ChaosBook.org chapter

cycle stability

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periodic orbits are topological invariants

a fixed point remains a fixed point for any choice of coordinates

a periodic orbit remains periodic in any representation of the dynamics

any continuous re-parametrization of a dynamical system preserves its topology and the topological relations between periodic orbits, such as their relative inter-windings and knots. So the mere existence of periodic orbits suffices to partially organize the spatial layout of a non–wandering set.

stability of periodic orbits are metric invariants

No less important: cycle stabilities are *metric* invariants: they determine the relative sizes of neighborhoods in a non–wandering set.

Note: Jacobian matrices multiply, so the Jacobian matrix for the rth repeat of a prime cycle p of period T is

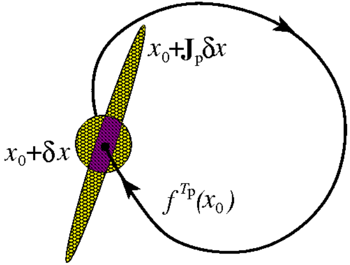
$$J^{rT}(x) = J^T(f^{(r-1)T}(x)) \cdots J^T(f^T(x))J^T(x) = J_p(x)^r \,,$$

where $J_p(x) = J^T(x)$ is the Jacobian matrix for a single traversal of the prime cycle p
$x \in \mathcal{M}_p$ is any point on the cycle
$f^{rT}(x) = x$ as $f^t(x)$ returns to x every multiple of the period T.

it suffices to study the stability of prime cycles

stretch / shrink along a periodic orbit

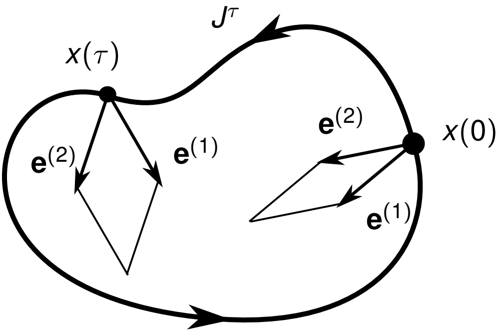
For a prime cycle p, Floquet matrix J_p returns an infinitesimal neighborhood of $x_0 \in \mathcal{M}_p$ stretched and/or shrunk, with overlap ratio along the eigendirection $\mathbf{e}^{(i)}$ of $J_p(x)$ given by the Floquet multiplier $|\Lambda_{p,i}|$



these ratios are invariant under smooth nonlinear reparametrizations of state space coordinates
intrinsic property of cycle p

Floquet eigenframe

the parallelepiped spanned by Floquet unit eigenvectors ('covariant vectors', 'covariant Lyapunov vectors') is transported along the orbit and deformed by Jacobian matrix

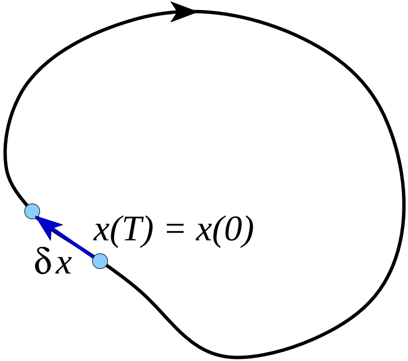


after one period T_p, the eigenframe maps into itself

Jacobian matrix is not self-adjoint
eigenvectors are not orthogonal

Jacobian matrix transports velocity

two points along a periodic orbit p are mapped into themselves after one cycle period T,



hence a longitudinal displacement $\delta x = v(x_0)\delta t$ is mapped into itself by the cycle Jacobian matrix J_p.

Jacobian matrix transports velocity

$J^t(x_0)$ transports the velocity vector

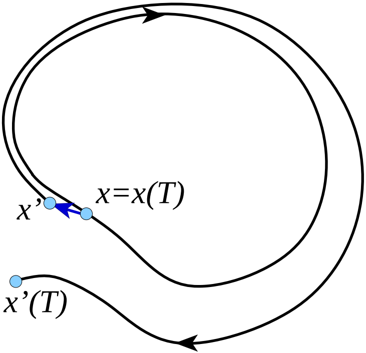
$$v(x(t)) = J^t(x_0)\, v(x_0)$$

For periodic orbit p, $x(T_p) = x(0)$, v is an eigenvector of the Jacobian matrix $J_p = J^{T_p}$ with unit eigenvalue,

$$J_p(x)\, v(x) = v(x)\,, \qquad x \in p$$

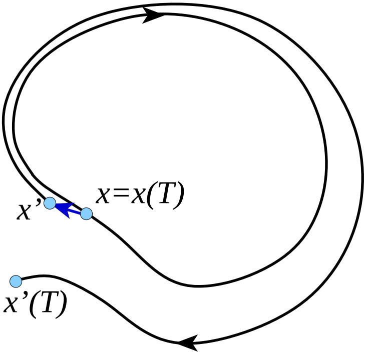
Jacobian matrix for a continuous time periodic orbit always has a marginal stability multiplier $\Lambda_k = 1$

cycle stability



an unstable periodic orbit repels every neighboring trajectory $x'(t)$, except those on its center and stable manifolds

cycle stability



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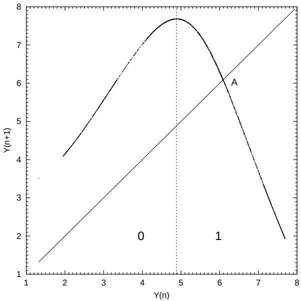
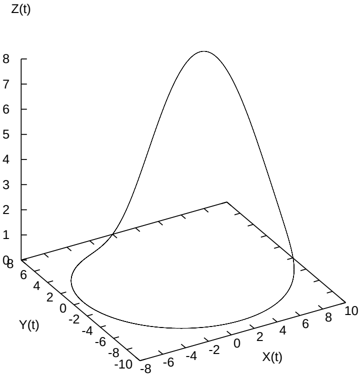
example : Rössler short cycles



(a)

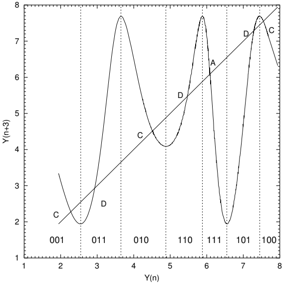


(b)

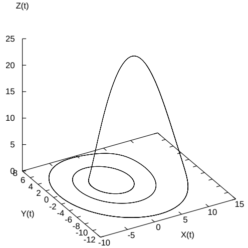
(a) $y \to P_1(y, z)$ return map for $x = 0$, $y > 0$ Poincaré section
(b) the $\overline{1}$-cycle found by Newton-Raphson, taking the fixed point $y_{k+n} = y_k$ as initial guess $(0, y(0), 0)$

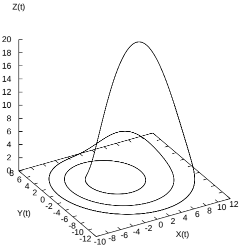
$$\overline{1}\text{-cycle:} \quad T_1 = 5.88108845586$$

$$(\Lambda_{1,e}, \Lambda_{1,m}, \Lambda_{1,c}) = (-2.40395353, 1, -1.29 \times 10^{-14})$$



$y_{k+3} = P_1^3(y_k, z_k)$, the third iterate of Poincaré return map is used to pick starting guesses for the Newton-Raphson searches for the two 3-cycles:





$\overline{001}$ and $\overline{011}$

Résumé

Link to full text