herding cats
a chaotic field theory

Predrag Cvitanović and Han Liang

ITS Symposium on Chaos and Quantum Field Theory

September 25, 2020

what is this? some background

this talk is an introduction to the

spatiotemporal cat[1]

the simplest example of

spatiotemporal turbulence[2]

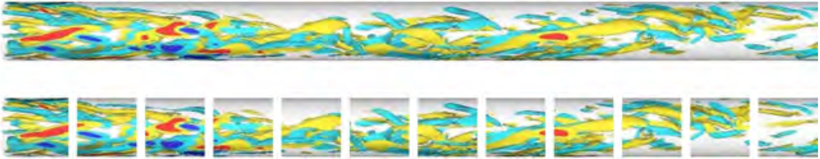
---

[1] P. Cvitanović and H. Liang, *Spatiotemporal cat: a chaotic field theory*, in preparation, 2020.

[2] M. N. Gudorf and P. Cvitanović, *Spatiotemporal tiling of the Kuramoto-Sivashinsky flow*, in preparation, 2020.

a motivation : need a theory of large fluid domains

pipe flow close to onset of turbulence [3]



we have a detailed theory of small turbulent fluid cells

can we can we construct the infinite pipe by coupling small turbulent cells ?

what would that theory look like ?

---

[3] M. Avila and B. Hof, Phys. Rev. **E 87** (2013)

build
a chaotic field theory

from
the simplest chaotic blocks

using

- time invariance
- space invariance

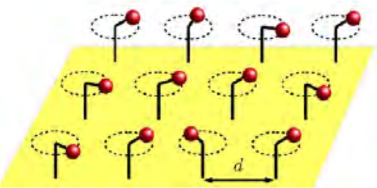
of the defining partial differential equations

take-home :

traditional field theory

Helmholtz

chaotic field theory



damped Poisson, Yukawa

Mephistopheles knocks at Faust's door and says, "Du mußt es dreimal sagen!"

. *"You have to say it three times"*

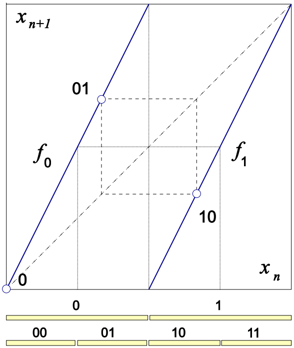
— Johann Wolfgang von Goethe

. *Faust I - Studierzimmer 2. Teil*

1. coin toss
2. temporal cat
3. spatiotemporal cat
4. bye bye, dynamics

fair coin toss (AKA Bernoulli map)

the essence of deterministic chaos



$$x_{t+1} = \begin{cases} f_0(x_t) = 2x_t \\ f_1(x_t) = 2x_t \pmod 1 \end{cases}$$

$\Rightarrow$ fixed point $\overline{0}$, 2-cycle $\overline{01}$, $\cdots$

a coin toss

the simplest example of deterministic chaos

what is (mod 1) ?

map with integer-valued 'stretching' parameter $s \geq 2$:

$$x_{t+1} = s\, x_t$$

(mod 1) : subtract the integer part $m_{t+1} = \lfloor s x_t \rfloor$
so fractional part ϕ_{t+1} stays in the unit interval $[0, 1)$

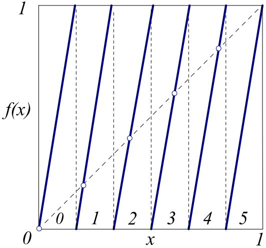
$$\phi_{t+1} = s\phi_t - m_{t+1}\,, \qquad \phi_t \in \mathcal{M}_{m_t}$$

m_t takes values in the s-letter alphabet

$$m \in \mathcal{A} = \{0, 1, 2, \cdots, s-1\}$$

a fair dice throw

slope 6 Bernoulli map



$$\phi_{t+1} = 6\phi_t - m_{t+1}\,,\ \phi_t \in \mathcal{M}_{m_t}$$

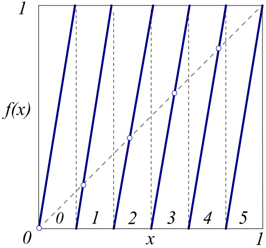
6-letter alphabet
$m_t \in \mathcal{A} = \{0, 1, 2, \cdots, 5\}$

6 subintervals $\{\mathcal{M}_{m_1}\}$

what is chaos ?

a fair dice throw

6 subintervals $\{\mathcal{M}_{m_1}\}$, 6^2 subintervals $\{\mathcal{M}_{m_1 m_2}\}, \cdots$



each subinterval contains a periodic point, labeled by $M = m_1 m_2 \cdots m_n$

$N_n = 6^n$ unstable orbits

definition : chaos is

positive Lyapunov ($\ln s$) - positive entropy ($\frac{1}{n} \ln N_n$)

definition : chaos is

positive Lyapunov ($\ln s$) - positive entropy ($\frac{1}{n} \ln N_n$)

the precise sense in which dice throw
is an example of deterministic chaos

lattice Bernoulli

now recast the time-evolution Bernoulli map

$$\phi_{t+1} = s\phi_t - m_{t+1}$$

as 1-step difference equation on the temporal lattice

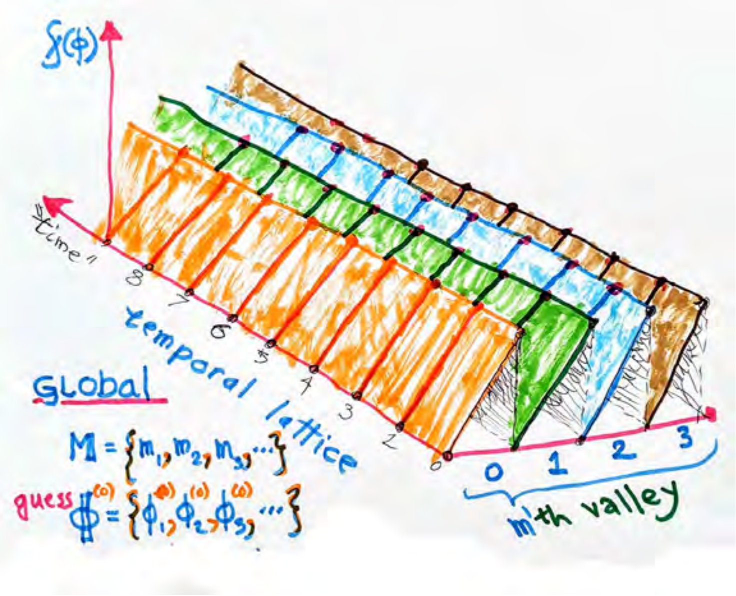
$$\phi_t - s\phi_{t-1} = -m_t \,, \qquad \phi_t \in [0,1)$$

field ϕ_t, source m_t
on each site t of a 1-dimensional lattice $t \in \mathbb{Z}$

write an n-sites lattice segment as
the lattice state and the symbol block

$$\Phi = (\phi_{t+1}, \cdots, \phi_{t+n}) \,, \quad \mathsf{M} = (m_{t+1}, \cdots, m_{t+n})$$

exponentially many distinct walks through Bernoulliland



think globally, act locally

Bernoulli equation at every instant t, local in time

$$\phi_t - s\phi_{t-1} = -m_t$$

is enforced by the global equation

$$\left(1 - s\sigma^{-1}\right) \Phi = -\mathrm{M}\,,$$

where the $[n \times n]$ matrix

$$\sigma_{jk} = \delta_{j+1,k}\,, \qquad \sigma = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & & \ddots & \\ & & & 0 & 1 \\ 1 & & & & 0 \end{pmatrix}$$

implements the 1-time step operation

think globally, act locally

solving the lattice Bernoulli equation

$$\mathcal{J}\Phi = -\mathsf{M}\,,$$

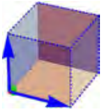
with the $[n\times n]$ matrix $\qquad \mathcal{J} = 1 - s\sigma^{-1}\,,$

can be viewed as a search for zeros of the function

$$F[\Phi] = \mathcal{J}\Phi + \mathsf{M} = 0$$

the entire global lattice state Φ_{M} is now

a single fixed point $(\phi_1, \phi_2, \cdots, \phi_n)$



in the n-dimensional unit hyper-cube $\qquad \Phi \in [0,1)^n$

orbit Jacobian matrix

solving a nonlinear $F[\Phi] = 0$ fixed point condition with Newton method requires evaluation of the $[n \times n]$ orbit Jacobian matrix

$$\mathcal{J}_{ij} = \frac{\delta F[\Phi]_i}{\delta \phi_j}$$

what does this global orbit Jacobian matrix do?

1. fundamental fact !
2. global stability of lattice state Φ, perturbed everywhere

(1) fundamental fact

to satisfy the fixed point condition

$$\mathcal{J}\Phi + \mathsf{M} = 0$$

the orbit Jacobian matrix $\mathcal{J}$

1. stretches the unit hyper-cube $\Phi \in [0,1)^n$ into the n-dimensional fundamental parallelepiped
2. maps each periodic point Φ_{M} into an integer lattice $\mathbb{Z}^n$ point
3. then translate by integers M into the origin

hence N_n, the total number of solutions = the number of integer lattice points within the fundamental parallelepiped

the fundamental fact[4] : Hill determinant counts solutions

$$N_n = |\mathrm{Det}\,\mathcal{J}|$$

integer points in fundamental parallelepiped = its volume

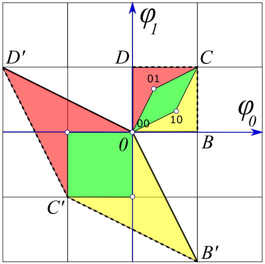
[4] M. Baake et al., J. Phys. A **30**, 3029–3056 (1997).

example : fundamental parallelepiped for $n = 2$

orbit Jacobian matrix, unit square basis vectors, their images :

$$\mathcal{J} = \begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix}; \quad \Phi_B = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \to \Phi_{B'} = \mathcal{J}\,\Phi_B = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \cdots ,$$

Bernoulli periodic points of period 2



$N_2 = 3$

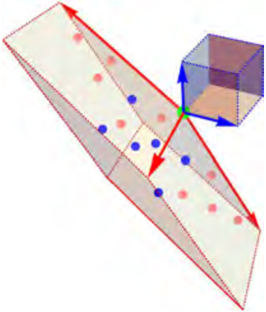
fixed point Φ_{00}
2-cycle Φ_{01}, Φ_{10}

square $[0BCD] \Rightarrow \mathcal{J} \Rightarrow$ fundamental parallelepiped $[0B'C'D']$

fundamental fact for any n

an $n = 3$ example

$\mathcal{J}$ [unit hyper-cube] = [fundamental parallelepiped]



unit hyper-cube $\Phi \in [0, 1)^n$

$n > 3$ cannot visualize

a periodic point $\rightarrow$ integer lattice point, $\bullet$ on a face, $\bullet$ in the interior

(2) orbit stability vs. temporal stability

orbit Jacobian matrix

$\mathcal{J}_{ij} = \frac{\delta F[\Phi]_i}{\delta \phi_j}$ stability under global perturbation of the whole orbit

for n large, a huge $[dn \times dn]$ matrix

temporal Jacobian matrix

J propagates initial perturbation n time steps

small $[d \times d]$ matrix

J and $\mathcal{J}$ are related by[5]

Hill's (1886) remarkable formula

$$|\mathrm{Det}\,\mathcal{J}_\mathrm{M}| = |\det(\mathbf{1} - J_\mathrm{M})|$$

$\mathcal{J}$ is huge, even ∞-dimensional matrix
J is tiny, few degrees of freedom matrix

[5] G. W. Hill, Acta Math. **8**, 1–36 (1886).

how come Hill determinant $\mathrm{Det}\,\mathcal{J}$ counts periodic points ?

in 1984 Ozorio de Almeida and Hannay[6] related the number of periodic points to a Jacobian matrix by their

principle of uniformity

"periodic points of an ergodic system, counted with their natural weighting, are uniformly dense in phase space"

where

'natural weight' of periodic orbit M

$$\frac{1}{|\det(1-J_M)|}$$

[6] A. M. Ozorio de Almeida and J. H. Hannay, J. Phys. A **17**, 3429 (1984).

how come a $\mathrm{Det}\,\mathcal{J}$ counts periodic points ?

"principle of uniformity" is in[7]

periodic orbit theory

known as the flow conservation sum rule :

$$\sum_{M} \frac{1}{|\det(1 - J_M)|} = \sum_{M} \frac{1}{|\mathrm{Det}\,\mathcal{J}_M|} = 1$$

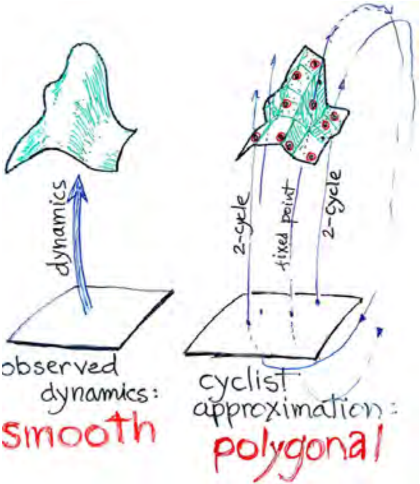
sum over periodic points Φ_M of period n

state space is divided into

neighborhoods of periodic points of period n

[7] P. Cvitanović, "Why cycle?", in *Chaos: Classical and Quantum*, edited by P. Cvitanović et al. (Niels Bohr Inst., Copenhagen, 2020).

tile the ergodic state space by recurrent neighborhoods



a fixed point

a 2-cycle, etc.

smooth dynamics (left frame)
tesselated by the skeleton of recurrent flows,
together with (right frame) their linearized neighborhoods

periodic orbit theory

how come a $\mathrm{Det}\,\mathcal{J}$ counts periodic points ?

flow conservation sum rule :

$$\sum_{\phi_i \in \mathrm{Fix} f^n} \frac{1}{|\mathrm{Det}\,\mathcal{J}_i|} = 1$$

Bernoulli system 'natural weighting' is simple :

the determinant $\mathrm{Det}\,\mathcal{J}_i = \mathrm{Det}\,\mathcal{J}$ the same for all periodic points, whose number thus verifies the fundamental fact

$$N_n = |\mathrm{Det}\,\mathcal{J}|$$

the number of Bernoulli periodic lattice states

$N_n = |\mathrm{Det}\,\mathcal{J}| = s^n - 1$ for any n

periodic orbit theory

how does 1-time step transition matrix T count periodic lattice states ? For any matrix $\ln \det = \mathrm{tr} \ln$, so

$\ln \det (1 - zT) = \mathrm{tr} \ln(1 - zT) =$ **sum over loops**

$$\det (1 - zT) = \exp \left(- \sum_{n=1} \frac{z^n}{n} \mathrm{tr}\, T^n \right)$$

AKA

'topological zeta function'

$$1/\zeta_{\mathrm{top}}(z) = \exp \left(- \sum_{n=1}^{\infty} \frac{z^n}{n} N_n \right)$$

1. weight $1/n$ as by (cyclic) translation invariance, n lattice states are equivalent
2. zeta function counts prime orbits, one per each set of equivalent lattice states

topological zeta function

counts prime orbits, one per each set of Bernoulli periodic states $N_n = s^n - 1$

$$1/\zeta_{\text{top}}(z) \,=\, \exp\left(-\sum_{n=1}^{\infty}\frac{z^n}{n}N_n\right) \,=\, \frac{1-sz}{1-z}$$

numerator $(1-sz)$ says that Bernoulli orbits are built from s fundamental primitive lattice states,

the fixed points $\{\phi_0, \phi_1, \cdots, \phi_{s-1}\}$

every other lattice state is built from their concatenations and repeats.

solved!

this is ‘periodic orbit theory’

And if you don’t know, now you know

think globally, act locally - summary

the problem of enumerating and determining all global solutions stripped to its essentials :

1. each solution is a zero of the global fixed point condition

$$F[\Phi] = 0$$

2. global stability : the orbit Jacobian matrix

$$\mathcal{J}_{ij} = \frac{\delta F[\Phi]_i}{\delta \phi_j}$$

3. fundamental fact : the number of period-n orbits

$$N_n = |\mathrm{Det}\,\mathcal{J}|$$

4. zeta function $1/\zeta_{\mathrm{top}}(z)$: all predictions of the theory

coin toss ? that's not physics !

a field theory should be Hamiltonian and energy conserving,
and Quantum Mechanics requires it

because that is physics !

need a system as simple as the Bernoulli, but mechanical

so, we move on from running in circles,

to a mechanical rotor to kick.

Du mußt es dreimal sagen!
— Mephistopheles

1. coin toss
2. kicked rotor
3. spatiotemporal cat
4. bye bye, dynamics

field theory in 1 spacetime dimension

we now define

the cat map in 1 spacetime dimension

then we generalize to

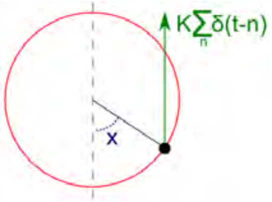
d-dimensional spatiotemporal cat

- cat map in Hamiltonian formulation
- cat map in Lagrangian formulation
  (so much more elegant!)

Hamiltonian formulation

example of a “small domain” dynamics : a single kicked rotor

an electron circling an atom, subject to
a discrete time sequence of angle-dependent kicks $F(x_t)$



Taylor, Chirikov and Greene standard map

$$\begin{aligned} x_{t+1} &= x_t + p_{t+1} \qquad \mod 1, \\ p_{t+1} &= p_t + F(x_t) \end{aligned}$$

$\rightarrow$ chaos in Hamiltonian systems

the simplest example : a cat map evolving in time

force $F(x) = Kx$ linear in the displacement x , $K \in \mathbb{Z}$

$$
\begin{aligned}
x_{t+1} &= x_t + p_{t+1} && \mod 1 \\
p_{t+1} &= p_t + Kx_t && \mod 1
\end{aligned}
$$

Continuous Automorphism of the Torus, or

Hamiltonian cat map

a linear, area preserving map of a 2-torus onto itself

$$
\begin{pmatrix} \phi_t \\ \phi_{t+1} \end{pmatrix} = J \begin{pmatrix} \phi_{t-1} \\ \phi_t \end{pmatrix} - \begin{pmatrix} 0 \\ m_t \end{pmatrix}, \qquad J = \begin{pmatrix} 0 & 1 \\ -1 & s \end{pmatrix}
$$

for integer "stretching" $s = \mathrm{tr}\, J > 2$ the map is
beloved by ergodicists :
hyperbolic $\rightarrow$ perfect chaotic Hamiltonian dynamical system

a cat is literally Hooke's wild, 'anti-harmonic' sister

for $s < 2$ Hooke rules

local restoring oscillations
around the sleepy z-z-z-zzz resting state

for $s > 2$ cats rule

exponential runaway
wrapped global around a phase space torus

cat is to chaos what harmonic oscillator is to order

there is no more fundamental example of chaos in mechanics

Lagrangian formulation

cat map in Lagrangian form

replace momentum by velocity

$$p_{t+1} = (\phi_{t+1} - \phi_t)/\Delta t$$

formulation on (ϕ_t, ϕ_{t-1}) temporal lattice is particularly pretty[8]

2-step difference equation

$$\phi_{t+1} - s\,\phi_t + \phi_{t-1} = -m_t$$

integer m_t ensures that

ϕ_t lands in the unit interval

$$m_t \in \mathcal{A}\,, \quad \mathcal{A} = \{\text{finite alphabet}\}$$

[8] I. Percival and F. Vivaldi, Physica D **27**, 373–386 (1987).

think globally, act locally

temporal cat at every instant t, local in time

$$\phi_{t+1} - s\,\phi_t + \phi_{t-1} = -m_t$$

is enforced by the global equation

$$\mathcal{J}\,\Phi = -\mathsf{M}\,,$$

where

orbit Jacobian matrix

$$\mathcal{J}\,\Phi + \mathsf{M} = 0$$

where

$$\Phi = (\phi_{t+1}, \cdots, \phi_{t+n}) \,, \quad \mathsf{M} = (m_{t+1}, \cdots, m_{t+n})$$

are a lattice state, and a symbol block

and $[n \times n]$ orbit Jacobian matrix $\mathcal{J}$ is

$$\sigma - s\mathbb{1} + \sigma^{-1} = \begin{pmatrix} -s & 1 & & & 1 \\ 1 & -s & 1 & & \\ & 1 & & \ddots & \\ & & & -s & 1 \\ 1 & & & & -s \end{pmatrix}$$

think globally, act locally

solving the temporal cat equation

$$\mathcal{J}\Phi = -\mathsf{M}\,,$$

with the $[n \times n]$ matrix $\quad \mathcal{J} = \sigma - s\mathbb{1} + \sigma^{-1}$

can be viewed as a search for zeros of the function

$$F[\Phi] = \mathcal{J}\Phi + \mathsf{M} = 0$$

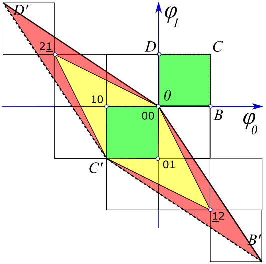
where the entire global lattice state Φ_{M} is

a single fixed point $\Phi_{\mathsf{M}} = (\phi_1, \phi_2, \cdots, \phi_n)$
in the n-dimensional unit hyper-cube $\Phi \in [0,1)^n$

fundamental fact in action

temporal cat fundamental parallelepiped for period 2

square $[0BCD] \Rightarrow \mathcal{J} = $ fundamental parallelepiped $[0B'C'D']$



$N_2 = |\mathrm{Det}\,\mathcal{J}| = 5$

fundamental parallelepiped
$=$ 5 unit area quadrilaterals

a periodic point per each unit volume

temporal cat zeta function

is the generating function that counts orbits

substituting the Hill determinant count of periodic lattice states

$$N_n = |\mathrm{Det}\,\mathcal{J}|$$

into the topological zeta function

$$1/\zeta_{\mathrm{top}}(z) = \exp\left(-\sum_{n=1} \frac{z^n}{n} N_n\right)$$

leads to the elegant explicit formula[9]

$$1/\zeta_{\mathrm{top}}(z) = \frac{1 - sz + z^2}{(1-z)^2}$$

solved!

[9] S. Isola, Europhys. Lett. **11**, 517–522 (1990).

what continuum theory is temporal cat discretization of?

have

2-step difference equation

$$\phi_{t+1} - s\,\phi_t + \phi_{t-1} = -m_t$$

discrete lattice

Laplacian in 1 dimension

$$\phi_{t+1} - 2\phi_t + \phi_{t-1} = \Box\,\phi_t$$

so temporal cat is an (anti)oscillator chain, known as

$d = 1$ damped Poisson (or Yukawa) equation (!)

$$(\Box - s + 2)\,\phi_t = -m_t$$

did you know that a cat map can be so cool?

a reminder slide, to skip : Helmholtz equation in continuum

inhomogeneous Helmoltz equation

is an elliptical equation of form

$$(\Box + k^2)\,\phi(x) = -m(x)\,, \qquad x \in \mathbb{R}^d$$

where $\phi(x)$ is a C^2 function, and $m(x)$ is a function with compact support

for the $\lambda^2 = -k^2 > 0$ (imaginary k), the equation is known as the screened Poisson equation[10], or the Yukawa equation

[10] A. L. Fetter and J. D. Walecka, *Theoretical Mechanics of Particles and Continua*, (Dover, New York, 2003).

that's it! for spacetime of 1 dimension

lattice damped Poisson equation

$$(\Box - s + 2)\phi_z = -m_z$$

solved completely and analytically!

think globally, act locally - summary

the problem of determining all global solutions stripped to its bare essentials :

1. each solution a zero of the global fixed point condition

$$F[\Phi] = 0$$

2. compute the orbit Jacobian matrix

$$\mathcal{J}_{ij} = \frac{\delta F[\Phi]_i}{\delta \phi_j}$$

3. fundamental fact $\quad N_n = |\mathrm{Det}\,\mathcal{J}| =$ period-n states

4. $\Rightarrow$ zeta function $1/\zeta_{\text{top}}(z)$

Du mußt es dreimal sagen!
— Mephistopheles

1. coin toss
2. kicked rotor
3. **spatiotemporal cat**
4. bye bye, dynamics

spatiotemporally infinite ‘spatiotemporal cat’

herding cats in d spacetime dimensions

start with

a cat at each lattice site

talk to neighbors

spacetime d-dimensional   spatiotemporal cat

- Hamiltonian formulation is awkward, fuggedaboutit!
- Lagrangian formulation is elegant

spatiotemporal cat

consider a 1 spatial dimension lattice, with field ϕ_{nt}
(the angle of a kicked rotor "particle" at instant t, at site n)

require

- each site couples to its nearest neighbors $\phi_{n\pm1,t}$
- invariance under spatial translations
- invariance under spatial reflections
- invariance under the space-time exchange

Gutkin & Osipov[11] obtain

2-dimensional coupled cat map lattice

$$\phi_{n,t+1} + \phi_{n,t-1} - 2s\,\phi_{nt} + \phi_{n+1,t} + \phi_{n-1,t} = -m_{nt}$$

[11] B. Gutkin and V. Osipov, Nonlinearity **29**, 325–356 (2016).

spatiotemporal cat : a strong coupling field theory

symmetries : translational and time-reversal, spatial reflections

the key assumption

- invariance under the space-time exchange

eliminates traditional, spatially weakly coupled map lattice models[12]

- spatiotemporal cat is a Euclidean field theory

[12] L. A. Bunimovich and Y. G. Sinai, Nonlinearity **1**, 491 (1988).

herding cats : a discrete Euclidean space-time field theory

write the spatial-temporal differences as discrete derivatives

Laplacian in $d = 2$ dimensions

$$\Box\,\phi_{nt} \;=\; \phi_{n,t+1} + \phi_{n,t-1} - 4\,\phi_{nt} + \phi_{n+1,t} + \phi_{n-1,t}$$

subtract 2-dimensional coupled cat map lattice equation

$$-m_{nt} = \phi_{n,t+1} + \phi_{n,t-1} - 2s\,\phi_{nt} + \phi_{n+1,t} + \phi_{n-1,t}$$

cat herd is thus governed by the law of

***d*-dimensional spatiotemporal cat**

$$(\Box - d(s-2))\phi_z = -m_z$$

where $\phi_z \in [0,1)\,,\quad m_z \in \mathcal{A}$ and $z \in \mathbb{Z}^d$ = integer lattice

discretized linear PDE

d-dimensional spatiotemporal cat

$$(\Box - d(s-2))\,\phi_z = -m_z$$

is linear and known as

- Helmholtz equation if stretching is weak, $s < 2$
  [oscillatory sine, cosine solutions]
- damped Poisson equation if stretching is strong, $s > 2$
  [hyperbolic sinches, coshes, 'mass' $m^2 = d(s-2)$]

nonlinearity is hidden in the "sources"

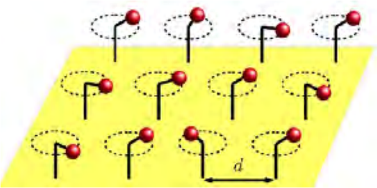
$$m_z \in \mathcal{A} \text{ at lattice site } z \in \mathbb{Z}^d$$

spring mattress vs field of rotors

traditional field theory

Helmholtz

chaotic field theory



damped Poisson

the simplest of all 'turbulent' field theories !

spatiotemporal cat

$$(\Box - d(s-2))\phi_z = -m_z$$

can be solved completely (?) and analytically (!)

assign to each site z a letter m_z from the alphabet $\mathcal{A}$.

a particular fixed set of letters m_z (a lattice state)

$$\mathsf{M} = \{m_z\} = \{m_{n_1 n_2 \cdots n_d}\}\,,$$

is a complete specification of the corresponding lattice state Φ

from now on work in $d=2$ dimensions, 'stretching parameter' $s=5/2$

think globally, act locally

solving the spatiotemporal cat equation

$$\mathcal{J}\Phi = -\mathsf{M}\,,$$

with the $[n\times n]$ matrix $\mathcal{J} = \sum_{j=1}^{2}\left(\sigma_j - s\mathbb{1} + \sigma_j^{-1}\right)$

can be viewed as a search for zeros of the function

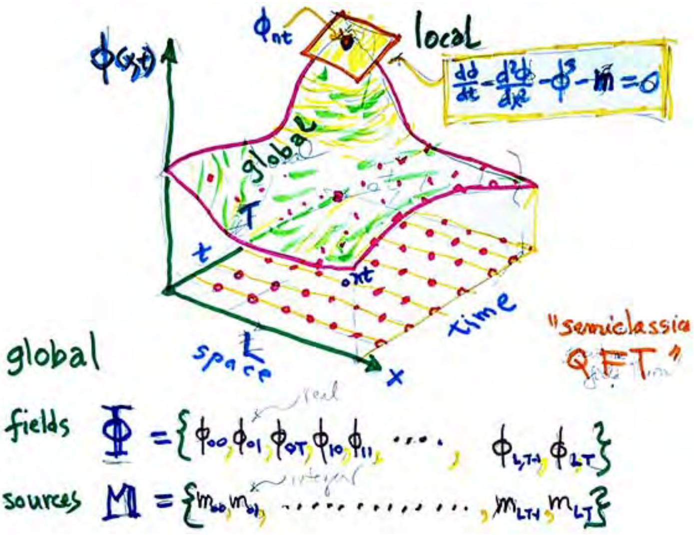
$$F[\Phi] = \mathcal{J}\Phi + \mathsf{M} = 0$$

where the entire global lattice state Φ_M is

a single fixed point $\Phi_\mathsf{M} = \{\phi_z\}$
in the LT-dimensional unit hyper-cube $\Phi \in [0,1)^{LT}$

L is the 'spatial', T the 'temporal' lattice period

think globally, act locally



for each symbol array M, a periodic lattice state Φ_M

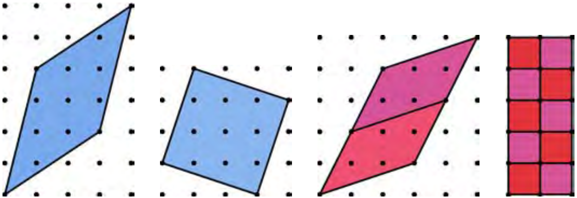
next, enumerate all periodic spacetime tilings of the integer lattice

each tile : 2-dimensional (sub)lattice, an infinite array of points

$$\Lambda = \{n_1\mathbf{a}_1 + n_2\mathbf{a}_2 \,|\, n_i \in \mathbb{Z}\}$$

with the defining tile spanned by a pair of basis vectors $\mathbf{a}_1, \mathbf{a}_2$

example : four tiles of area 10



The two blue tiles appear ‘prime’, i.e., not tiled by smaller tiles.
False! all four big tiles can tilled by smaller ones.

tricky!

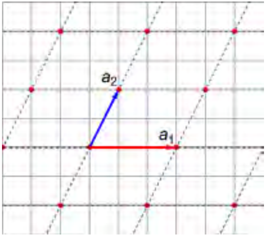
2-dimensional lattice tilings

2-dimensional *lattice* is defined by a $[2 \times 2]$ fundamental parallelepiped matrix whose columns are basis vectors

$$\mathbf{A} = [\mathbf{a}_1 \mathbf{a}_2] = \begin{bmatrix} L & S \\ 0 & T \end{bmatrix},$$

L, T : spatial, temporal lattice periods
'tilt' $0 \leq S < L$ imposes the *relative-periodic* (*'helical'*, *'toroidal'*, *'twisted'*, *'screw'*, $\cdots$) bc's

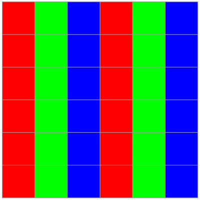
example : $[3 \times 2]_1$ tile



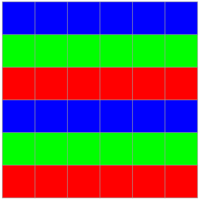
basis vectors

$$\mathbf{a}_1 = \begin{pmatrix} 3 \\ 0 \end{pmatrix}, \mathbf{a}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

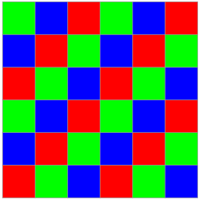
exponentially many periodic lattice states in Felinestan



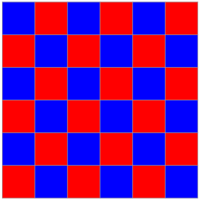
$[3\times1]_0$



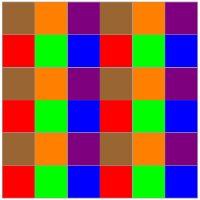
$[1\times3]_0$



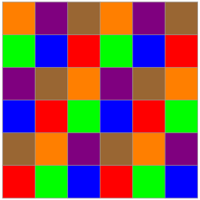
$[3\times1]_1$



$[2\times1]_1$



$[3\times2]_0$



$[3\times2]_1$

tile color = value of symbol m_z

note : spatiotemporal cat dances over a parquet floor

(so far) latticization of spacetime continuum :
field $\phi(x,t)$ over spacetime coordinates (x,t)
for *any* field theory

$$\Rightarrow$$

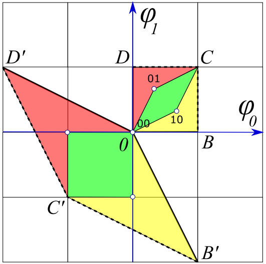
set of lattice site values $\phi_z = \phi(n\Delta L, t\Delta T)$.
Subscript $z = (n,t) \in \mathbb{Z}^d$ is a discrete d-dimensional spacetime *coordinate* over which the field ϕ lives

distinct spacetime tiles have tilted shapes $[L \times T]_S$

(next) spatiotemporal cat *field* ϕ_z is confined to $[0,1)$
That imparts a $\mathbb{Z}^1$ lattice structure on fundamental parallelepiped $\mathcal{J}$ basis vectors ; fundamental fact then counts all periodic lattice *states* Φ_M for a given spacetime tile $[L \times T]_S$

fundamental fact works over a spacetime lattice (!)

recall Bernoulli fundamental fact example ?



unit hyper-cube $\Phi \in [0,1)^2$

$$\Rightarrow \mathcal{J} \Rightarrow$$

fundamental parallelepiped

spacetime fundamental parallelepiped basis vectors $\Phi^{(j)}$
$=$ columns of the orbit Jacobian matrix

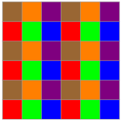
$$\mathcal{J} = (\Phi^{(1)}|\Phi^{(2)}|\cdots|\Phi^{(LT)})$$

example : spacetime periodic $[3\times 2]_0$ lattice state

$$F[\Phi] = \mathcal{J}\Phi + \mathsf{M} = 0$$

6 field values, on 6 lattice sites $z = (n, t)$, $[3\times 2]_0$ tile :

$$\Phi_{[3\times2]_0} = \left[\begin{array}{ccc} \phi_{01} & \phi_{11} & \phi_{21} \\ \phi_{00} & \phi_{10} & \phi_{20} \end{array}\right], \qquad 6\,\mathsf{M}_{[3\times2]_0} =$$



where the region of symbol plane shown is tiled by 6 repeats of the $\mathsf{M}_{[3\times2]_0}$ block, and tile color = value of symbol m_z

'stack up' vectors and matrices, vectors as 1-dimensional arrays,

$$\Phi_{[3\times2]_0} = \left(\begin{array}{c} \phi_{01} \\ \phi_{00} \\ \hline \phi_{11} \\ \phi_{10} \\ \hline \phi_{21} \\ \phi_{20} \end{array}\right), \qquad \mathsf{M}_{[3\times2]_0} = \left(\begin{array}{c} m_{01} \\ m_{00} \\ \hline m_{11} \\ m_{10} \\ \hline m_{21} \\ m_{20} \end{array}\right)$$

with the $[6\times 6]$ orbit Jacobian matrix in block-matrix form

$$
\mathcal{J}_{[3\times 2]_0} = \left(\begin{array}{cc|cc|cc}
-2s & 2 & 1 & 0 & 1 & 0 \\
2 & -2s & 0 & 1 & 0 & 1 \\
\hline
1 & 0 & -2s & 2 & 1 & 0 \\
0 & 1 & 2 & -2s & 0 & 1 \\
\hline
1 & 0 & 1 & 0 & -2s & 2 \\
0 & 1 & 0 & 1 & 2 & -2s
\end{array}\right)
$$

fundamental parallelepiped basis vectors $\Phi^{(j)}$ are the columns of the orbit Jacobian matrix

$$\mathcal{J}_{[3\times2]_0} = \left(\begin{array}{c|c|c|c|c|c} -2s & 2 & 1 & 0 & 1 & 0 \\ 2 & -2s & 0 & 1 & 0 & 1 \\ 1 & 0 & -2s & 2 & 1 & 0 \\ 0 & 1 & 2 & -2s & 0 & 1 \\ 1 & 0 & 1 & 0 & -2s & 2 \\ 0 & 1 & 0 & 1 & 2 & -2s \end{array}\right)$$

the 'fundamental fact' now yields the number of solutions for any half-integer s as Hill determinant

$$N_{[3\times2]_0} = |\mathrm{Det}\,\mathcal{J}_{[3\times2]_0}| = 4(s-2)s(2s-1)^2(2s+3)^2$$

can count spatiotemporal cat states for any $\Lambda = [L \times T]_S$

Λ	$N_\Lambda(s)$	$M_\Lambda(s)$
$[1\times1]_0$	$2(s-2)$	$2(s-2)$
$[2\times1]_0$	$2(s-2)2s$	$2(s-2)\frac{1}{2}(2s-1)$
$[2\times1]_1$	$2(s-2)2(s+2)$	$2(s-2)\frac{1}{2}(2s+3)$
$[3\times1]_0$	$2(s-2)(2s-1)^2$	$2(s-2)\frac{4}{3}(s-1)s$
$[3\times1]_1$	$2(s-2)4(s+1)^2$	$2(s-2)\frac{1}{3}(2s+1)(2s+3)$
$[4\times1]_0$	$2(s-2)8(s-1)^2s$	$2(s-2)\frac{1}{2}(2s-3)(2s-1)s$
$[4\times1]_1$	$2(s-2)8s^2(s+2)$	$2(s-2)\frac{1}{2}(s+2)(2s-1)(2s+1)$
$[4\times1]_2$	$2(s-2)8(s+1)^2s$	$2(s-2)\frac{1}{2}(2s+3)(2s+1)s$
$[4\times1]_3$	$2(s-2)8s^2(s+2)$	$2(s-2)\frac{1}{2}(s+2)(2s-1)(2s+1)$
$[5\times1]_0$	$2(s-2)\left(4s^2-6s+1\right)^2$	$2(s-2)\frac{4}{5}(s-1)(2s-3)(2s-1)s$
$[5\times1]_1$	$2(s-2)16\left(s^2+s-1\right)^2$	$2(s-2)\frac{1}{5}(2s-1)(2s+3)(4s^2+4s-5)$
$[2\times2]_0$	$2(s-2)8s^2(s+2)$	$2(s-2)\frac{1}{2}(2s-1)(2s^2+5s+1)$
$[2\times2]_1$	$2(s-2)8s(s+1)^2$	$2(s-2)\frac{1}{2}(2s+1)(2s+3)s$
$[3\times2]_0$	$2(s-2)2s(2s-1)^2(2s+3)^2$	$2(s-2)\frac{2}{3}(2s-1)(4s^3+10s^2+3s-5)s$
$[3\times2]_1$	$2(s-2)32s^3(s+1)^2$	$2(s-2)\frac{1}{6}(2s-1)(2s+1)(8s^3+16s^2+10s+$
$[3\times3]_0$	$2(s-2)16(s+1)^4(2s-1)^4$	

we can count !

1. can construct all spacetime tilings, from small tiles to as large as you wish
2. for each spacetime tile $[L \times T]_S$, can evaluate # of doubly-periodic lattice states for a tile

$$N_{[L \times T]_S}$$

3. # of prime orbits for a tile

$$M_{[L \times T]_S}$$

zeta function for a field theory ???

'periodic orbits' are now invariant 2-tori (tiles)

each a spacetime lattice tile p of area $A_p = L_p T_p$
that cover the phase space with 'natural weight'

$$\sum_p \frac{e^{-A_p s}}{|\mathrm{Det}\,\mathcal{J}_p|}$$

at this time :

- $d = 1$ cat map zeta function works like charm
- $d = 2$ spatiotemporal cat works
- $d \geq 2$ Navier-Stokes zeta is still but a dream

spatiotemporal cat topological zeta function

know how to evaluate the number of doubly-periodic lattice states

$$N_{[L\times T]_S}\,,$$

for a given $[L\times T]_S$ finite spacetime tile

now substitute these numbers of lattice states into the topological zeta function

$$1/\zeta_{\text{top}}(z_1,z_2) \;=\; 1-\frac{2\,(s-2)}{z_1+z_2-4+z_1^{-1}+z_2^{-1}}$$

??

but that's just a guess - we currently have no generating function that presents all solutions in a compact form

funky... not solved :(

2.15 Integer lattices literature

There are many reasons why one needs to compute an "orbit Jacobian matrix" Hill determinant $|\mathrm{Det}\,\mathcal{J}|$, in fields ranging from number theory to engineering, and many methods to accomplish that:

discretizations of Helmholtz [58] and screened Poisson [59, 80, 96, 97] (also known as Klein–Gordon or Yukawa) equations

Green's functions on integer lattices [5, 8, 24, 33, 37, 40, 63, 67, 78, 92, 93, 115–117, 135, 140, 143, 149, 150, 159, 180, 196]

Gaussian model [71, 111, 139, 172]

linearized Hartree-Fock equation on finite lattices [121]

quasilattices [29, 69]

circulant tensor systems [33, 37, 146, 164, 166, 200]

Ising model [19, 88, 89, 98, 100, 103–105, 128, 136, 141, 153, 161, 199], transfer matrices [154, 199]

lattice field theory [108, 144, 148, 151, 168, 175, 176, 192]

modular transformations [34, 205]

lattice string theory [77, 157]

random walks, resistor networks [9, 25, 49, 50, 60, 81, 86, 99, 122, 163, 183, 188, 198]

spatiotemporal stability in coupled map lattices [4, 75, 203]

Van Vleck determinant, Laplace operator spectrum, semiclassical Gaussian path integrals [47, 125, 126, 187]

Hill determinant [26, 47, 137]; discrete Hill's formula and the Hill discriminant [186]

Lindstedt-Poincaré technique [189–191]

heat kernel [38, 61, 64, 110, 114, 143, 159, 201]

lattice points enumeration [15, 16, 20, 56]

primitive parallelogram [10, 30, 152, 193]

difference equations [55, 68, 181]

digital signal processing [62, 130, 197]

generating functions, Z-transforms [64, 194]

integer-point transform [20]

graph Laplacians [41, 79, 134, 162]

graph zeta functions [7, 13, 18, 27, 42–44, 57, 61, 83, 87, 94, 101, 123, 124, 162, 165, 169, 171, 179, 184, 185, 204]

zeta functions for multi-dimensional shifts [12, 132, 133, 147]

zeta functions on discrete tori [38, 39, 201]

but, is this

chaos?

yes, short tiles are exponentially good ‘shadows’ of the larger ones, so can attain any desired accuracy

is spatiotemporal cat ‘chaotic’?

in time-evolving deterministic chaos any chaotic trajectory is shadowed by shorter periodic orbits

in spatiotemporal chaos, any unstable lattice state is shadowed by smaller invariant 2-tori (Gutkin *et al.*[13,14])

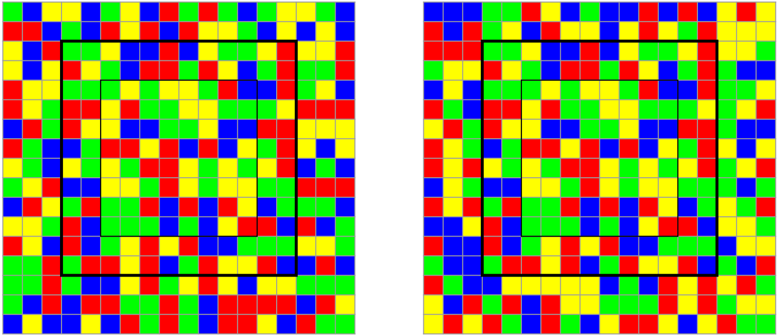
next figure : code the M symbol block ϕ_{nt} at the lattice site *nt* with (color) alphabet

$$m_{t\ell} \in \mathcal{A} = \{\underline{1}, 0, 1, 2, \cdots\} = \{red, green, blue, yellow, \cdots\}$$

[13] B. Gutkin and V. Osipov, Nonlinearity **29**, 325–356 (2016).

[14] B. Gutkin et al., *Linear encoding of the spatiotemporal cat map*, 2019.

shadowing, symbolic dynamics space

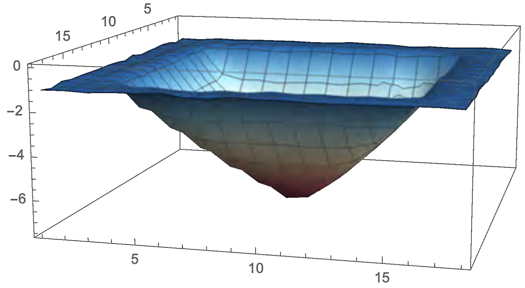


2d symbolic representation M_j of two lattice states Φ_j shadowing each other within the shared block $\mathsf{M}_{\mathcal{R}}$

- border $\mathcal{R}$ (thick black)
- symbols outside $\mathcal{R}$ differ

$s = 7/2$ Adrien Saremi 2017

shadowing



the logarithm of the average of the absolute value of site-wise distance

$$\ln |\phi_{2,z} - \phi_{1,z}|$$

averaged over 250 solution pairs

note the exponential falloff of the distance away from the center of the shared block $\mathcal{R}$

$\Rightarrow$ within the interior of the shared block,

shadowing is exponentially close

1. coin toss
2. kicked rotor
3. spatiotemporal cat
4. bye bye, dynamics

summary

spatiotemporal cat

insight 1 : how is turbulence described?

not by the evolution of an initial state

exponentially unstable system have finite (Lyapunov) time and space prediction horizons

but

by enumeration of admissible field configurations

and their natural weights

insight 2 : symbolic dynamics for turbulent flows

applies to all PDEs with d translational symmetries

a d-dimensional spatiotemporal field configuration

$$\{\phi_z\} = \{\phi_z, z \in \mathbb{Z}^d\}$$

is labelled by a d-dimensional spatiotemporal block of symbols

$$\{m_z\} = \{m_z, z \in \mathbb{Z}^d\},$$

rather than a single temporal symbol sequence

(as is done when describing a small coupled few-“body” system, or a small computational domain).

insight 3 : description of turbulence by invariant 2-tori

1 time, 0 space dimensions

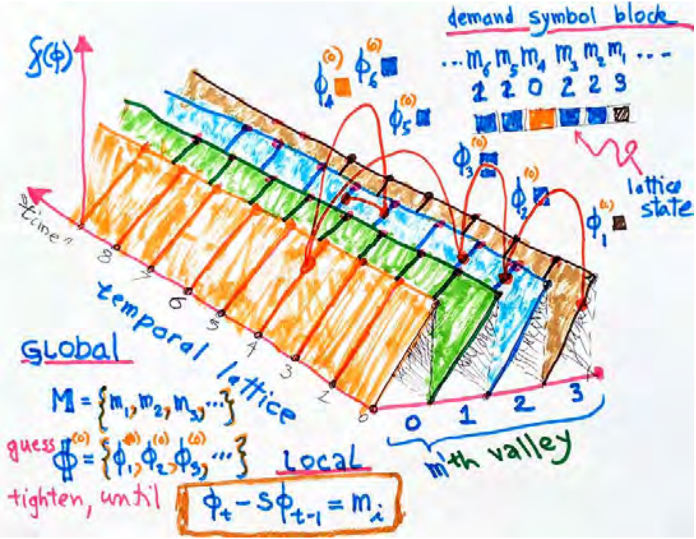
a phase space point is *periodic* if its orbit returns to itself after a finite time T; such orbit tiles the time axis by infinitely many repeats

1 time, d-1 space dimensions

a phase space point is *spatiotemporally periodic* if it belongs to an invariant d-torus $\mathcal{R}$,
i.e., a block $\mathsf{M}_{\mathcal{R}}$ that tiles the lattice state M,
with period ℓ_j in jth lattice direction

insight 4 : can compute ‘all’ solutions

Bernoulliland - rough initial guesses converge



no exponential instabilities

bye bye, dynamics

1. goal : describe states of turbulence in infinite spatiatemporal domains
2. theory : classify, enuremate all spatiotemporal tilings
3. example : spatiotemporal cat, the simplest model of “turbulence”

there is no more time

there is only enumeration of admissible spacetime field configurations time is dead !

in future there will be no future

goodbye

to long time and/or space integrators

they never worked and could never work

miaw

the stage is set for the quantum field theory of spatiotemporal cat, the details of which we leave to our always trustworthy friends Jon Keating and Marcos Saraceno