P. Cvitanovic 2010-06-16 MPI-DS Goettingen chalkboard seminar
(this is supposed to be 8 blackboards, written over 2 vertical pairs of physical blackboards)

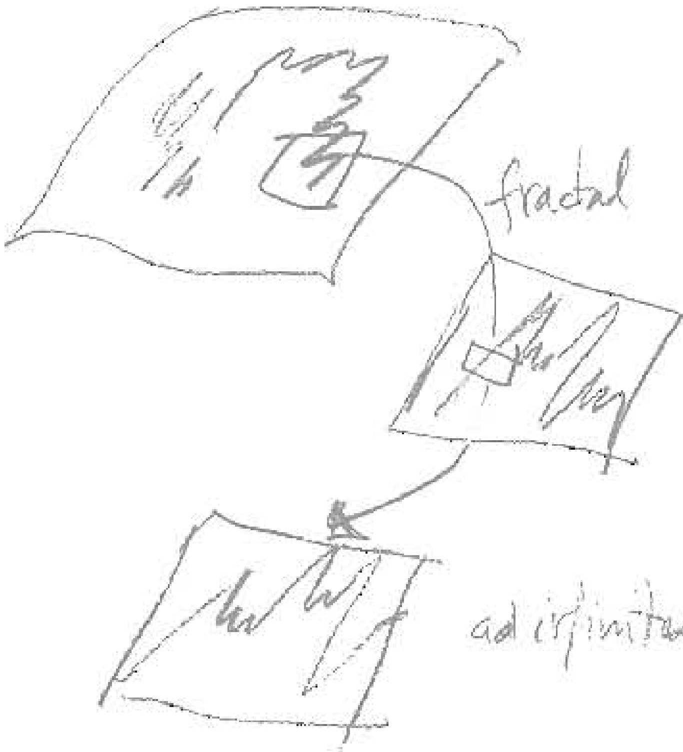
1

SVEN-POINT

Deterministic chaos

$x_{n+1}=f(x_n)$

for $t \to \infty$



1

Real World

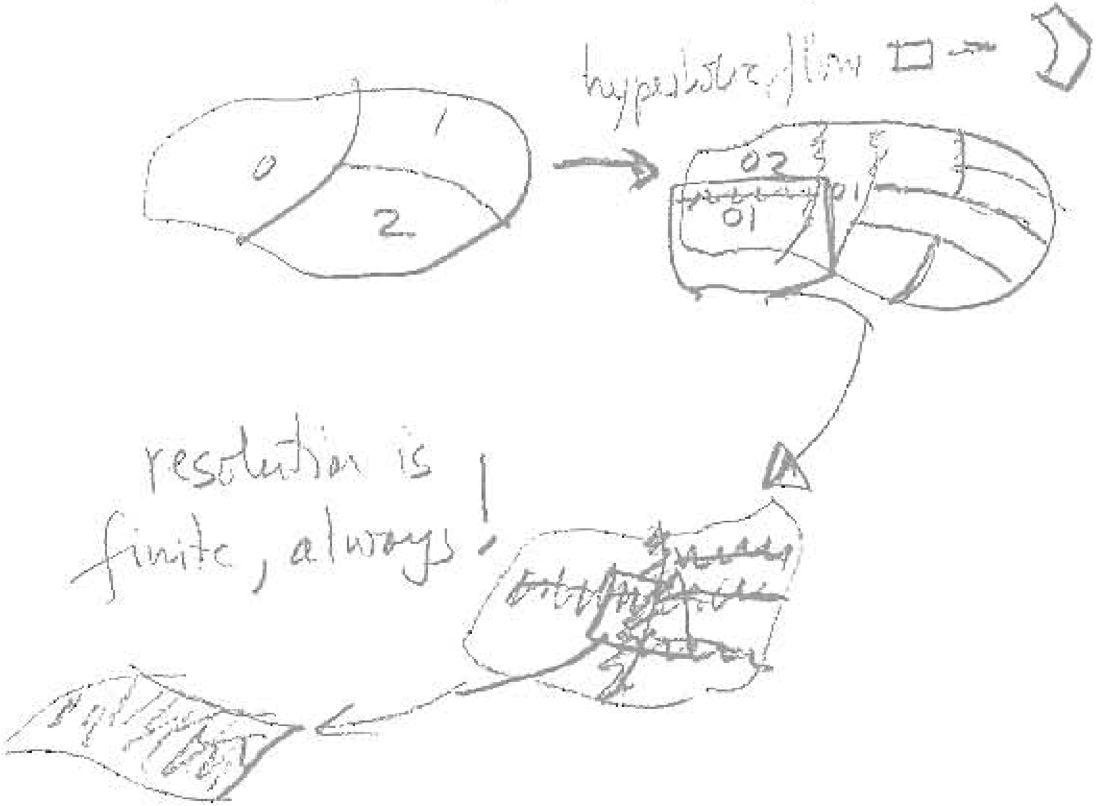
$$x_{n+1} = f(x_n) + \xi_n$$

background
observational
intrinsic
measurement
modelling
numerical, ...

$\langle \xi_{j,n} \rangle = 0$,

$$\langle \xi_{j,n}\, \xi_{k,m} \rangle = \Delta_{jk}\, \delta_{nm}$$

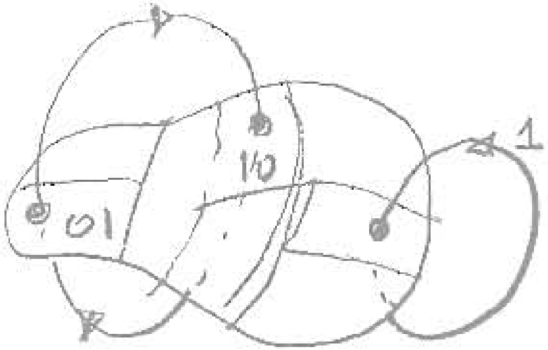
hyperbolic flow



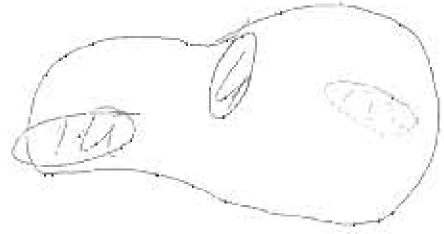
resolution is finite, always!

chaos = local stretching + recurrence

exact recurrence:
periodic orbits



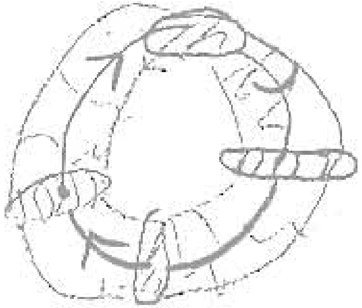
optimal partition



= overlapping cigars!

local stretching + noise:

periodic orbit + noise



point ---> 'cigar'

compute it!

1 3

$\boxed{3}$

Noisy trajectories: densities

Brownian motion, 1d

$$\mathcal{L}(x'|x) = \frac{1}{2\pi\sigma^2} e^{-\frac{(x'-x)^2}{2\sigma^2}}$$

mean 0, variance $\sigma^2 = 2D$

stand. dev $\sqrt{2D}$ diffusion const

Fokker-Planck operator

evolve + kick

$$\xi_n = f(x_n) - x_{n+1}$$

$$\mathcal{L}(x'|x) = \frac{1}{N} e^{-\frac{1}{2}(x'-f(x))^T \frac{1}{\Delta}(x'-f(x))}$$

example:

isotropic noise $\Delta = 2D\mathbb{1}$

$$\rho(x_k,k) = \int [dx_{k-1}\, dx_{k-2} \cdots dx_0]\, e^{-\frac{1}{4D}\sum_n (x_{n+1}-f(x_n))^2}\, \rho(x_0,0)$$

wrong

2

2 4

Cigor, ~~[illegible]~~ computed

Laplace 1810, Langevin 19—, Ornstein-Uhlenbeck 1930,

linearize: $x = x_a + z_a$
← deterministic trajectory point

$$f(x_a + z_a) = x_{a+1} + \mathbb{M}_a z_a + \cdots$$

$$\mathcal{L}(z_{a+1}, z_a) = \frac{1}{N} e^{-\frac{1}{2}(z_{a+1} - \mathbb{M}_a z_a)^T \cdot \frac{1}{\Delta} \cdot (z_{a+1} - M z_a)}$$

gaussian, so take initial $\rho_0(z_0)$ gaussian, covariance Q

$$\rho_{a+1}(z_{a+1}) = \frac{1}{C_a} \int [dz_a] \, e^{-\frac{1}{2}(\quad)^T \cdot \frac{1}{\Delta_a} \cdot (z_{a+1} - M z_a)} e^{-\frac{1}{2} z_a^T \cdot Q_a \cdot z_a} = \frac{1}{C_{a+1}} e^{-\frac{1}{2} z_{a+1}^T \cdot Q_{a+1} z_{a+1}^T}$$

new covariance = "sum of squares", deterministically transported

$$\boxed{Q_{a+1} = \mathbb{M}_a Q_a \mathbb{M}^T + \Delta}$$

noise

Jacobian, Floquet, monodromy matrix, ...

5 Fixed points

if $z_{a+1} = z_a = 0$ ~~fixed point~~

$$Q = MQM^T + \Delta$$

$$= \Delta + M\Delta M^T + M^2\Delta(M^2)^T + \cdots$$

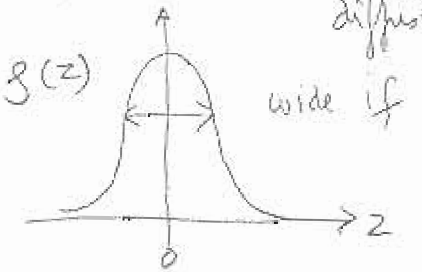
for isotropic case $\Delta = 2D\mathbb{1}$

$$= \frac{2D}{1 - MM^T}$$

1-d example $\mathbb{M} \to \Lambda$, $|\Lambda| < 1$

$$Q = \frac{2D}{1 - \Lambda^2}$$

natural measure = ~~balance of diff~~ balance diffusive spreading − deterministic contraction



wide if $\Lambda \to 1$:

$\Lambda = 1$ Brownian motion

erase (2), (4), (2.1)

Periodic points $\{z_1, z_2, \cdots z_n\}$

$$f^n(z_k) = z_k$$

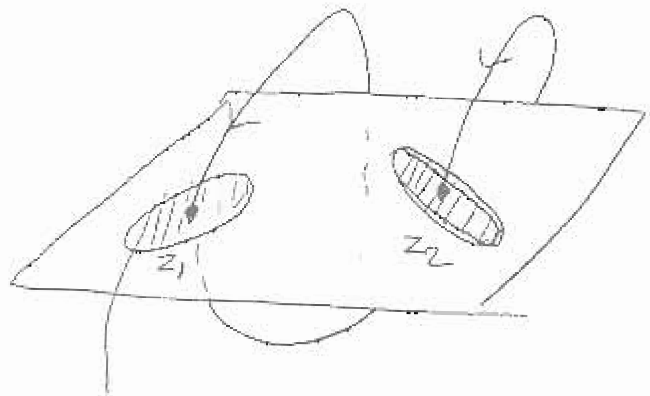
$$Q_a = \Delta_a + M_a \Delta_a M_a^T + M_a^2 \Delta_a (M_a^2)^T + \cdots$$

accumulated noise

$$\Delta_a = \Delta + M_{a-1} \Delta M_{a-1}^T + \cdots$$

noise + nonlinear flow

never isotropic, homogenous



cigar: ellipsoid semiaxes

eigenvectors of Q_a

computable, already know x_a, $\mathbb{M}(x_a)$

6 What if fixed point unstable?

erase (3) (1) (1) (3.1)

Ornstein-Uhlenbeck process

Langevin $\dot{x} = \lambda x + \hat{\xi}(t)$, $\lambda < 0$

1-d map $z_{n+1} = \Lambda z_n + \xi_n$ $0 < \Lambda < 1$

Hermite polynomials orthogonal wrt gaussian kernel

$$\frac{1}{2^n n! \sqrt{2\pi}} \int dx \, H_m(x) \, e^{-x^2} \, H_n(x) = \delta_{mn}$$

rewrite

$$\int dx \, \underbrace{H_m(x)} \, \underbrace{e^{-\frac{x^2}{2}}} \underbrace{\left(\frac{H_n(x) \, e^{-\frac{x^2}{2}}}{2^n n! \sqrt{2\pi}} \right)} = \delta_{mm}$$

Full spectrum analytically!

case $0 < \Lambda < 1$

Fokker Planck $e^{-\frac{(x-\Lambda y)^2}{4D}}$

right eigenfunctions concentrated

left eigenfunctions spread out

case $1 < \Lambda < \infty$

adjoint F.P. eigenfunctions concentrated

$\Longrightarrow$ For unstable directions look back in time: adjoint FP

$\boxed{7}$

erase (1) (3,1) (3,1) (1,1)

Cigar; back in time:

Adjoint F.P. yields

$$\boxed{M_{a-1} Q_{a-1} M_{a-1}^T = Q_a + \Delta}$$

$\Lambda^2 Q = Q + 2D$

strictly expanding;

noisy neighborhood of repelling fixed point

1-d:

$$Q = \frac{2D}{\Lambda^2 - 1}$$

unstable periodic orbits

$$Q_a = \frac{2D}{1-\Lambda_p^{-2}}\left(\frac{1}{(f_a')^2} + \cdots + \frac{1}{\Lambda_p^2}\right)$$

stable/unstable

dominant

local: cigar computed

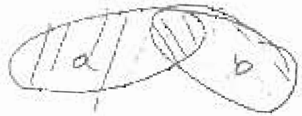
$\boxed{8}$

Optimal partition hypothesis (global)

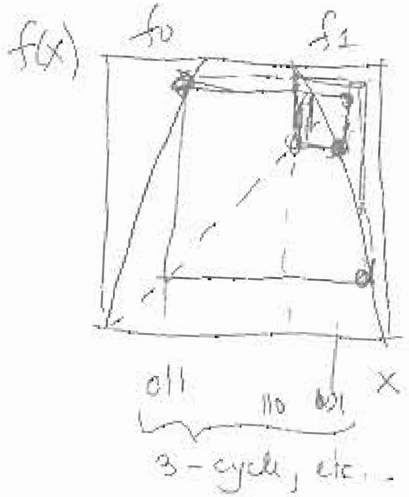
erase 3,1 , 1,1 — 1,1 / 3,2

~~stepzeben~~ determinism → exponentially # periodic points "a"

noise → with widths $\sqrt{Q_a} \sim \sqrt{\frac{2D}{1-\Lambda_p^{-2}}\left(\frac{1}{(f_a')^2} + \cdots\right)}$

if  overlap, stop resolving

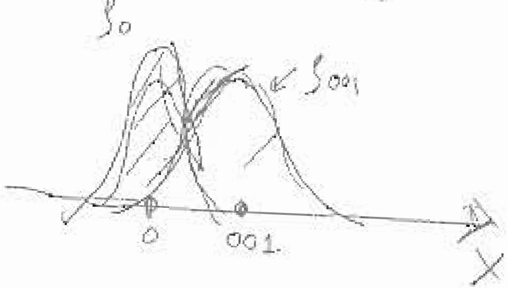
1-d example



for $2D = 0.002$

$\mathcal{M}_{000} \cap \mathcal{M}_{001} \simeq \mathcal{M}_{00}$

stop resolving



optimal partition = 7 neighborhoods

$\{\mathcal{M}_{00}, \mathcal{M}_{011}, \mathcal{M}_{010}, \cdots, \mathcal{M}_{100}\}$

FP finite, [7 x 7]

Markov graph (non-zero F.P. elements)