herding cats
a chaotic field theory

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ChaosBook.org/overheads/spatiotemporal
→ Chaotic field theory slides
→ QM^3 video channel

"QM^3 Quantum Matter meets Maths"
Lisbon

December 6, 2021

Q. what is a chaotic field theory?

A. it is a field theory

field configuration Φ probability

$$p(\Phi) = \frac{1}{Z} e^{-S[\Phi]} , \qquad Z = Z[0]$$

partition function = sum over configurations

$$Z[\mathsf{M}] = \int [d\phi] \, e^{-S[\Phi]+\Phi\cdot\mathsf{M}} , \qquad [d\phi] = \prod_{z}^{\mathcal{L}} \frac{d\phi_z}{\sqrt{2\pi}}$$

example : Euclidean ϕ^4 theory action

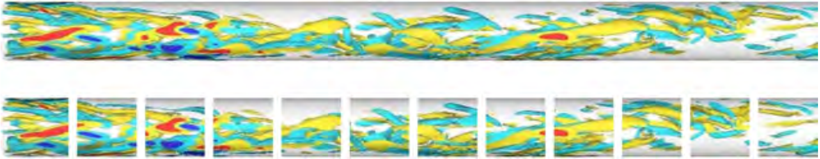
$$S[\Phi] = \int dx^d \left\{ \frac{1}{2} \sum_{i=1}^{d} (\partial_\mu \phi(x))^2 + \frac{\mu^2}{2} \phi(x)^2 + \frac{g}{4!} \phi(x)^4 \right\}$$

Q. why a "chaotic" field theory?

turbulence !

a motivation : need a theory of large turbulent domains

pipe flow close to onset of turbulence [1]



we have a detailed theory of small turbulent fluid cells

can we can we construct the infinite pipe by coupling small turbulent cells ?

what would that theory look like ?

[1] M. Avila and B. Hof, Phys. Rev. **E 87** (2013)

the goal

build
a chaotic field theory

from
the simplest chaotic blocks

using

- time invariance
- space invariance

of the defining partial differential equations

Dreams of Grand Schemes : solve

Navier- Stokes

$$\rho \frac{\partial u_i}{\partial t} + \rho u_j \frac{\partial u_i}{\partial x_j} = \rho X_i - \frac{\partial p}{\partial x_i} + \mu \nabla^2 u_i$$

Einstein

$$R_{ik} - \frac{1}{2} g_{ik} R = \frac{8\pi k}{c^4} T_{ik}$$

$$R^i_{\ klm} = \frac{\partial \Gamma^i_{km}}{\partial x^l} - \frac{\partial \Gamma^i_{kl}}{\partial x^m} + \Gamma^i_{nl}\Gamma^n_{km} - \Gamma^i_{nm}\Gamma^n_{kl}$$

Yang- Mills

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F_a^{\mu\nu}$$

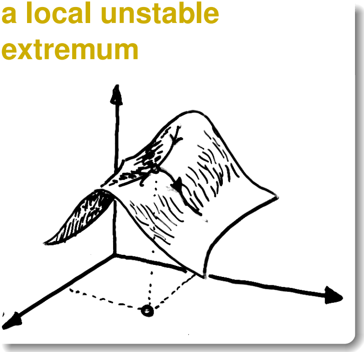
$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\, c_{abc} A^b_\mu A^c_\nu$$

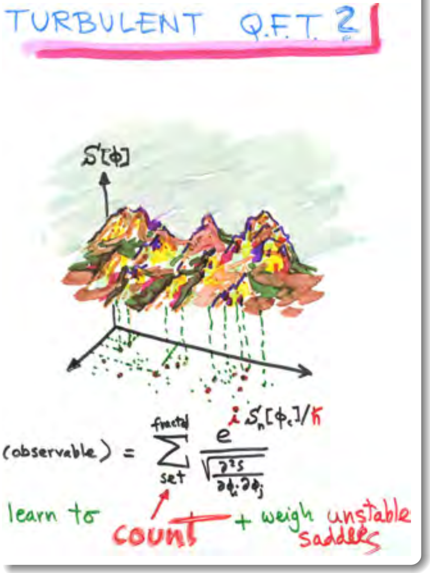
QUANTUM?

QFT path integrals : semi-classical WKB quantization

a fractal set of saddles

a local unstable extremum





Q. what is a chaotic field theory?

A. say it three times

coin flip

serves here as an introduction to the

spatiotemporal cat[a]

which is the simplest example of

spatiotemporally chaotic field theory[b]

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[a] P. Cvitanović and H. Liang, *Spatiotemporal cat: A chaotic field theory*, In preparation, 2021.

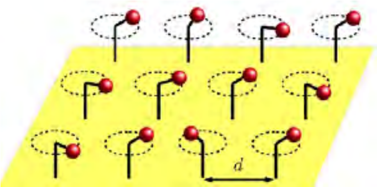
[b] M. N. Gudorf et al., *Spatiotemporal tiling of the Kuramoto-Sivashinsky flow*, In preparation, 2021.

take-home :

harmonic field theory

tight-binding model
(Helmholtz)

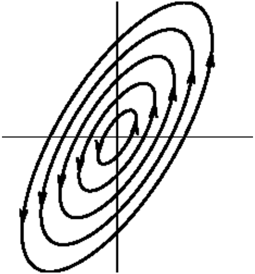
chaotic field theory



Euclidean Klein-Gordon
(damped Poisson)

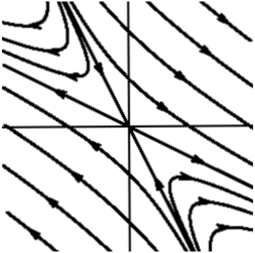
take-home :

harmonic field theory



oscillatory eigenmodes

chaotic field theory



hyperbolic instabilities

the very short answer : POT

if you win : I teach you how

(for details, see ChaosBook.org)

Mephistopheles knocks at Faust's door and says, “Du mußt es dreimal sagen!"

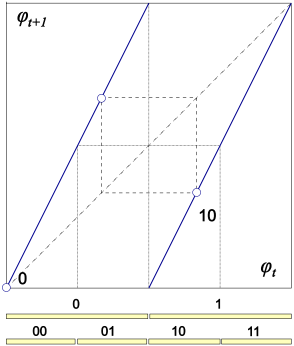
“You have to say it three times"

— Johann Wolfgang von Goethe
Faust I - Studierzimmer 2. Teil

time-evolution formulation

fair coin toss

Bernoulli map



$$\phi_{t+1} = \begin{cases} 2\phi_t \\ 2\phi_t \pmod 1 \end{cases}$$

$\Rightarrow$ fixed point $\overline{0}$, 2-cycle $\overline{01}$, $\cdots$

a coin toss

the essence of deterministic chaos

what is (mod 1) ?

map with integer-valued 'stretching' parameter $s \geq 2$:

$$x_{t+1} = s\, x_t$$

(mod 1) : subtract the integer part $m_t = \lfloor s x_t \rfloor$
so fractional part ϕ_{t+1} stays in the unit interval $[0, 1)$

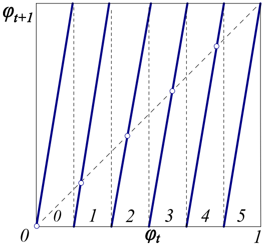
$$\phi_{t+1} = s\phi_t - m_t \,, \qquad \phi_t \in \mathcal{M}_{m_t}$$

m_t takes values in the s-letter alphabet

$$m \in \mathcal{A} = \{0, 1, 2, \cdots, s-1\}$$

a fair dice throw

slope 6 Bernoulli map



$$\phi_{t+1} = 6\phi_t - m_t\,,\ \phi_t \in \mathcal{M}_{m_t}$$

6-letter alphabet

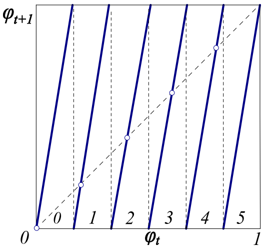
$m_t \in \mathcal{A} = \{0, 1, 2, \cdots, 5\}$

6 subintervals $\{\mathcal{M}_0, \mathcal{M}_1, \cdots, \mathcal{M}_5\}$

what is chaos ?

a fair dice throw

6 subintervals $\{\mathcal{M}_{m_t}\}$, 6^2 subintervals $\{\mathcal{M}_{m_1 m_2}\}, \cdots$



each subinterval contains a periodic point, labeled by $\mathsf{M} = m_1 m_2 \cdots m_n$

$N_n = 6^n - 1$ unstable orbits

definition : chaos is

positive Lyapunov ($\ln s$) - positive entropy ($\frac{1}{n} \ln N_n$)

definition : chaos is

positive Lyapunov ($\ln s$) - positive entropy ($\frac{1}{n} \ln N_n$)

- Lyapunov : how fast is local escape?
- entropy : how many ways of getting back?

$\Rightarrow$ ergodicity

the precise sense in which dice throw
is an example of deterministic chaos

lattice formulation

lattice Bernoulli

recast the time-evolution Bernoulli map

$$\phi_{t+1} = s\phi_t - m_t$$

as 1-step difference equation on the temporal lattice

$$-\phi_{t+1} + s\phi_t = m_t\,, \qquad \phi_t \in [0,1)$$

field ϕ_t, source m_t
on each site t of a 1-dimensional lattice $t \in \mathbb{Z}$

write an n-sites lattice segment as
the field configuration and the symbol block

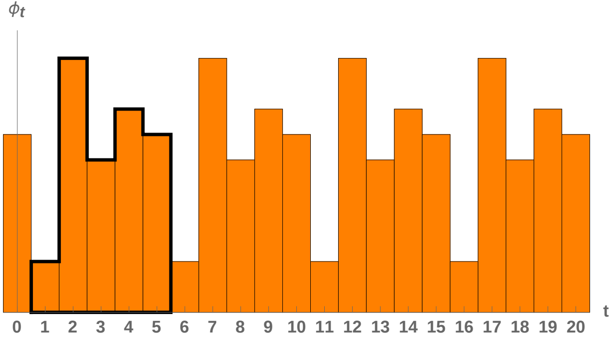
$$\Phi = (\phi_{t+1}, \cdots, \phi_{t+n})\,, \quad \mathsf{M} = (m_{t+1}, \cdots, m_{t+n})$$

'M' for 'marching orders' : come here, then go there, $\cdots$

scalar field theory on 1-dimensional lattice

write a periodic field over *n*-sites Bravais cell as
the field configuration and the symbol block (sources)

$$\Phi = (\phi_{t+1}, \cdots, \phi_{t+n}), \quad \mathsf{M} = (m_{t+1}, \cdots, m_{t+n})$$



'M' for 'marching orders' : come here, then go there, $\cdots$

think globally, act locally

Bernoulli condition at every lattice site t, local in time

$$-\phi_{t+1} + s\phi_t = m_t$$

is enforced by the global equation

$$(-r + s\,1)\;\Phi = \mathrm{M}\,,$$

$[n \times n]$ shift matrix

$$rjk = \delta_{j+1,k}\,, \qquad r = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & & \ddots & \\ & & & 0 & 1 \\ 1 & & & & 0 \end{pmatrix}$$

compares the neighbors

think globally, act locally

solving the lattice Bernoulli system

$$\mathcal{J}\Phi = \mathsf{M}\,,$$

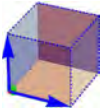
$[n \times n]$ Hill matrix $\qquad \mathcal{J} = -r + s\,1\,,$

is a search for zeros of the function

$$F[\Phi] = \mathcal{J}\Phi - \mathsf{M} = 0$$

the entire global lattice state Φ_{M} is now

a single fixed point $(\phi_1, \phi_2, \cdots, \phi_n)$



in the n-dimensional unit hyper-cube $\qquad \Phi \in [0,1)^n$

orbit stability

orbit Jacobian matrix

solving a nonlinear

$$F[\Phi] = 0 \quad \text{fixed point condition}$$

with Newton method requires evaluation of the $[n \times n]$

orbit Jacobian matrix

$$\mathcal{J}_{ij} = \frac{\delta F[\Phi]_i}{\delta \phi_j}$$

what does this global orbit Jacobian matrix do?

1. fundamental fact !
2. global stability of lattice state Φ, perturbed everywhere

fundamental fact

(1) fundamental fact

to satisfy the fixed point condition

$$\mathcal{J}\Phi - \mathsf{M} = 0$$

the orbit Jacobian matrix $\mathcal{J}$

1. stretches the unit hyper-cube $\Phi \in [0,1)^n$ into the n-dimensional fundamental parallelepiped
2. maps each periodic point Φ_M $\Rightarrow$ integer lattice $\mathbb{Z}^n$ point
3. then translate by integers M $\Rightarrow$ into the origin

hence $N_n =$ total ♯ solutions = ♯ integer lattice points within the fundamental parallelepiped

the fundamental fact[2] : Hill determinant counts solutions

$$N_n = \mathrm{Det}\,\mathcal{J}$$

♯ integer points in fundamental parallelepiped = its volume

[2] M. Baake et al., J. Phys. A **30**, 3029–3056 (1997).

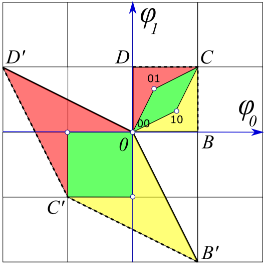
example : fundamental parallelepiped for $n = 2$

orbit Jacobian matrix for $s = 2$;

unit square basis vectors ; their images :

$$\mathcal{J} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} ; \quad \Phi_B = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \to \Phi_{B'} = \mathcal{J}\,\Phi_B = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \cdots ,$$

Bernoulli periodic points of period 2



$N_2 = 3$

fixed point Φ_{00}

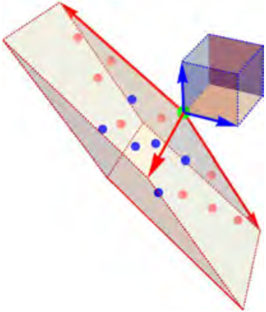
2-cycle Φ_{01}, Φ_{10}

square $[0BCD] \Rightarrow \mathcal{J} \Rightarrow$ fundamental parallelepiped $[0B'C'D']$

fundamental fact for any n

an $n = 3$ example

$\mathcal{J}$ [unit hyper-cube] = [fundamental parallelepiped]



unit hyper-cube $\Phi \in [0, 1)^3$

$n > 3$ cannot visualize

a periodic point $\Rightarrow$ integer lattice point : $\bullet$ on a face, $\bullet$ in the interior

(2)

orbit stability

(2) orbit stability vs. temporal stability

orbit Jacobian matrix

$\mathcal{J}_{ij} = \frac{\delta F[\Phi]_i}{\delta \phi_j}$ stability under global perturbation of the whole orbit

for n large, a huge $[dn \times dn]$ matrix

temporal Jacobian matrix

J propagates initial perturbation n time steps

small $[d \times d]$ matrix

J and $\mathcal{J}$ are related by[3]

Hill's 1886 remarkable formula

$$|\mathrm{Det}\,\mathcal{J}_\mathrm{M}| = |\det(\mathbf{1} - J_\mathrm{M})|$$

$\mathcal{J}$ is huge, even ∞-dimensional matrix

J is tiny, few degrees of freedom matrix

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[3] G. W. Hill, Acta Math. **8**, 1–36 (1886).

definition : chaos is

expanding	Hill determinants	$\mathrm{Det}\,\mathcal{J}$
exponential $\sharp$	field configurations	N_n

the precise sense in which
a (discretized) field theory is deterministically chaotic

note : there is no 'time' in this definition

periodic orbit theory

volume of a periodic orbit

Ozorio de Almeida and Hannay[4] 1984 :
$\sharp$ of periodic points is related to a Jacobian matrix by

principle of uniformity

"periodic points of an ergodic system, counted with their natural weighting, are uniformly dense in phase space"

where

'natural weight' of periodic orbit M

$$\frac{1}{|\det(1 - J_M)|}$$

[4] A. M. Ozorio de Almeida and J. H. Hannay, J. Phys. A **17**, 3429 (1984).

periodic orbits partition lattice states into neighborhoods

how come Hill determinant $\mathrm{Det}\,\mathcal{J}$ counts periodic points ?

'principle of uniformity' is in[5]

periodic orbit theory

known as the flow conservation sum rule :

$$\sum_{\mathsf{M}} \frac{1}{|\det(1 - J_{\mathsf{M}})|} = \sum_{\mathsf{M}} \frac{1}{|\mathrm{Det}\,\mathcal{J}_{\mathsf{M}}|} = 1$$

sum over periodic points Φ_{M} of period n

state space is divided into

neighborhoods of periodic points of period n

[5] P. Cvitanović, "Why cycle?", in *Chaos: Classical and Quantum*, edited by P. Cvitanović et al. (Niels Bohr Inst., Copenhagen, 2020).

periodic orbit counting

how come a $\mathrm{Det}\,\mathcal{J}$ counts periodic points ?

flow conservation sum rule :

$$\sum_{\Phi_\mathsf{M}\in \mathsf{Fix} f^n} \frac{1}{|\mathrm{Det}\,\mathcal{J}_\mathsf{M}|} = 1$$

Bernoulli system 'natural weighting' is simple :

the determinant $\mathrm{Det}\,\mathcal{J}_\mathsf{M} = \mathrm{Det}\,\mathcal{J}$ the same for all periodic points, whose number thus verifies the fundamental fact

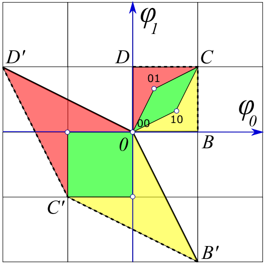
$$N_n = |\mathrm{Det}\,\mathcal{J}|$$

the number of Bernoulli periodic lattice states

$N_n = |\mathrm{Det}\,\mathcal{J}| = s^n - 1$ for any n

remember the fundamental fact?

period 2 example



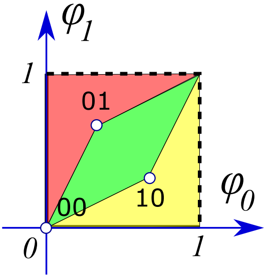
fixed point Φ_{00}
2-cycle Φ_{01}, Φ_{10}

$\mathcal{J}$ [unit hyper-cube] = [fundamental parallelepiped]

look at preimages of the fundamental parallelepiped :

example : lattice states of period 2

unit hypercube, partitioned



fixed point Φ_{00}

2-cycle Φ_{01}, Φ_{10}

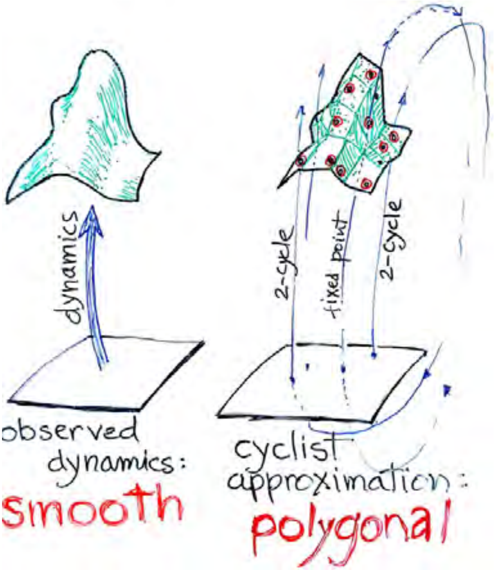
flow conservation sum rule

$$\frac{1}{|\mathrm{Det}\,\mathcal{J}_{00}|} + \frac{1}{|\mathrm{Det}\,\mathcal{J}_{01}|} + \frac{1}{|\mathrm{Det}\,\mathcal{J}_{10}|} = 1$$

sum over periodic points Φ_M of period $n = 2$

state space is divided into

neighborhoods of periodic points of period n



tessellate the state space by recurrent flows

zeta function

periodic orbit theory : counting lattice states

topological zeta function

$$1/\zeta_{\mathrm{top}}(z) = \exp\left(-\sum_{n=1}^{\infty} \frac{z^n}{n} N_n\right)$$

1. weight $1/n$ as by (cyclic) translation invariance, n lattice states are equivalent
2. zeta function counts orbits, one per each set of equivalent lattice states

Bernoulli topological zeta function

counts orbits, one per each set of lattice states $N_n = s^n - 1$

$$1/\zeta_{\text{top}}(z) = \exp\left(-\sum_{n=1}^{\infty} \frac{z^n}{n} N_n\right) = \frac{1 - sz}{1 - z}$$

numerator $(1 - sz)$ says that Bernoulli orbits are built from
s fundamental primitive lattice states,
the fixed points $\{\phi_0, \phi_1, \cdots, \phi_{s-1}\}$

every other lattice state is built from their concatenations and repeats.

solved!

this is ‘periodic orbit theory’
And if you don’t know, now you know

summary : think globally, act locally

the problem of enumerating and determining all lattice states stripped to its essentials :

1. each solution is a zero of the global fixed point condition

$$F[\Phi] = 0$$

2. global stability : the orbit Jacobian matrix

$$\mathcal{J}_{ij} = \frac{\delta F[\Phi]_i}{\delta \phi_j}$$

3. fundamental fact : the number of period-n orbits

$$N_n = |\mathrm{Det}\, \mathcal{J}|$$

4. zeta function $1/\zeta_{\mathrm{top}}(z)$: all predictions of the theory

Du mußt es dreimal sagen!
— Mephistopheles

coin toss ? that's not physics !

Field Theory should be Hamiltonian and energy conserving
Quantum Mechanics requires it

because that is physics !

need a system as simple as the Bernoulli, but mechanical

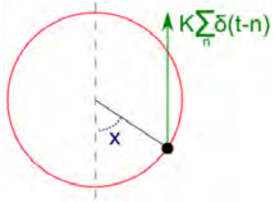
so, we move on from running in circles,

to a mechanical rotor to kick.

time-evolution formulation

example of a "small domain" dynamics : a single kicked rotor

an electron circling an atom, subject to
a discrete time sequence of angle-dependent kicks $F(x_t)$



Taylor, Chirikov and Greene standard map

$$\begin{aligned} x_{t+1} - x_t &= p_{t+1} \qquad \bmod 1 \\ p_{t+1} - p_t &= F(x_t) \end{aligned}$$

→ chaos in Hamiltonian systems

the simplest example : a cat map evolving in time

force $F(x) = Kx$ linear in the displacement x , $K \in \mathbb{Z}$

$$\begin{aligned} x_{t+1} &= x_t + p_{t+1} && \text{mod } 1 \\ p_{t+1} &= p_t + Kx_t && \text{mod } 1 \end{aligned}$$

Continuous Automorphism of the Torus, or

time-evolution cat map

a linear, area preserving map of a 2-torus onto itself

$$\begin{bmatrix} \phi_t \\ \phi_{t+1} \end{bmatrix} = J \begin{bmatrix} \phi_{t-1} \\ \phi_t \end{bmatrix} - \begin{bmatrix} 0 \\ m_t \end{bmatrix}, \qquad J = \begin{bmatrix} 0 & 1 \\ -1 & s \end{bmatrix}$$

for integer 'stretching' $s = \mathrm{tr}\, J > 2$ the map is
beloved by ergodicists :
hyperbolic $\Rightarrow$ perfect chaotic Hamiltonian dynamical system

a cat is literally Hooke's wild, 'anti-harmonic' sister

for $s < 2$ Hooke rules

local restoring oscillations
around the sleepy z-z-z-zzz resting state

for $s > 2$ cats rule

exponential runaway
wrapped global around a phase space torus

cat is to chaos what harmonic oscillator is to order

there is no more fundamental example of chaos in mechanics

lattice formulation

cat map in lattice formulation

replace momentum by velocity

$$p_{t+1} = (\phi_{t+1} - \phi_t)/\Delta t$$

obtain

$$\begin{bmatrix} \phi_t \\ \phi_{t+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & s \end{bmatrix} \begin{bmatrix} \phi_{t-1} \\ \phi_t \end{bmatrix} - \begin{bmatrix} 0 \\ m_t \end{bmatrix}$$

temporal lattice formulation is pretty[6] :

2-step difference equation

$$-\phi_{t+1} + s\,\phi_t - \phi_{t-1} = m_t$$

integer m_t ensures that

ϕ_t lands in the unit interval

$$m_t \in \mathcal{A}\,, \quad \mathcal{A} = \{\text{finite alphabet}\}$$

[6] I. Percival and F. Vivaldi, Physica D **27**, 373–386 (1987).

think globally, act locally

spatiotemporal cat at every instant t, local in time

$$-\phi_{t+1} + s\,\phi_t - \phi_{t-1} = m_t$$

is enforced by the global equation

$$\mathcal{J}\,\Phi = \mathsf{M}\,,$$

where

orbit Jacobian matrix

$$\mathcal{J}\,\Phi - \mathsf{M} = 0$$

with

$$\Phi = (\phi_{t+1}, \cdots, \phi_{t+n})\,, \quad \mathsf{M} = (m_{t+1}, \cdots, m_{t+n})$$

a lattice state, and a symbol block

and $[n\times n]$ orbit Jacobian matrix $\mathcal{J}$ is

$$-r + s\,\mathbb{1} - r^{-1} = \begin{pmatrix} s & -1 & & & -1 \\ -1 & s & -1 & & \\ & -1 & & \ddots & \\ & & & s & -1 \\ -1 & & & -1 & s \end{pmatrix}$$

think globally, act locally

solving the spatiotemporal cat equation

$$\mathcal{J}\Phi = \mathsf{M}\,,$$

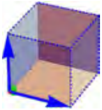
with the $[n\times n]$ matrix $\qquad \mathcal{J} = -r + s\,\mathbb{1} - r^{-1}$

can be viewed as a search for zeros of the function

$$F[\Phi] = \mathcal{J}\Phi - \mathsf{M} = 0$$

where the entire global lattice state Φ_M is

a single fixed point $\Phi_\mathsf{M} = (\phi_1, \phi_2, \cdots, \phi_n)$

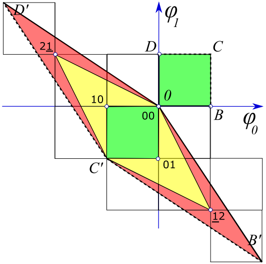


in the n-dimensional unit hyper-cube $\qquad \Phi \in [0,1)^n$

fundamental fact in action

temporal cat fundamental parallelepiped for period 2

square $[0BCD] \Rightarrow \mathcal{J} \; = \;$ fundamental parallelepiped $[0B'C'D']$



$N_2 = |\mathrm{Det}\,\mathcal{J}| = 5$

fundamental parallelepiped
$=$ 5 unit area quadrilaterals

a periodic point per each unit volume

spatiotemporal cat zeta function

is the generating function that counts orbits

substituting the Hill determinant count of periodic lattice states

$$N_n = \mathrm{Det}\,\mathcal{J}$$

into the topological zeta function

$$1/\zeta_{\mathrm{top}}(z) = \exp\left(-\sum_{n=1} \frac{z^n}{n} N_n\right)$$

leads to the elegant explicit formula[7]

$$1/\zeta_{\mathrm{top}}(z) = \frac{1 - sz + z^2}{1 - 2z + z^2}$$

solved!

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[7] S. Isola, Europhys. Lett. **11**, 517–522 (1990).

what continuum theory is temporal cat discretization of?

have

2-step difference equation

$$-\phi_{t+1} + s\,\phi_t - \phi_{t-1} = m_t$$

discrete lattice

Laplacian in 1 dimension

$$\phi_{t+1} - 2\phi_t + \phi_{t-1} = \Box\,\phi_t$$

so temporal cat is an (anti)oscillator chain, known as

$d = 1$ Klein-Gordon (or damped Poisson) equation (!)

$$(-\Box + \mu^2)\,\phi_t = m_t\,, \qquad \mu^2 = s - 2$$

did you know that a cat map can be so cool?

that's it! for spacetime of any dimension

lattice Klein-Gordon equation

$$(-\Box + \mu^2)\,\phi_t = m_t$$

solved completely and analytically!

summary : think globally, act locally

the problem of determining all global solutions stripped to its bare essentials :

1. each solution a zero of the global fixed point condition

$$F[\Phi] = 0$$

2. compute the orbit Jacobian matrix

$$\mathcal{J}_{ij} = \frac{\delta F[\Phi]_i}{\delta \phi_j}$$

3. fundamental fact $\quad N_n = |\mathrm{Det}\, \mathcal{J}| =$ period-n states

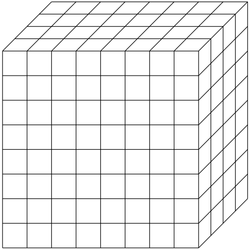
4. $\Rightarrow$ zeta function $1/\zeta_{\mathrm{top}}(z)$

chaotic field theory

Euclidean lattice field theory

scalar *field* $\phi(x)$

evaluated on lattice points



$\phi_z = \phi(x)$
$x = a\,z =$ lattice point
$z \in \mathbb{Z}^d$

a periodic point per each unit cell

8

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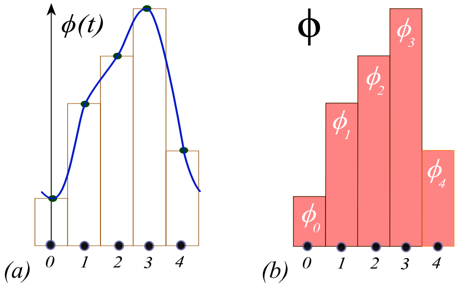
[8] G. Münster and M. Walzl, *Lattice gauge theory - A short primer*, 2000.

example : discretization of a $1d$ field

scalar *field* $\phi(x)$ evaluated on lattice points



periodic field $\phi(t)$ is a function of continuous coordinate t

corresponding discretized period-5 lattice state $\Phi = \overline{\phi_0\phi_1\phi_2\phi_3\phi_4}$,

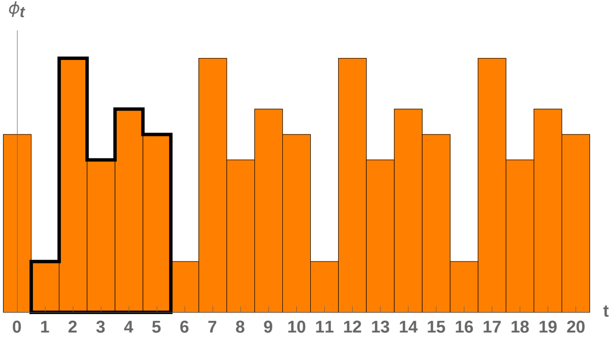
Horizontal: t coordinate, lattice sites marked by dots, labelled by $t \in \mathbb{Z}$

the value of the discretized field $\phi_t \in \mathbb{R}$ is plotted as a bar centred at lattice site t

Bravais cell lattice tiling

write a periodic field over n-sites Bravais cell as the lattice state and the symbol block (sources)

$$\Phi = (\phi_{t+1}, \cdots, \phi_{t+n}), \quad \mathsf{M} = (m_{t+1}, \cdots, m_{t+n})$$



'M' for 'marching orders' : come here, then go there, $\cdots$

field theory is defined by its action

field theory

field configuration Φ occurs with probability

$$p(\Phi) = \frac{1}{Z} e^{-S[\Phi]}, \qquad Z = Z[0]$$

partition function $=$ sum over all configurations

$$Z[\mathrm{M}] = \int [d\phi]\, e^{-S[\Phi]+\Phi\cdot\mathrm{M}}, \qquad [d\phi] = \prod_z^{\mathcal{L}} \frac{d\phi_z}{\sqrt{2\pi}}$$

'source' M

example : Euclidean ϕ^4 theory

continuum action

$$S \;=\; \int dx^d \left\{ \frac{1}{2} \sum_{i=1}^{d} (\partial_\mu \phi(x))^2 + \frac{\mu^2}{2} \phi(x)^2 + \frac{g}{4!} \phi(x)^4 \right\}$$

lattice action

$$S[\Phi] \;=\; \sum_{z,z'} \frac{1}{2} \left\{ \phi_z \left(-\Box + \mu^2 \right)_{zz'} \phi_{z'} \right\} + \sum_z \frac{g}{4!} \phi_z^4 \,.$$

in ‘lattice units’ : $a = 1$

QFT path integrals : semi-classical WKB quantization

WKB backbone

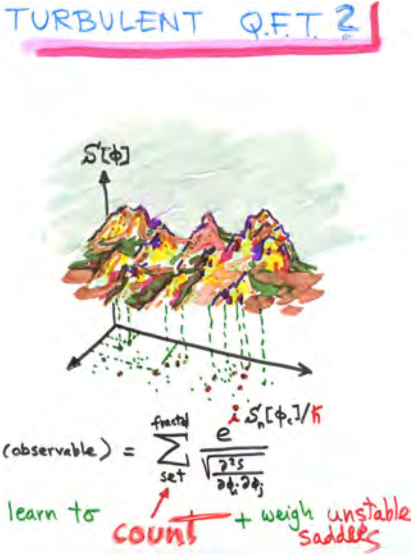
classical field theory

extremal condition $\rightarrow$ eqs

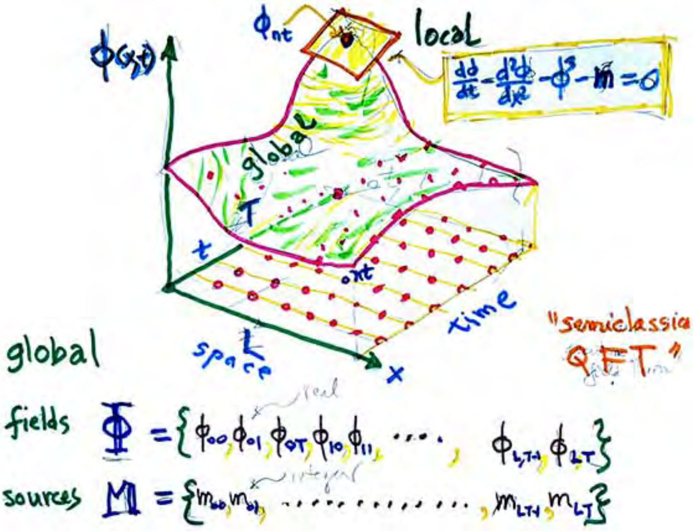
$$\frac{\delta S[\Phi]}{\delta \phi_z} = m_z$$

classical solution Φ satisfies the extremal condition on every lattice site

a fractal set of saddles



think globally, act locally



for each symbol array M, a periodic lattice state Φ_M

orbit Jacobian (Hill, Hessian, ...) matrix

each lattice state has its own

$$\mathcal{J}[\Phi] = \begin{pmatrix} s_0 & -1 & 0 & 0 & \cdots & 0 & 0 & -1 \\ -1 & s_1 & -1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -1 & s_2 & -1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -1 & s_{n-2} & -1 \\ -1 & 0 & 0 & 0 & \cdots & 0 & -1 & s_{n-1} \end{pmatrix},$$

stretching factor $s_t = V''[\phi_t]$ is
function of the site field ϕ_t for the given lattice state Φ

1. can compute Hill determinant $\mathrm{Det}\,\mathcal{J}$
2. Hill-Lindstedt-Poincaré :
   all calculations should be done on reciprocal lattice
3. toolbox : discrete Fourier transforms, irreps of D_n

popular 1d lattice field theories

spatiotemporal lattice field theory

$$-\phi_{t+1} + V'[\phi_t] - \phi_{t-1} \quad = \quad m_t$$

spatiotemporal Bernoulli

$$-\phi_{t+1} + s\,\phi_t \quad = \quad m_t$$

spatiotemporal cat

$$-\phi_{t+1} + s\,\phi_t - \phi_{t-1} \quad = \quad m_t$$

spatiotemporal Hénon

$$-\phi_{t+1} + a\,\phi_t^2 - \phi_{t-1} \quad = \quad m_t$$

spatiotemporal ϕ^4 theory

$$-\phi_{t+1} + \frac{g}{3!}\phi_t^3 - \phi_{t-1} \quad = \quad m_t$$

There are only two 1-dimensional space groups G:
$p1$ *infinite cyclic group* C_∞ of all lattice translations,

$$\mathrm{C}_\infty = \{\cdots, r_{-2}, r_{-1}, 1, r_1, r_2, r_3, \cdots\}$$

$p1m$ *infinite dihedral group* D_∞ of all translations and reflections[9],

$$\mathrm{D}_\infty = \{\cdots, r_{-2}, \sigma_{-2}, r_{-1}, \sigma_{-1}, 1, \sigma, r_1, \sigma_1, r_2, \sigma_2, \cdots\}$$

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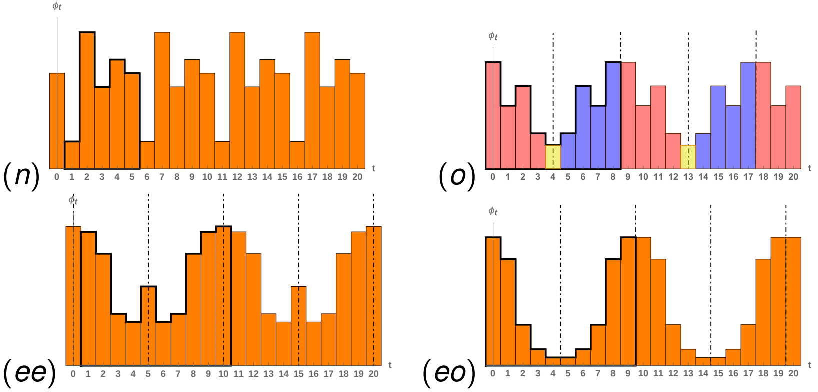
[9] Y.-O. Kim et al., Pacific J. Math. **209**, 289–301 (2003).

4 kinds of Bravais lattice states



(n) *no reflection symmetry:* H_5 invariant period-5 lattice state

(o) *odd period, symmetric:* an $H_{9,8}$ invariant period-9

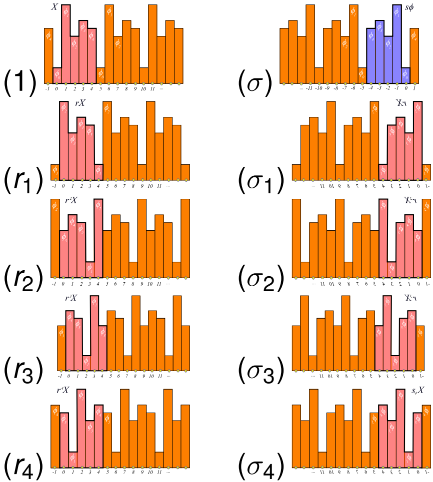
(ee) *even period, even symmetric:* $H_{10,0}$ invariant period-10

(eo) *even period, odd symmetric:* $H_{10,9}$ invariant period-10 group multiplication $g_i g_j$

	r_j	σ_j
r_i	r_{i+j}	σ_{j-i}
σ_i	σ_{i+j}	r_{j-i}

either adds up translations,
or shifts and then reverses their direction

D_∞ orbit of a generic lattice state



lattice state $\Phi = \overline{\phi_0\phi_1\phi_2\phi_3\phi_4}$, no reflection symmetry

D_∞-orbit is isomorphic to D_5 : 10 distinct lattice states

zeta functions unlike 1980's

periodic orbit theory : counting lattice states[10]

Lind zeta function

$$\zeta_{Lind}(t) = \exp\left(\sum_{H} \frac{N_H}{|G/H|} t^{|G/H|}\right)$$

sum is over all subgroups H of space group G
N_H is the number of fixed points of H
$|G/H|$ is the number of states in H orbit

1. Lind zeta function counts group orbits, one per each set of equivalent lattice states

[10] D. A. Lind, "A zeta function for Z^d-actions", in *Ergodic Theory of Z^d Actions*, edited by M. Pollicott and K. Schmidt (Cambridge Univ. Press, 1996), pp. 433–450.

zeta functions unlike 1980's

periodic orbit theory :
counting lattice states for reflection-symmetric systems[11,12]

Kim-Lee-Park zeta function

$$\zeta_\sigma(t) = \sqrt{\zeta_{top}(t^2)}\, e^{h(t)},$$

where ζ_{top} is the Artin-Mazur zeta function, and the counts of the 3 kinds of symmetric orbits are

$$h(t) = \sum_{m=1}^{\infty} \left\{ N_{2m-1,0}\, t^{2m-1} + \left(N_{2m,0} + N_{2m,1}\right) \frac{t^{2m}}{2} \right\}$$

[11] M. Artin and B. Mazur, Ann. Math. **81**, 82–99 (1965).

[12] Y.-O. Kim et al., Pacific J. Math. **209**, 289–301 (2003).

1. coin toss
2. kicked rotor
3. spatiotemporal cat
4. bye bye, dynamics

insight 1 : how is turbulence described?

not by the evolution of an initial state

exponentially unstable system have finite (Lyapunov) time and space prediction horizons

but

by enumeration of admissible field configurations

and their natural weights

insight 2 : description of turbulence by d-tori

1 time, 0 space dimensions

a phase space point is *periodic* if its orbit returns to itself after a finite time T; such orbit tiles the time axis by infinitely many repeats

1 time, d-1 space dimensions

a phase space point is *spatiotemporally periodic* if it belongs to an invariant d-torus $\mathcal{R}$,
i.e., a block $\mathsf{M}_{\mathcal{R}}$ that tiles the lattice state M,
with period ℓ_j in jth lattice direction

insight 3 : can compute ‘all’ solutions

Bernoulliland - rough initial guesses converge

no exponential instabilities

reciprocal lattice

what we still do not understand today

1. solved so far only 1-dimensional spatiotemporal lattice, point group D_1
2. should all time-reversal symmetric systems be analyzed this way ?
3. should all dynamical systems should be solved on reciprocal lattice ?
4. for 2-dimensional spatiotemporal chaotic field theory, still have to do this for square lattice point group D_4
5. then, solve the problem of turbulence (Navier-Stokes, Yang-Mills, general relativity)

Verbrechen des Jahrhunderts : das Ende der Zeit

die Zeit ist tot

also, an die Arbeit!

bye bye, dynamics

1. goal : describe states of turbulence in infinite spatiatemporal domains
2. theory : classify, enuremate all spatiotemporal tilings
3. example : spatiotemporal cat, the simplest model of “turbulence”

there is no more time

there is only enumeration of

admissible spacetime field configurations

time is dead

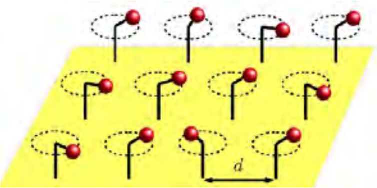
now, get to work

take-home :

traditional field theory

Helmholtz

chaotic field theory



damped Poisson, Yukawa