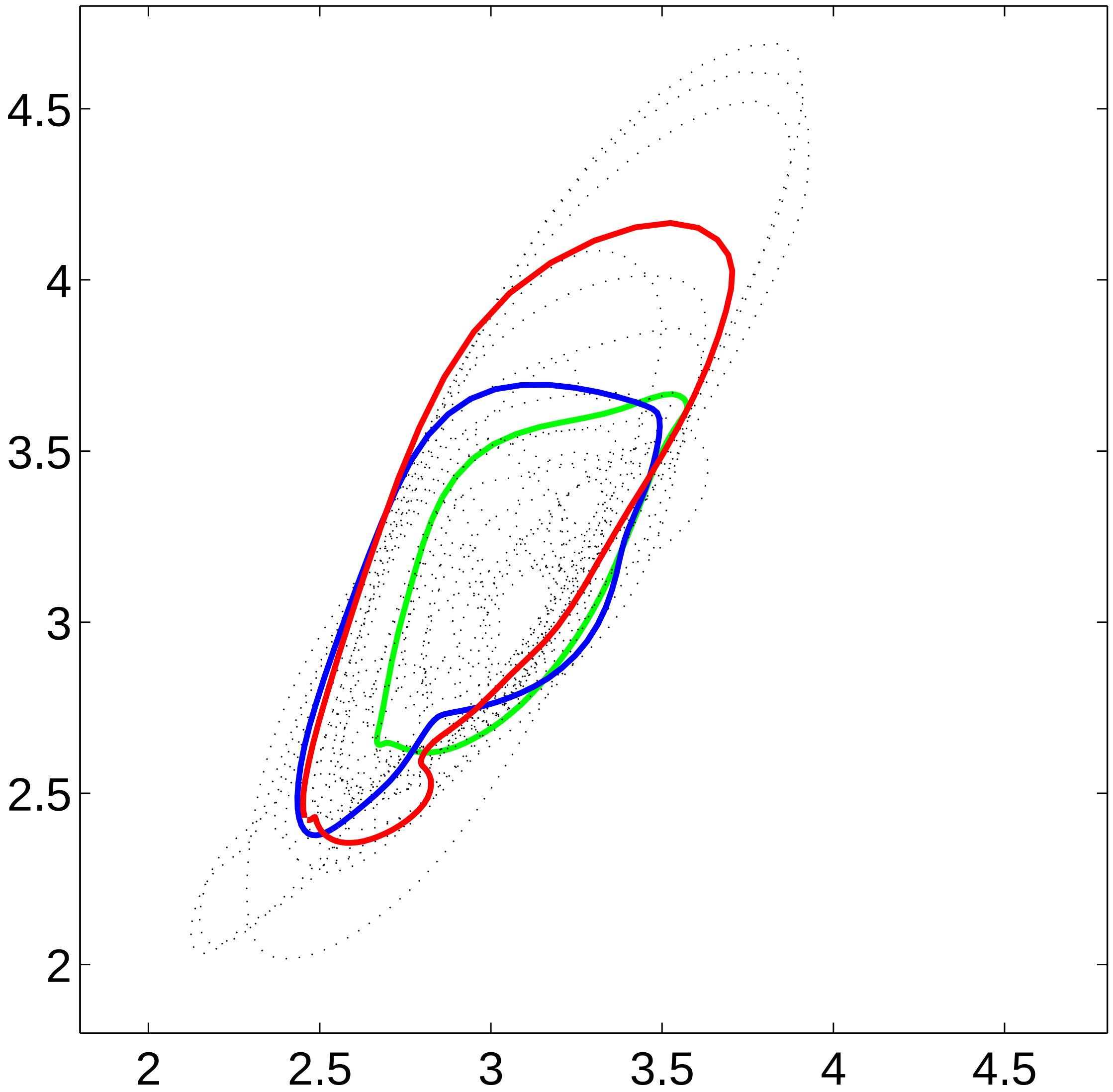
Unstable manifolds of $\mathbf{u}_{LB}$ and its half-cell translations, and a 2d portion of the $\mathbf{u}_{NB}$ unstable manifold, projected from 61,506 dimensions to 3 in the state space global basis



A transiently turbulent trajectory in the $\mathbf{u}_{NB}$ unstable manifold, in isolation.



A transiently turbulent trajectory in the $\mathbf{u}_{NB}$ unstable manifold, within the cage formed by $\mathbf{u}_{LB}$, $\mathbf{u}_{NB}$, $\mathbf{u}_{UB}$, their half-cell translations, and their unstable manifolds. The final decay to laminar of several other trajectories in the unstable manifolds of $\mathbf{u}_{NB}$ and $\mathbf{u}_{UB}$ are also shown.



Three periodic orbits: (green) T = 74.348. (red) T = 102.286 (may be a close recurrence). (blue) T = 88.905.

Conclusions: geometry of Navier Stokes

dual ODE / CFD representations of Navier-Stokes

State space portraits

Computed eigenvalues, eigenfunctions of equilibrium states, w,w/o symmetry

Heteroclinic connections between equilibria

Turbulent dynamics around upper branch

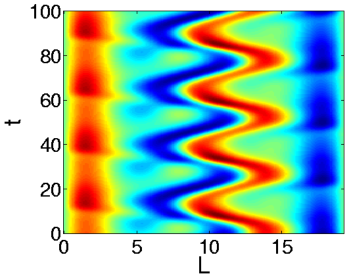
www.channelflow.org public domain software

Future looks bright

Kuramoto-Sivashinsky: Hopf's vision

A long time series:

jumps between



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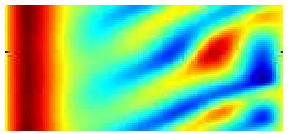

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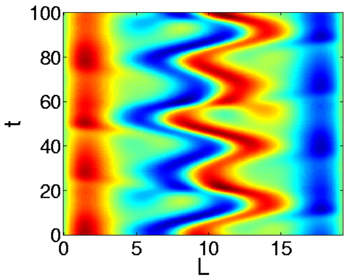
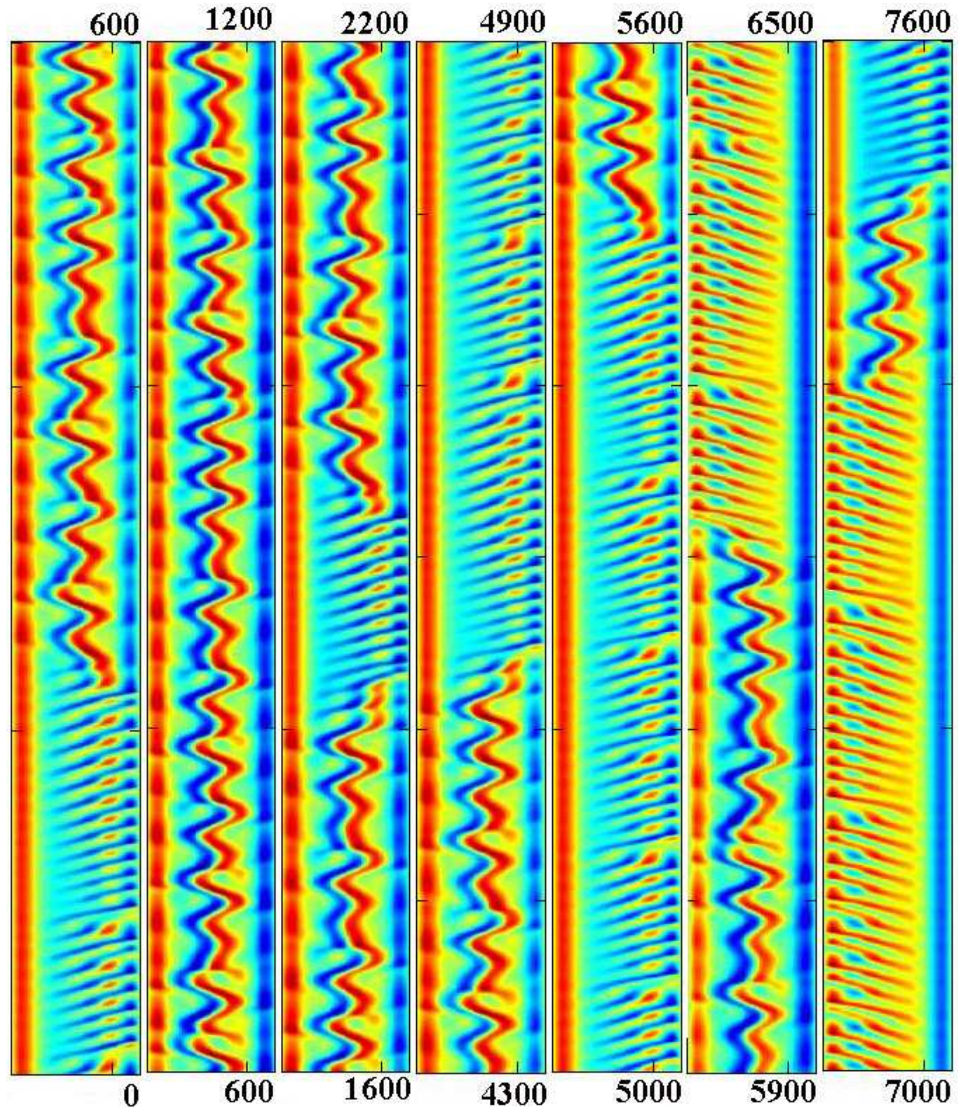


→ etc.



Moral of the story

If you raise a group of plumbers, you shouldn't be upset if they can't do theoretical physics. [6]

A retired Army two-star general [who requested anonymity]

[6] Fred Kaplan, ``Challenging the Generals'', New York Times Sunday Magazine (August 26, 2007).