
Revolution will not be Twitterized!
Spatiotemporal Cat
A Theory of Turbulence
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Matt Gudorf

a spatiotemporal theory of
turbulence
computational challenges

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working notes
Georgia Tech

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overview

1. what this talk is about
2. turbulence in large domains
3. space is time
4. bye bye, dynamics

how do clouds solve PDEs?

do clouds integrate Navier-Stokes equations?

?

 $\Longrightarrow$ other swirls $\Longrightarrow$ 

are clouds Navier-Stokes supercomputers in the sky?

1. turbulence in large domains
2. space is time
3. spacetime
4. bye bye, dynamics

goal : enumerate the building blocks of turbulence

Navier-Stokes equations

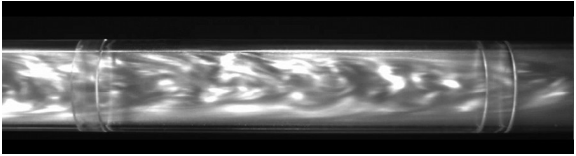
$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = \frac{1}{R}\nabla^2\mathbf{v} - \nabla p + \mathbf{f}\,, \qquad \nabla \cdot \mathbf{v} = 0,$$

velocity field $\mathbf{v} \in \mathbb{R}^3$; pressure field p ; driving force $\mathbf{f}$

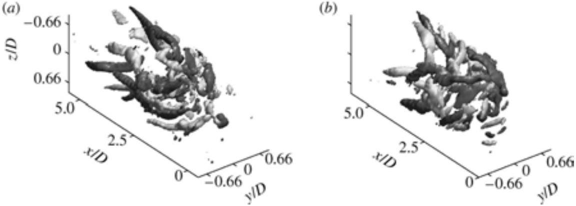
describe turbulence

starting from the equations (no statistical assumptions)

challenge : experiments are amazing

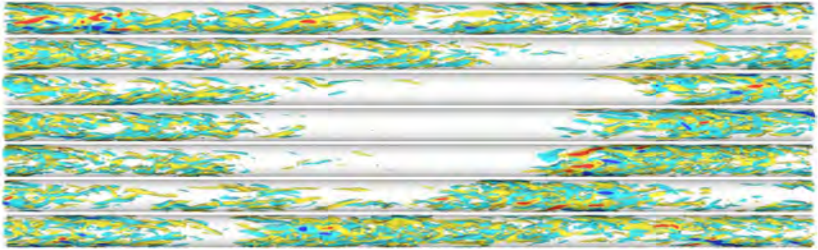


T. Mullin lab



B. Hof lab

can simulate **large** computational domains



pipe flow close to onset of turbulence [1]

but we have **hit a wall**:

exact coherent structures are too unstable to compute

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[1] M. Avila and B. Hof, Phys. Rev. **E 87** (2013)

goal : we can do 3D turbulence, but for this presentation

Navier-Stokes equations

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = \frac{1}{R}\nabla^2 \mathbf{v} + (\cdots)$$

velocity field $\mathbf{v}(\mathbf{x}; t) \in \mathbb{R}^3$

not helpful for developing intuition

we cannot visualize 3D velocity field at every 3D spatial point

look instead at 1D ‘flame fronts’

(1+1) spacetime dimensional “Navier-Stokes”

Navier-Stokes equations **(1822)**

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = \frac{1}{R}\nabla^2\mathbf{v} + (\cdots)$$

▼
▼
▼

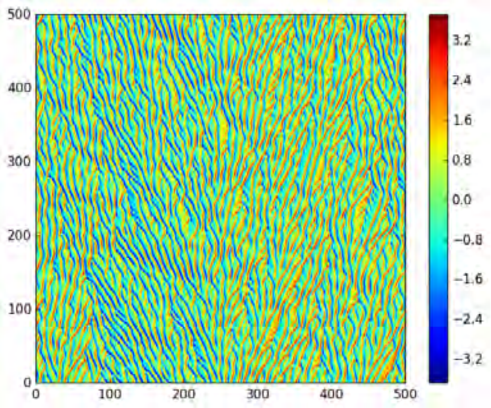
Kuramoto-Sivashinsky (1+1)-dimensional PDE **(1975)**

$$u_t + u\nabla u = -\nabla^2 u - \nabla^4 u \,, \qquad x \in \mathbb{R} \,,$$

describes spatially extended systems such as

- flame fronts in combustion
- reaction-diffusion systems
- ...

an example : Kuramoto-Sivashinsky on a large domain

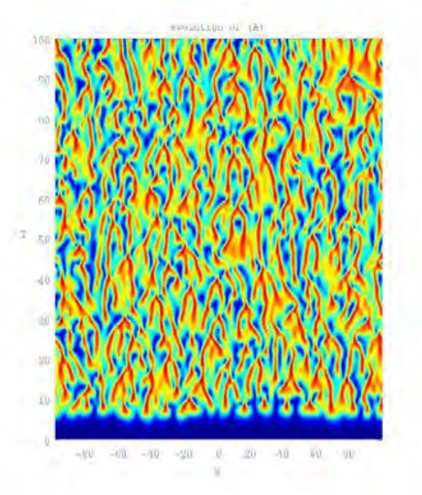


[horizontal] space $x \in [0, L]$ [up] time evolution

- turbulent behavior
- simpler physical, mathematical and computational setting than Navier-Stokes

another example of large spacetime domain simulation

complex Ginzburg-Landau

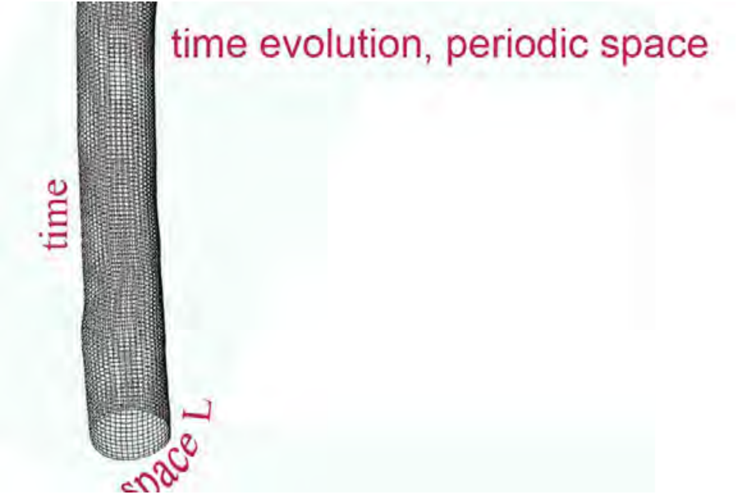


(will return to this)

[horizontal] space $x \in [-L/2, L/2]$ [up] time evolution

codeinthehole.com/static/tutorial/coherent.html

compact space, infinite time cylinder



so far : Navier-Stokes on compact spatial domains, all times

compact space, infinite time

Kuramoto-Sivashinsky equation

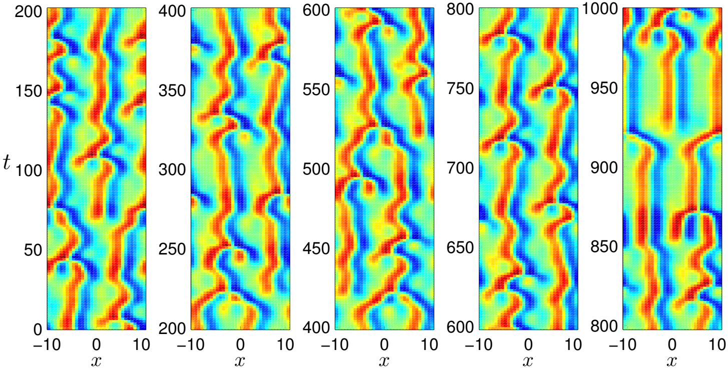
$$u_t = -(+\nabla^2 + \nabla^4)u - u\nabla u\,, \qquad x \in [-L/2, L/2]\,,$$

in terms of discrete spatial Fourier modes

N ordinary differential equations (ODEs) in time

$$\dot{\tilde{u}}_k(t) = (q_k^2 - q_k^4)\,\tilde{u}_k(t) - i\frac{q_k}{2}\sum_{k'=0}^{N-1}\tilde{u}_{k'}(t)\tilde{u}_{k-k'}(t)\,.$$

evolution of Kuramoto-Sivashinsky on small $L = 22$ cell



horizontal: $x \in [-11, 11]$
vertical: time
color: magnitude of $u(x, t)$

part 2

yes, but

is space time?

compact time, infinite space

rewrite Kuramoto-Sivashinsky

$$u_t = -uu_x - u_{xx} - u_{xxxx}$$

as 4-fields vector

$$\mathbf{u}^\top = (u, u', u'', u''')$$
$$\text{where } u' \equiv u_x\,,\ u'' \equiv u_{xx}\,,\ u''' \equiv u_{xxx}$$

equation $\frac{d}{dx}\mathbf{u}(x) = \mathbf{v}(x)$ now 1st order in spatial derivative

Kuramoto-Sivashinsky = four coupled 1st order PDEs

$$\frac{d\,u}{dx} = u'\,, \qquad \frac{d\,u'}{dx} = u''$$
$$\frac{d\,u''}{dx} = u'''\,, \qquad \frac{d\,u'''}{dx} = -u_t - u'' - u\,u'$$

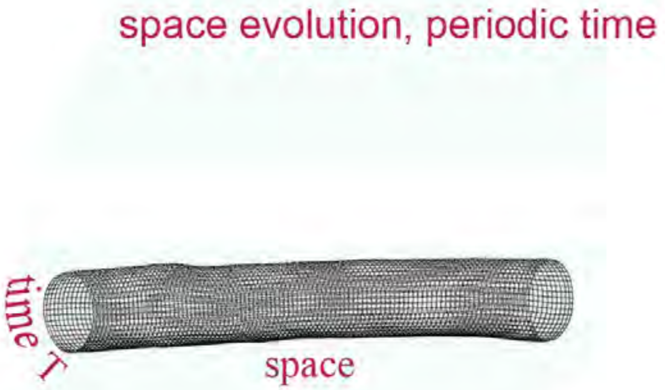
compact time, infinite space

1st order in spatial derivative

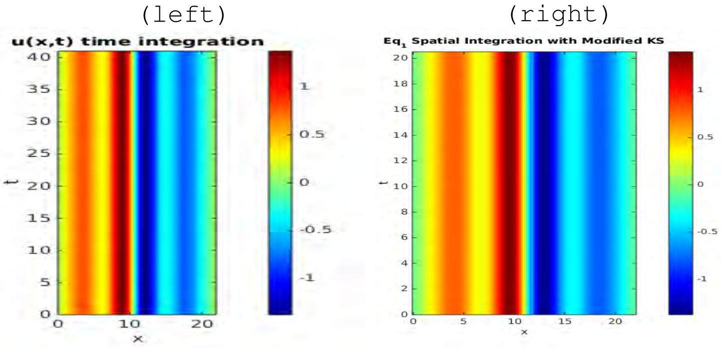
evolve four 1st order PDEs for $\mathbf{u}(x)$ in x,

- $$\frac{d}{dx}\mathbf{u}(x) = \mathbf{v}(x)$$
- compact in time, periodic boundary condition
$$u(x,t) = u(x,t+T)$$
- initial data
$$\mathbf{u}_0^\top = (u(x_0,t), u'(x_0,t), u''(x_0,t), u''(x_0,t))$$
specified for all $t \in [0, T)$, at a fixed space point x_0

can do : compact time, infinite space cylinder



a time-invariant equilibrium, spatial periodic orbit

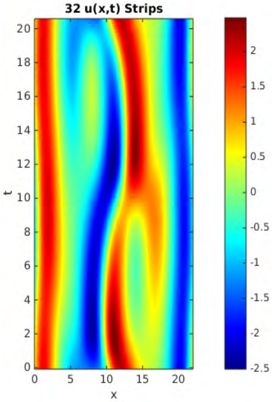


evolution of EQ_1 : (left) in time, (right) in space
initial condition for the spatial integration is the time strip $u(x_0, t)$, $t = [0, T)$, where time period $T = 0$, spatial x period is $L = 22$.

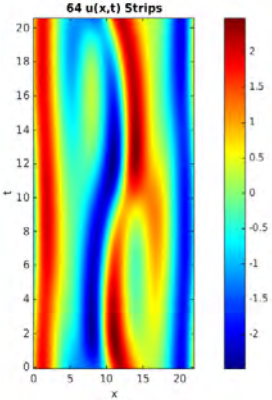
Michelson 1986

a spacetime invariant 2-torus integrated in either time or space

(left)



(right)



(left) old : time evolution. (right) new : space evolution

$x = [0, L]$ initial condition : time periodic line $t = [0, T]$

Gudorf 2016

but integrations are uncontrollably unstable

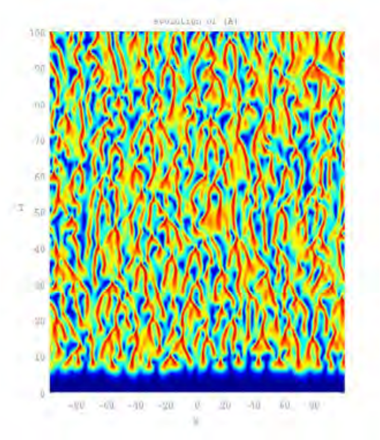
neither time nor space integration works
for large domains

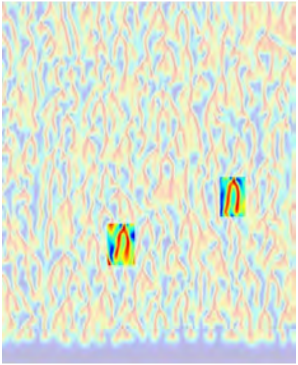
rethink the calculation

part 3

complex Ginzburg-Landau on a large spacetime domain

goal : enumerate nearly recurrent chronotopes

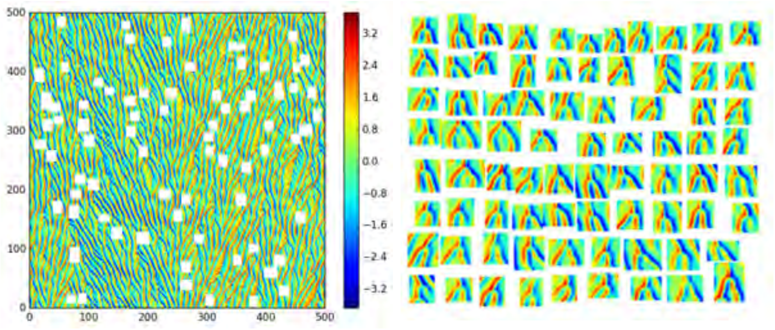




[left-right] space $x \in [-L/2, L/2]$ [up] time $t \in [0, T]$

Kuramoto-Sivashinsky on a large spacetime domain

the same small tile recurs often in a turbulent pattern



goal : define, enumerate nearly recurrent tiles

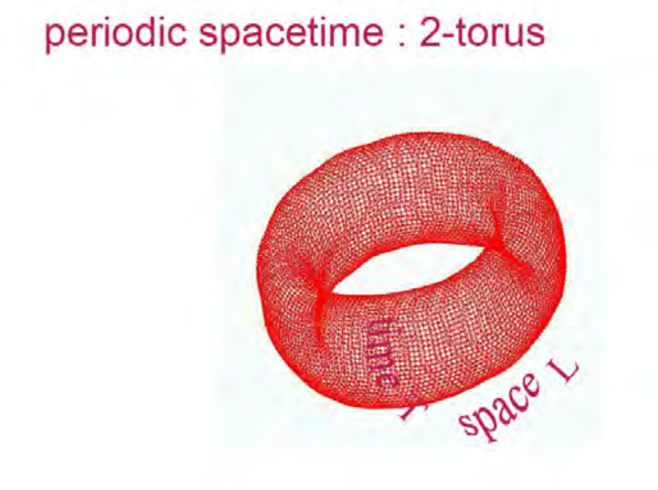
chronotope[2]

> In literary theory and philosophy of language, the chronotope is how configurations of time and space are represented in language and discourse.
>
> — Wikipedia : Chronotope

- Mikhail Mikhailovich Bakhtin (1937)

[2] S. Lepri et al., J. Stat. Phys. **82**, 1429–1452 (1996).

use spatiotemporally compact solutions as chronotopes



this ‘exact coherent structure’
shadows a small patch of spacetime solution $u(x, t)$

periodic orbits generalize to *d*-tori

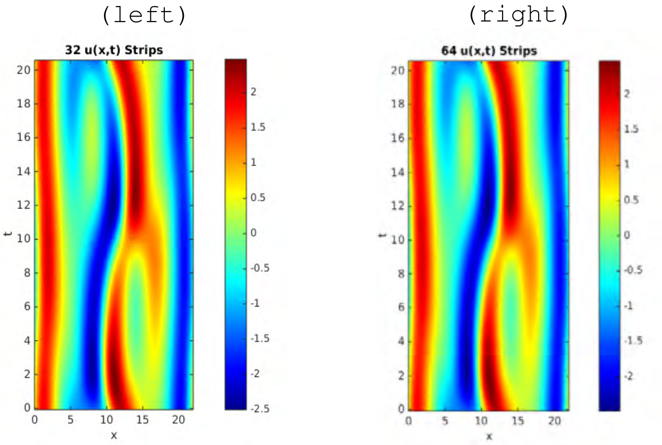
1 time, 0 space dimensions

a state space point is *periodic* if its orbit returns to it after a finite time T ;
such orbit tiles the time axis by infinitely many repeats

1 time, *d*-1 space dimensions

a state space point is *spatiotemporally periodic* if it belongs to an invariant *d*-torus $\mathcal{R}$;
such torus tiles the spacetime by infinitely many repeats

a spacetime invariant 2-torus integrated in either time or space



(left) old : time evolution $t = [0, T]$

initial condition : space periodic line $x = [0, L]$

(right) new : space evolution $x = [0, L]$

initial condition : time periodic line $t = [0, T]$

Gudorf 2016

every compact solution is a fixed point on a discrete lattice

discretize $u_{nm} = u(x_n, t_m)$ over NM points of spatiotemporal periodic lattice $x_n = nL/N$, $t_m = mT/M$, Fourier transform :

$$\tilde{u}_{k\ell} = \frac{1}{NM} \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} u_{nm} \, e^{-i(q_k x_n + \omega_\ell t_m)} \,, \quad q_k = \frac{2\pi k}{L} \,, \; \omega_\ell = \frac{2\pi \ell}{T}$$

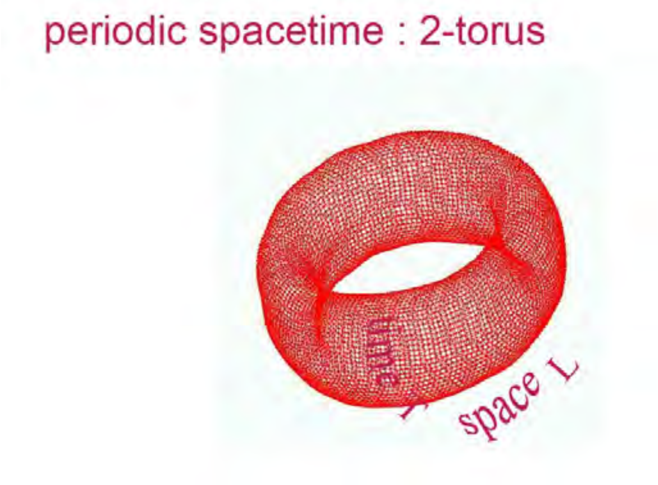
Kuramoto-Sivashinsky is no more a PDE,
but an algebraic $[N \times M]$-dimensional problem
of determining global solution $\mathbf{u}$ to

fixed point condition

$$\left(-i\omega_\ell - (q_k^2 - q_k^4)\right) \tilde{u}_{k\ell} + i\frac{q_k}{2} \sum_{k'=0}^{N-1} \sum_{m'=0}^{M-1} \tilde{u}_{k' m'} \tilde{u}_{k-k', m-m'} = 0$$

every calculation is a spatiotemporal lattice calculation

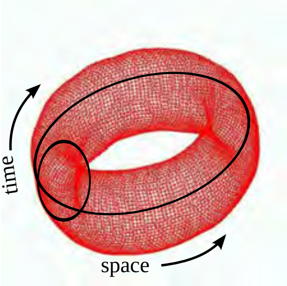
field is discretized as $\tilde{u}_{k\ell}$ values
over NM points of a periodic lattice



professor Zweistein forgets to take his meds

statement : HA!

You are imposing by hand the space & time periods L, T !



answer : NO!

nature chooses L & T, they are free parameters.

there is no more time evolution

solution to Kuramoto-Sivashinsky is now given as

condition that

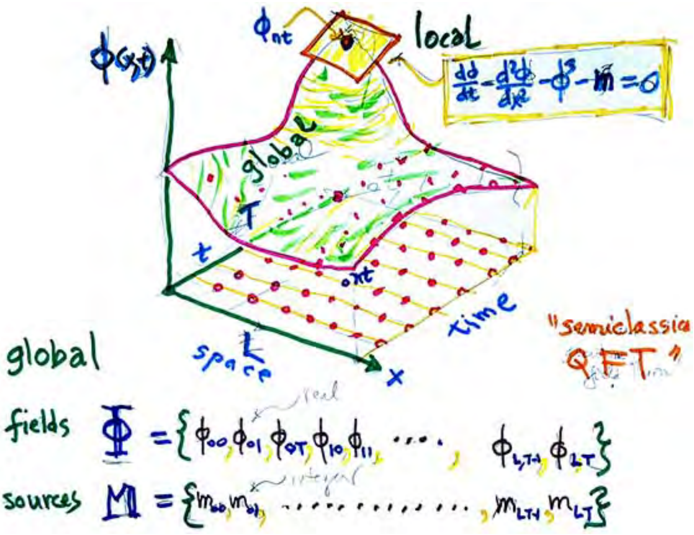
at each lattice point $k\ell$
the tangent field at $\tilde{u}_{k\ell}$

satisfies the equations of motion

$$\left[-i\omega_\ell - (q_k^2 - q_k^4)\right]\tilde{u}_{k\ell} + i\frac{q_k}{2}\sum_{k'=0}^{N-1}\sum_{m'=0}^{M-1}\tilde{u}_{k'm'}\tilde{u}_{k-k',m-m'} = 0$$

this is a local tangent field constraint on a global solution

think globally, act locally



for each symbol array M, a periodic lattice state X_M

unexpected gift from nature

robust : no exponential instabilities

as there are no finite time / space integrations

no need for $\sim 10^{-11}$ accuracies,

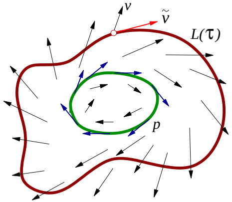
SO

accuracy to a few % suffices,

you only need to get the shape of a solution right

how to find solutions ? an ODE example

the law of motion : $\dot{x} = v(x)$



guess loop tangent $\tilde{v}(\tilde{x}) \neq v(\tilde{x})$

periodic orbit $\tilde{v}(\tilde{x})$, $v(\tilde{x})$ aligned

cost function

$$F^2[\tilde{x}] = \oint_L ds\, (\tilde{v} - v)^2 \,; \quad \tilde{v} = \tilde{v}(\tilde{x}(s,\tau))\,, \;\; v = v(\tilde{x}(s,\tau))\,,$$

penalize[3] misorientation of the loop tangent $\tilde{v}(\tilde{x})$ relative to the true dynamical flow tangent field $v(\tilde{x})$

[3] Y. Lan and P. Cvitanović, Phys. Rev. E **69**, 016217 (2004).

how do clouds solve PDEs?

clouds do not **NOT** integrate Navier-Stokes equations

$\Longrightarrow$ other swirls $\Longrightarrow$

do clouds satisfy Navier-Stokes equations?

yes!

they satisfy them locally, everywhere and at all times

the equations imposed as local constraints

Kuramoto-Sivashinsky equation

$$F(u) = u_t + u_{xx} + u_{xxxx} + uu_x = 0$$

for example, minimize over the entire 2-torus

cost function

$$G \equiv \frac{1}{2}|F(u)|^2_{L\times T}$$

need your help !

adjoint descent

cost function

$$G = \frac{1}{2}\mathbf{F}^\top \mathbf{F}\,.$$

introduce fictitious time (τ) flow by differentiation of cost function.

$$\partial_\tau G = (J^\top \mathbf{F})^\top (\partial_\tau \mathbf{x})$$

“adjoint descent” method defined by chosing[4]

$$\partial_\tau \mathbf{x} = -(J^\top \mathbf{F})$$

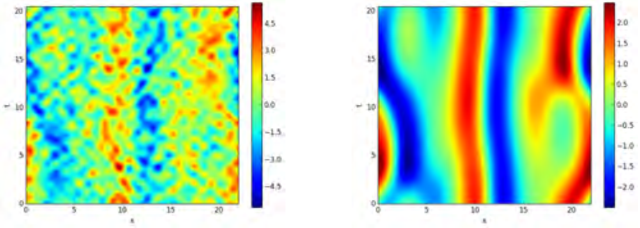
[4] M. Farazmand, J. Fluid M. **795**, 278–312 (2016).

does it work at all ?

add strong noise to a *known* solution,
twice the typical amplitude

only the first test

(not how we actually generate guesses)



(left) initial guess: a known invariant 2-torus

$(L_0, T_0) = (22.0, 20.5057459345)$ + strong random noise

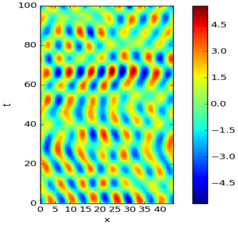
(right) the resulting adjoint descent converged invariant 2-torus

$(L_f, T_f) = (21.95034935834641, 20.47026321555662)$

initial guess generation ?

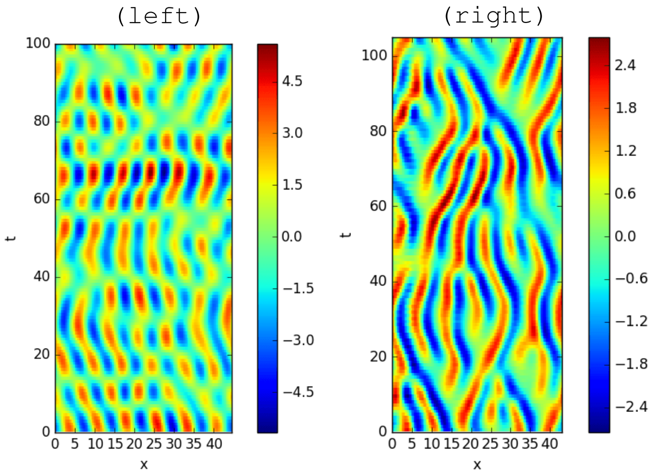
the time scale : the shortest ‘turnover’ scale characterized by the period of the shortest periodic orbit? Or perhaps the Lyapunov time?

the spatial scale : $\bar{L} = 2\pi\sqrt{2}$, the most unstable spatial wavelength of the Kuramoto-Sivashinsky



initial : spatial $\bar{L}$-modulated random guess

KS invariant 2-torus found variationally



(left) initial : $\bar{L} = 2\pi\sqrt{2}$ spatially modulated “noisy” guess
(right) adjoint descent : converged invariant 2-torus

Gudorf 2018

initial guesses, embedded in ergodic sea?

Historically,

guesses extracted from close recurrences
observed in long turbulent simulations

1. inefficient, finds only the shortest, least unstable orbits[5,6]
2. can integrate only not far in time

need spatiotemporal guesses

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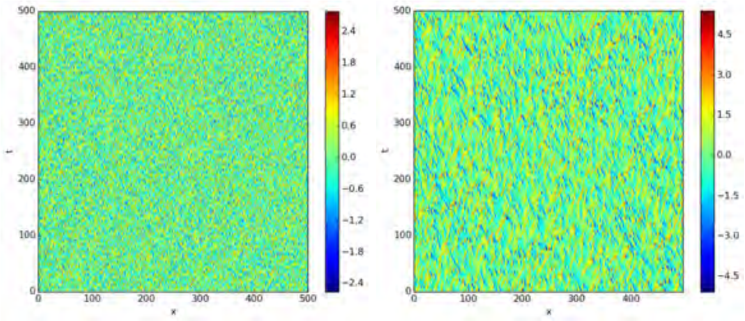
[5] D. Auerbach et al., Phys. Rev. Lett. **58**, 2387–2389 (1987).

[6] J. F. Gibson et al., J. Fluid Mech. **611**, 107–130 (2008).

guesses extracted from large spacetime domains

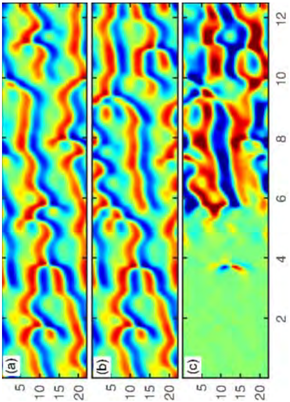


(left) random initial state on $(L, T) = (500, 500)$
(right) adjoint descent $\rightarrow$ typical Kuramoto-Sivashinsky state

finite windows are our starting guesses for invariant 2-tori

another, much twittered : machine learning guesses

“reservoir computing” example[7]



(a) data: Kuramoto-Sivashinsky simulation

(b) reservoir computing prediction

(c) two subtracted agree to ~ 5 Lyapunov times

Q : how would you learn this data?

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[7] J. Pathak et al., Phys. Rev. Lett. **120**, 024102 (2018).

what to do?

Matthew N. Gudorf

has 1 000's of such invariant 2-tori

1. turbulence in large domains
2. space is time
3. spacetime
4. **fundamental tiles**
5. bye bye, dynamics

how do we recognize a cloud?

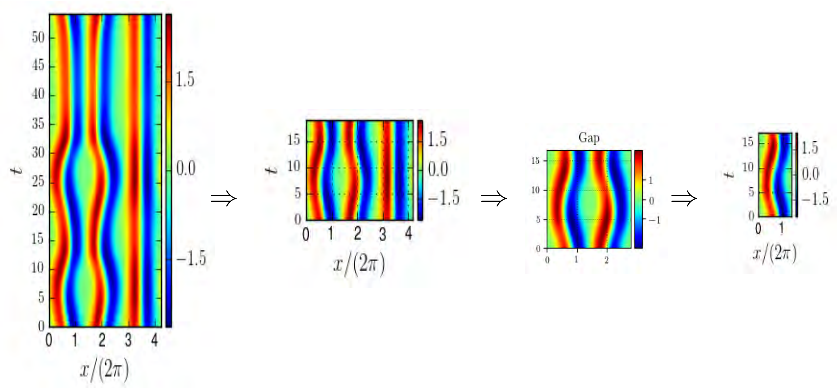
WATCH

⟹ other swirls ⟹

by recurrent shapes!

so, construct an alphabet of possible shapes

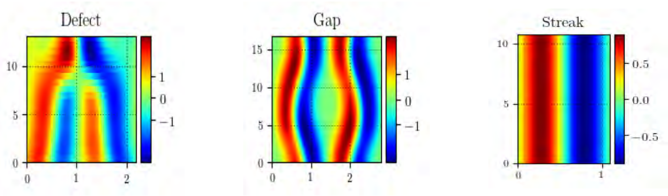
extracting a fundamental tile



1) invariant 2-torus
2) invariant 2-torus computed from initial guess cut out from 1)
3) “gap" invariant 2-torus, initally cut out from 2)
4) the “gap" prime invariant 2-torus fundamental domain

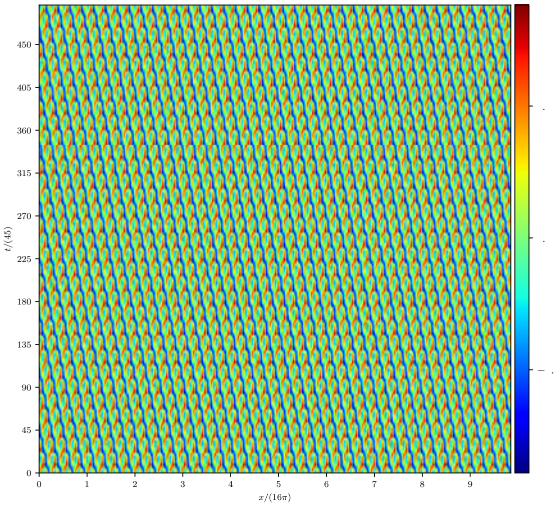
a trial set of prime (rubber) tiles

an alphabet of Kuramoto-Sivashinsky fundamental tiles



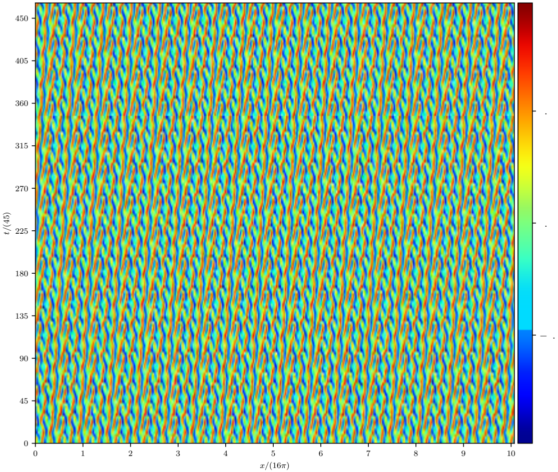
utilize also discrete symmetries :
spatial reflection, spatiotemporal shift-reflect, $\cdots$

Kuramoto-Sivashinsky tiled by a small tile



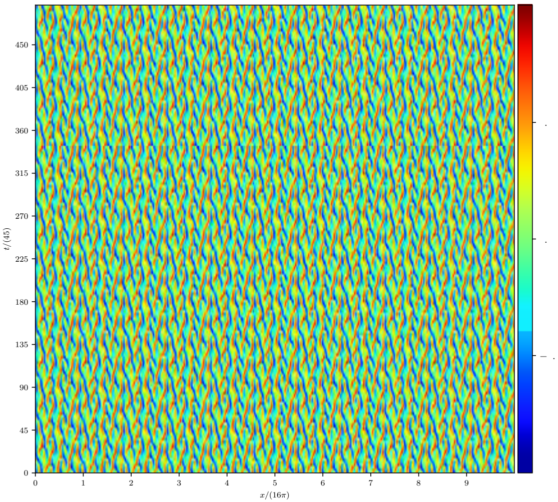
tiling by relative periodic invariant 2-torus
$(L, T) = (13.02, 15)$

spacetime tiled by a larger tile



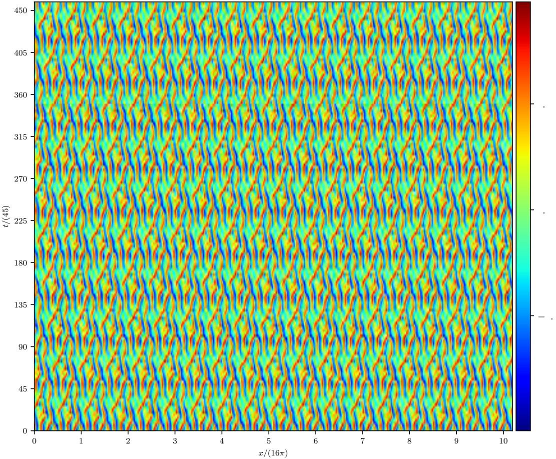
tiling by relative periodic invariant 2-torus
$(L, T) = (33.73, 35)$

spacetime tiled by a tall tile



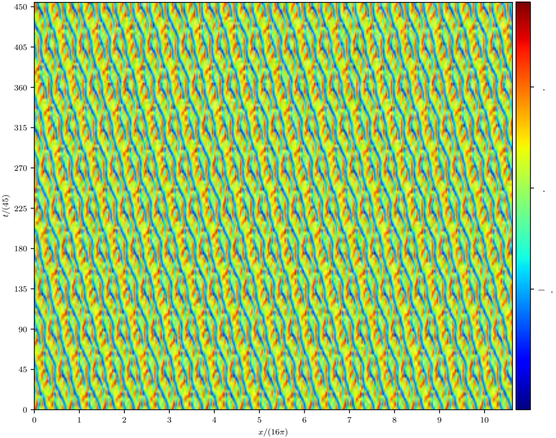
tiling by shift-reflect invariant 2-torus
$(L, T) = (55.83, 24)$

spacetime tiled by a larger tile



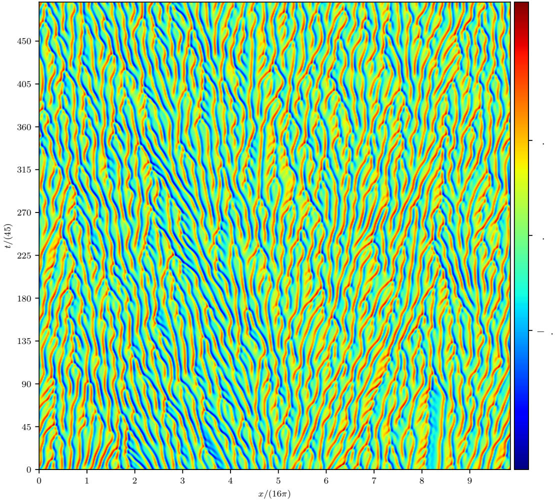
tiling by relative periodic invariant 2-torus
$(L, T) = (32.02, 51)$

spacetime tiled by a larger tile



tiling by relative periodic invariant 2-torus
$(L, T) = (44.48, 50)$

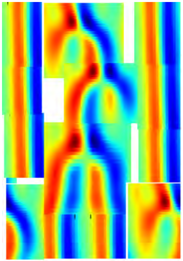
any particular tiling looks nothing like turbulent Kuramoto-Sivashinsky!

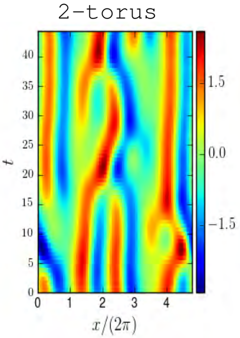


[horizontal] space $x \in [-L/2, L/2]$ [up] time evolution

a qualitative tiling guess

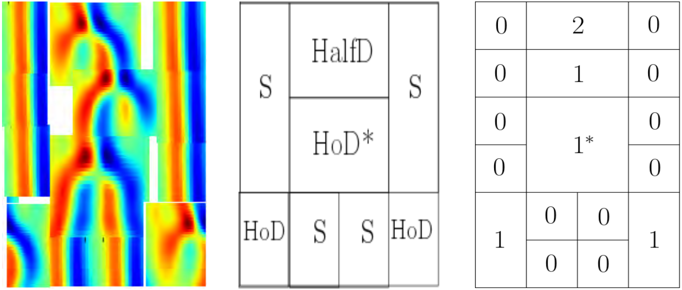
a tiling and the resulting solution





turbulence.zip : each solution has a unique symbolic name

symbolic dynamics is 2-dimensional!



- each symbol indicates a corresponding spatiotemporal tile
- these are “rubber” tiles

1. turbulence in large domains
2. space is time
3. bye bye, dynamics

goodbye

to long time and/or space integrators

they never worked and could never work

life outside of time

the trouble:
forward time-integration codes too unstable

multishooting inspiration: replace a guess that a point is on the periodic orbit by a guess of the entire orbit.

$\rightarrow$

spatio-temporally periodic solutions of classical field theories can be found by variational methods

the equations solved as global optimization problems

impose the equations as local constraints

$$F(u) = u_t + u_{xx} + u_{xxxx} + uu_x = 0$$

minimize globally

perhaps using cost function

$$G \equiv \frac{1}{2} |F(u)|^2_{L \times T}$$

do this ?

the answer is

scalability

compute locally, adjust globally

Navier-Stokes codes

- T. M. Schneider : developing a matrix-free variational Navier-Stokes code, machine learning initial guesses
- D. Lasagna and A. Sharma : developing variational adjoint solvers to find periodic orbits with long periods
- Q. Wang : parallelizing spatiotemporal computation is FLOPs intensive, but more robust than integration forward in time

it's rocket science[8,9,10]

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[8] T. M. Schneider, *Variational adjoint methods coupled with machine learning*, private communication, 2019.

[9] D. Lasagna et al., *Periodic shadowing sensitivity analysis of chaotic systems*, 2018.

[10] Q. Wang et al., Phys. Fluids **25**, 110818 (2013).

towards scalable parallel-in-time turbulent flow simulations

future :

processor speed $\to$ limit

number of cores $\to 10^6 \to \cdots$

Wang et al (2013)[11] *:*
next-generation : spacetime parallel simulations,
on discretized $4D$ spacetime computational domains,
with each computing core handling a spacetime lattice cell

compared to time-evolution solvers: significantly higher level of concurrency, reduction the ratio of inter-core communication to floating point operations

$\Rightarrow$ a path towards exascale DNS of turbulent flows

[11] Q. Wang et al., Phys. Fluids **25**, 110818 (2013).

enumerate hierarchically spatiotemporal patterns

2D symbolic encoding ⇒ admissible solutions

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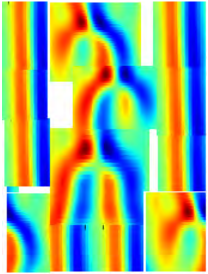
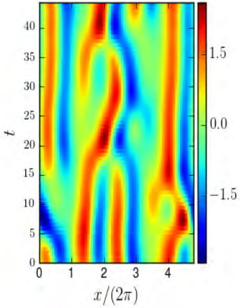
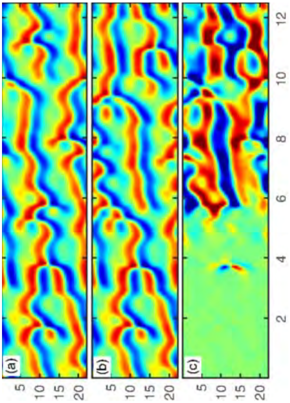




- each symbol indicates a minimal spatiotemporal tile
- glue them in all admissible ways

machine learning will be needed

"reservoir computing" example[12]



(a) data: Kuramoto-Sivashinsky simulation

(b) reservoir computing prediction

(c) two subtracted agree to ~ 5 Lyapunov times

Q : how would you learn this data?

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[12] J. Pathak et al., Phys. Rev. Lett. **120**, 024102 (2018).

take home : clouds do not integrate PDEs

do clouds integrate Navier-Stokes equations?

NO!

$\Longrightarrow$ other swirls $\Longrightarrow$

at any spacetime point Navier-Stokes equations describe the local tangent space

they satisfy them locally, everywhere and at all times

summary

1. study turbulence in infinite spatiatemporal domains
2. theory : classify all spatiotemporal tilings
3. numerics : future is spatiotemporal

there is no more time

there is only enumeration of spacetime solutions

spatiotemporally infinite spatiotemporal cat

a theory of turbulence ?

1. (semi-)classical field theories
2. state space
3. symbolic dynamics

Dreams of Grand Schemes : solve

Navier- Stokes

$$\rho \frac{\partial u_i}{\partial t} + \rho u_j \frac{\partial u_i}{\partial x_j} = \rho X_i - \frac{\partial p}{\partial x_i} + \mu \nabla^2 u_i$$

Einstein

$$R_{ik} - \frac{1}{2} g_{ik} R = \frac{8\pi k}{c^4} T_{ik}$$

$$R^i_{\ klm} = \frac{\partial \Gamma^i_{km}}{\partial x^l} - \frac{\partial \Gamma^i_{kl}}{\partial x^m} + \Gamma^i_{nl}\Gamma^n_{km} - \Gamma^i_{nm}\Gamma^n_{kl}$$

Yang- Mills

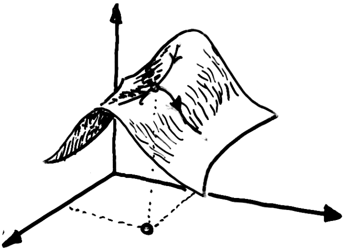
$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F_a^{\mu\nu}$$

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\, c_{abc} A^b_\mu A^c_\nu$$

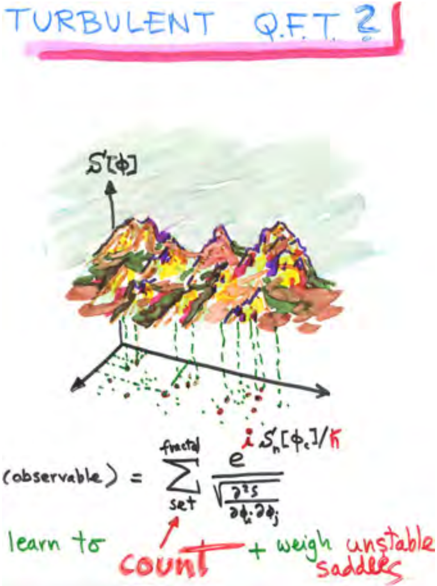
QUANTUM?

QFT path integrals : semi-classical quantization

a local unstable extremum



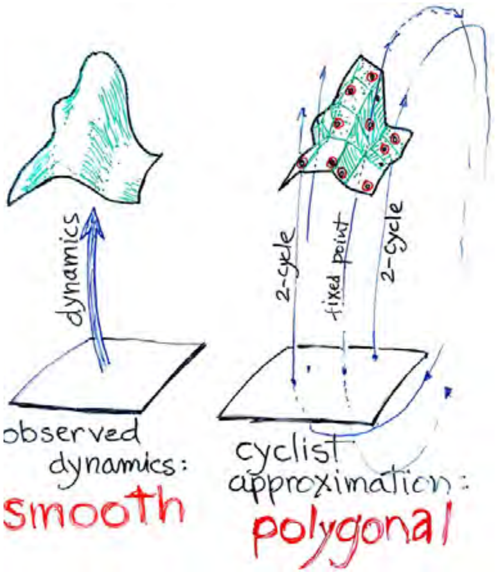
a fractal set of saddles



the very short answer : POT

if you win : I teach you how

(for details, see ChaosBook.org)



tessellate the state space by recurrent flows

classical trace formula for continuous time flows

$$\sum_{\alpha=0}^{\infty} \frac{1}{s - s_\alpha} = \sum_p T_p \sum_{r=1}^{\infty} \frac{e^{r(\beta A_p - s T_p)}}{\left|\det\left(\mathbf{1} - M_p^r\right)\right|}$$

relates the spectrum of the evolution operator

$$\mathcal{L}(x', x) = \delta\big(x' - f^t(x)\big) \; e^{\beta A(x,t)}$$

to the unstable periodic orbits p of the flow $f^t(x)$.

classical trace formula for averaging over 2-tori

something like

$$\sum_{\alpha=0}^{\infty} \frac{1}{s - s_\alpha} = \sum_p V_p \sum_{r=1}^{\infty} \frac{e^{r(\beta A_p - s V_p)}}{|\det \mathcal{J}_{p^r}|}$$

weighs the unstable relative prime (all symmetries quotiented) d-torus p by the inverse of its Hill determinant, the determinant (state space volume) of its orbit Jacobian matrix $\mathcal{J}_p$

$$\det \mathcal{J}_p$$

and V_p is the volume

$$V_p = T_p L_p$$

of the prime spacetime tile p

extras

speculation : code discrete Lagrangian methods?

the idea : construct a discrete counterpart to the considered system

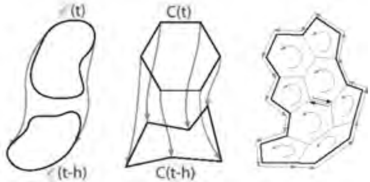
variational integrator : evolution map that corresponds to the discrete Euler–Lagrange equations

Discrete Lagrangian methods

action $S(q) = \int_0^T dt\, L(q, \dot{q})$ + Hamilton's principle $\delta S(q) = 0$

$$\text{discretize} \int_{t_k}^{t_{k+1}} L(q, \dot{q})\, dt \approx \Delta t L(q_k, q_{k+1}) \,.$$

symplectic methods preserve phase-space areas[13]



(left) Kelvin's circulation advected by the flow is constant
(middle) the discrete version, on a Voronoi loop
(right) circulation is constant on any discrete loop.

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[13] J. E. Marsden and M. West, Acta Numerica **10**, 357–514 (2001).

Discrete Lagrangian codes ?

so far, no codes for
discretized spatiotemporal action / Lagrangian density

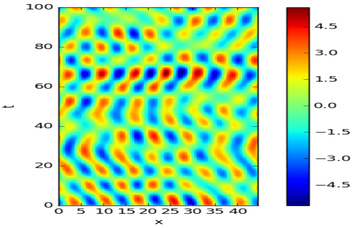
$$S = \int dq^d \, \mathcal{L}(q)$$

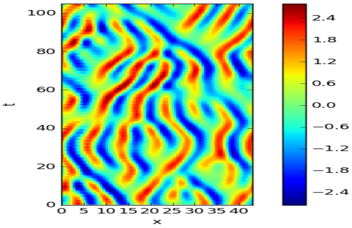
symplectic Euler incompressible fluid dynamics time-evolution codes exist[14]
claim : can apply to non-conservative system

Navier-Stokes?

[14] D. Pavlov et al., Physica D **240**, 443–458 (2011).

an intermediate spacetime domain



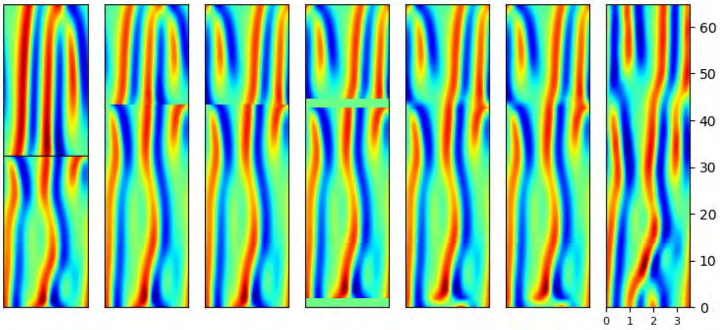


(left) $\bar{L} = 2\pi\sqrt{2}$ modulated initial random guess
$(L_0, T_0) = (5\bar{L}, 100) = (44.4, 100)$

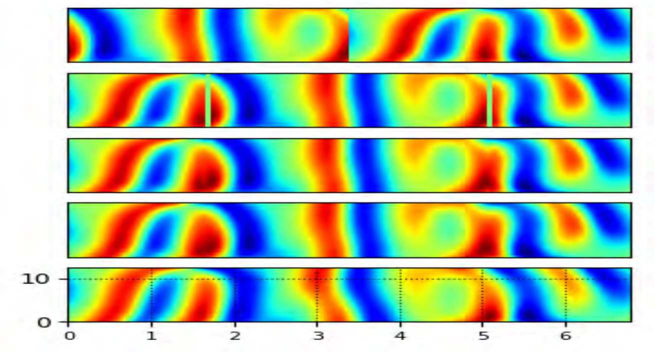
(right) Resulting invariant 2-torus
$(L_f, T_f) = (43.066, 105.08) = (L_0 - 1.363, T_0 + 5.08)$

Adjoint descent took only 7 laptop CPU seconds

temporally glued Frankenstein

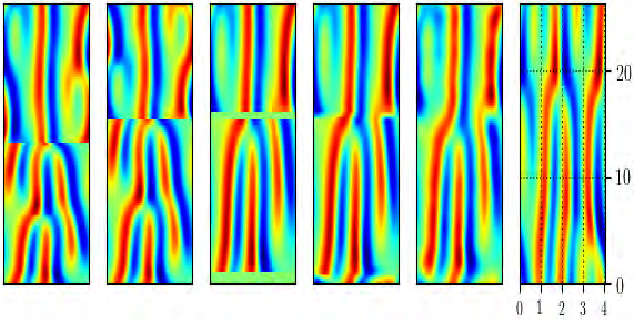


spatial gluing of two invariant 2-tori



1) two invariant 2-tori side by side
2) initial invariant 2-tori split into smaller tiles
3) a guess invariant 2-torus obtained by gluing / smoothing
4) converges to a larger invariant 2-torus

temporal gluing of two invariant 2-tori



1) an invariant 2-torus atop another invariant 2-torus
2) initial invariant 2-tori split into smaller tiles
3) a guess invariant 2-torus obtained by gluing / smoothing
4) converges to a larger invariant 2-torus

KS invariant 2-tori found by rocket science

[15]

the initial guess

the converged solution $u(x, t)$

[15] Q. Wang et al., Phys. Fluids **25**, 110818 (2013).