ChaosBook.org chapter

discrete time dynamics

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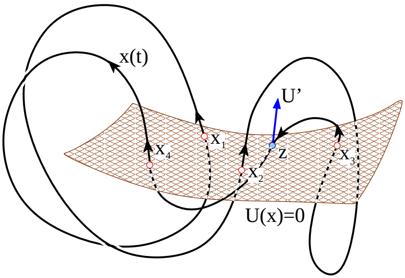
Poincaré sections



successive trajectory intersections with a Poincaré section, a $(d-1)$-dimensional set of hypersurfaces $\mathcal{P}_\alpha$ embedded in the d-dimensional phase space $\mathcal{M}$

Poincaré section

(or, just ‘section’)

is *not* a projection onto a lower-dimensional space:
it is a local change of coordinates to a direction along the flow, and the remaining coordinates (spanning the section) transverse to it.

No information about the flow is lost: the full space trajectory can always be reconstructed by integration from the nearest point in the section.

transverse Poincaré sections

a hypersurface $\mathcal{P}$ can be specified implicitly by a function $U(x)$ that is zero whenever a point x is on the section

transversality condition:

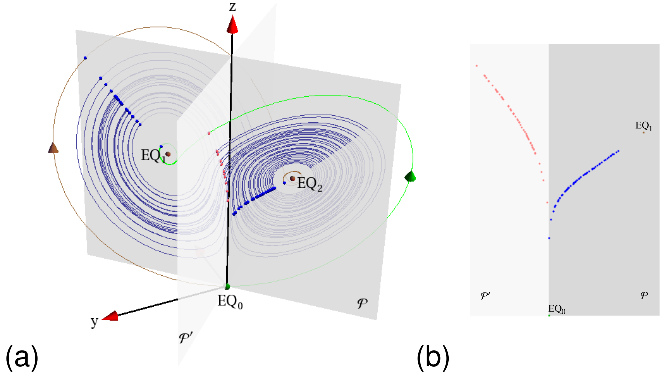
$$(v \cdot \nabla U) = v_j(x)\nabla_j U(x) \neq 0\,, \quad \nabla_j U(x) = \frac{d}{dx_j}U(x)\,, \quad x \in \mathcal{P}\,.$$

gradient $\nabla_j U \to$ orientation of $\mathcal{P}$

continuous time t flow $f^t(x)$ is reduced to the

discrete time n sequence x_n of successive oriented trajectory traversals of $\mathcal{P}$

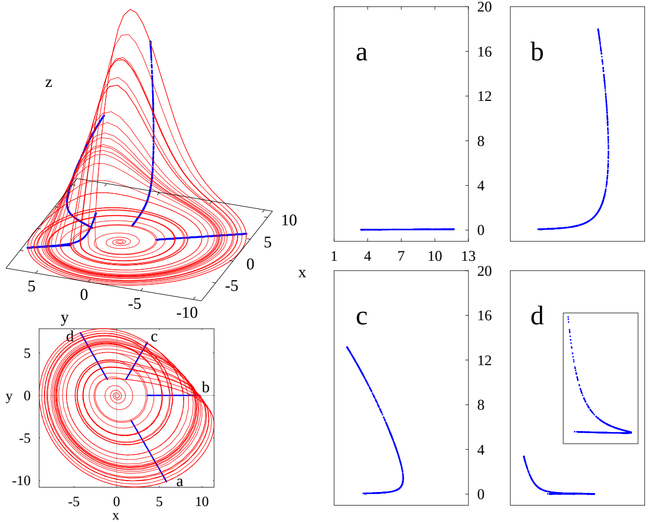
example : two sections of Lorenz flow



(a) $y = x$ section plane $\mathcal{P}$ through the z axis and both $EQ_{1,2}$ equilibria. The second section $\mathcal{P}'$, through $y = -x$ and the z axis includes the EQ_0 equilibrium

(b) sections $\mathcal{P}$ and $\mathcal{P}'$ laid side-by-side. Note the singularity close to EQ_0

example : Poincaré sections, Rössler strange attractor



planes at angles (a) -60^o (b) 0^o, (c) 60^o, (d) 120^o

Rössler stretch and mix

A line segment $[A, B]$ starts close to the x-y plane
stretching (a) $\to$ (b)

flow is expanding

followed by the folding (c) $\to$ (d):
the folded segment returns close to the x-y plane
C from the interior mapped into the outer edge
edge point B lands in the interior

flow is mixing

In one Poincaré return the $[A, B]$ interval is stretched, folded and mapped onto itself

section → section dynamics

The dynamically important transverse dynamics –description of how nearby trajectories attract / repeal each other– is reveled by a section

dynamics along orbits is of secondary (if any) importance

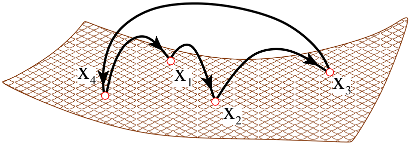
return maps

Poincaré return map

$$x' = P(x) = f^{\tau(x)}(x)\,, \qquad x', x \in \mathcal{P}\,.$$

first return function $\tau(x)$ - time of flight to the next section

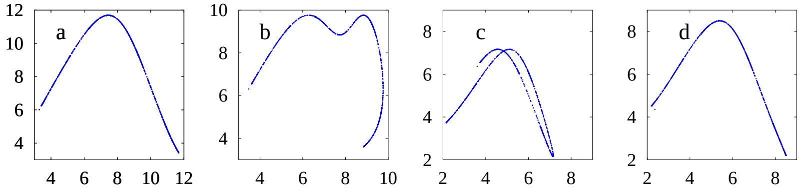
example



A flow $x(t)$ reduced to a Poincaré return map that maps points in the Poincaré section $\mathcal{P}$ as $x_{n+1} = f(x_n)$. In this example the orbit of x_1 is periodic and consists of the four periodic points (x_1, x_2, x_3, x_4)

example : Rössler return maps

Poincaré sections projected onto radial distance $R_n \to R_{n+1}$



(a) and (d) nice 1-to-1 return maps
(b) and (c) appear multimodal and non-invertible artifacts of projections $(R_n, z_n) \to (R_{n+1}, z_{n+1})$ onto a 1-dimensional subspace $R_n \to R_{n+1}$

a strange attractor???

no proof that this - or any attractor of interest - is asymptotically aperiodic - it might well be that what we see is but a long transient on a way to an attractive periodic orbit.

pragmatist: I accept that is a “strange attractor”

discrete time dynamics

beyond Poincaré sections:

there are many settings in which dynamics is inherently discrete, naturally described by repeated iterations of a map

$$f : \mathcal{M} \to \mathcal{M} \,,$$

with time incremented by 1 in each iteration

replacing flows by maps

multinomial approximations

$$P_k(x) = a_k + \sum_{j=1}^{d+1} b_{kj} x_j + \sum_{i,j=1}^{d+1} c_{kij} x_i x_j + \ldots\,, \qquad x \in \mathcal{P}$$

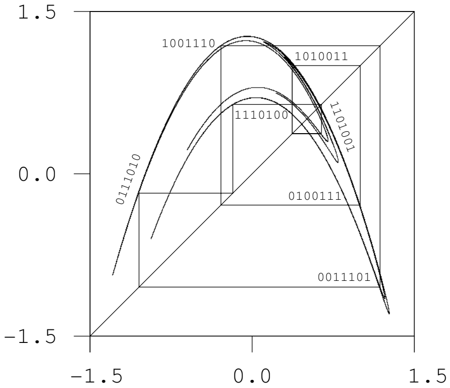
to Poincaré return maps

$$\begin{pmatrix} x_{1,n+1} \\ x_{2,n+1} \\ \ldots \\ x_{d,n+1} \end{pmatrix} = \begin{pmatrix} P_1(x_n) \\ P_2(x_n) \\ \ldots \\ P_d(x_n) \end{pmatrix}, \qquad x_n, x_{n+1} \in \mathcal{P}$$

motivate model mappings such as the Hénon map

example : Hénon map

$$x_{n+1} = 1 - ax_n^2 + by_n$$
$$y_{n+1} = x_n$$



strange attractor and an unstable period 7 cycle of the Hénon map, $a = 1.4$, $b = 0.3$

determinism can be given up

for vanishingly small b the Hénon map $\rightarrow$

parabola:

$$x_{n+1} = 1 - a x_n^2 \, .$$

determinism (or, at least, reversibility) is lost: the inverse of map has two preimages $\{x_{n-1}^{+}, x_{n-1}^{-}\}$ for most x_n

summary

reduction

continuous time flow $\rightarrow$ Poincaré section

is a powerful visualization tool

it is also a fundamental tool of dynamics - to fully unravel the geometry of a chaotic flow, one has to quotient all of its symmetries, and evolution in time is one of these