
CHAOS THEORY

chaos
classical and quantum

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available at: ChaosBook.org
cover design: P. Cvitanovic
'Clouds over Croatia'

Chapter 1
Prelude

All you need to know about chaos is contained in the introduction of ChaosBook. However, in order to understand the introduction you will first have to read the rest of the book.

---Gary Morriss

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WHAT IS CHAOS?

Newtonian straightjacket

infinitesimal time:

$$\frac{d}{dt} x_i = v_i(x)$$

long time outcome:

$$x_i(t) = \text{no clue}$$

Navier-Stokes

$$\rho \frac{\partial u_i}{\partial t} + \rho u_j \frac{\partial u_i}{\partial x_j} = \rho X_i - \frac{\partial p}{\partial x_i} + \mu \nabla^2 u_i$$

Einstein

$$R_{ik} - \frac{1}{2} g_{ik} R = \frac{8\pi k}{c^4} T_{ik}$$

$$R^i_{klm} = \frac{\partial \Gamma^i_{km}}{\partial x^l} - \frac{\partial \Gamma^i_{kl}}{\partial x^m} + \Gamma^i_{nl}\Gamma^n_{km} - \Gamma^i_{nm}\Gamma^n_{kl}$$

Yang-Mills

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F_a^{\mu\nu}$$

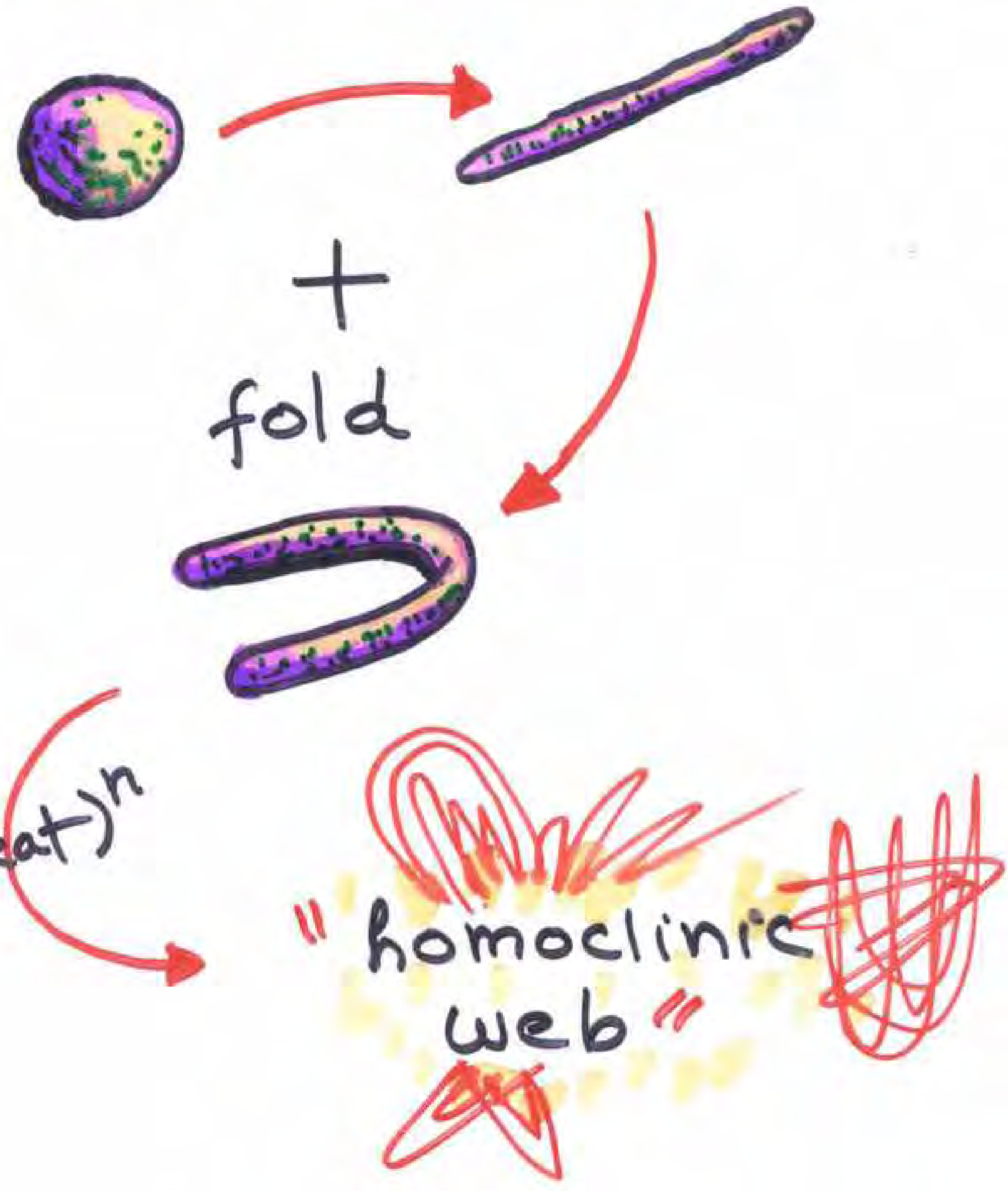
$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\, C_{abc} A^b_\mu A^c_\nu$$

Quantum?

(Poincaré 1882)

Chaos is everywhere

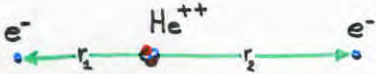
Chaos = stretch



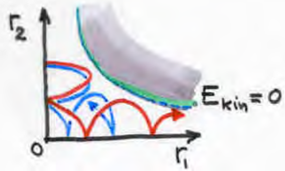
EXAMPLES OF CHAOS

pinball

colinear helium



$$H = \frac{p_1^2}{2} + \frac{p_2^2}{2} - \frac{2}{r_1} - \frac{2}{r_2} + \frac{1}{r_1 + r_2}$$



What is chaos?

if deterministic system is

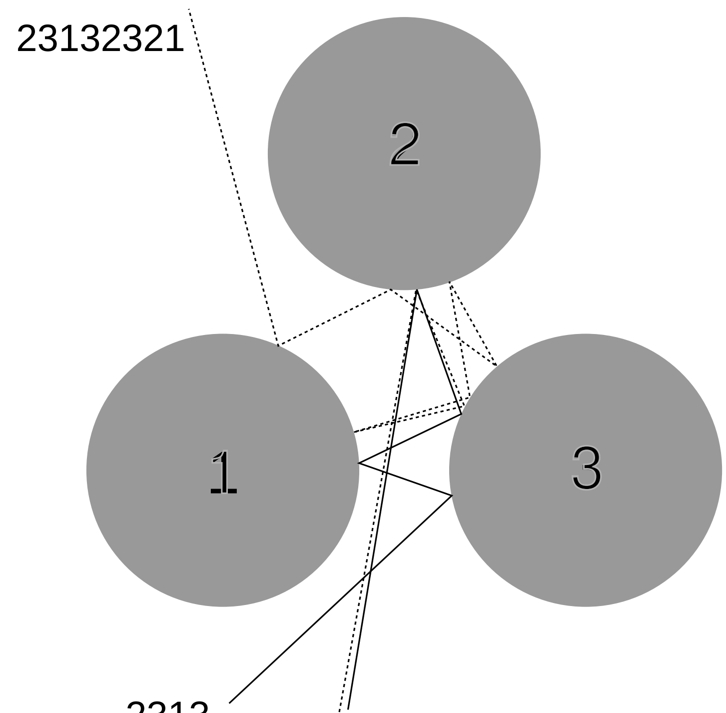
locally unstable + globally mixing

$\lambda > 0$ and $h > 0$

[unmeasurable] [unmeasurable]

definition "chaos = Lyapunov + entropy" useless in practice

Sensitivity to initial conditions

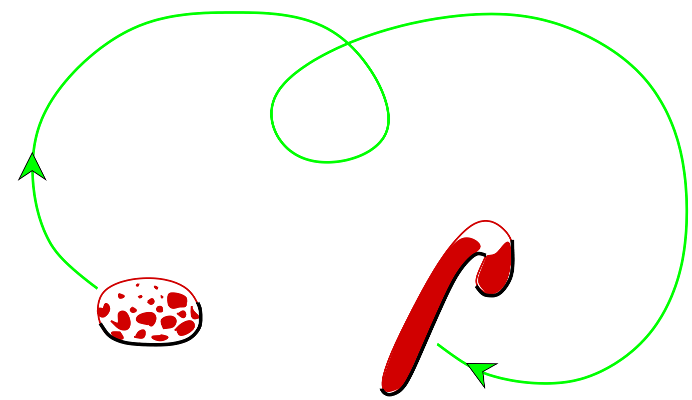


2 trajectories start close, separate exponentially with time

quantified by Lyapunov exponent λ

$$|\delta \mathbf{x}(t)| \approx e^{\lambda t} |\delta \mathbf{x}(0)|$$

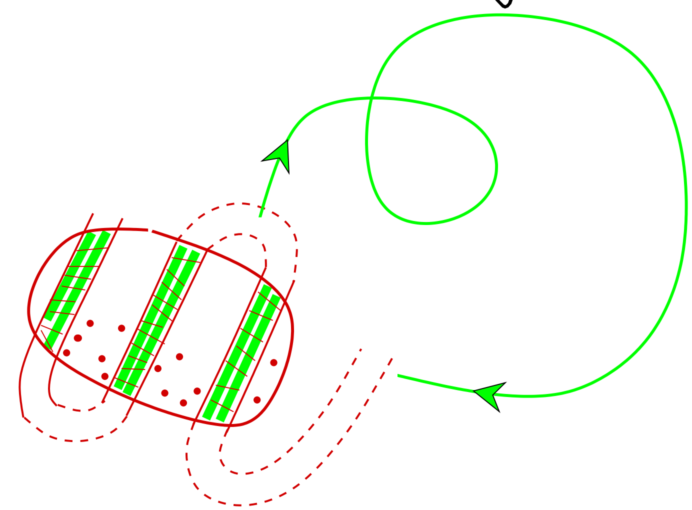
Global mixing



local stretching

topologically distinct

$N(t) \approx e^{ht}$ $\rightarrow$ topological entropy h



global folding

measurement of λ, h

asymptotic $t \rightarrow \infty$

beyond reach

for systems observed in nature.

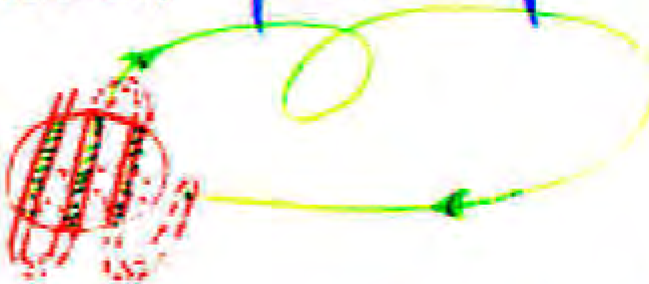
CLASSICAL CHAOS

Poincaré (1890) + ...

1. sensitivity to initial conditions

Lyapunov $\frac{1}{T} \ln \left| \text{stretching factor} \right| > 0$

2. bounded phase space



entropy $\frac{\ln \left| \# \text{ pieces} \right|}{T} > 0$

WHAT IS THE PROBLEM ?

"CHAOS" =

⇒ DETAILED PREDICTION IMPOSSIBLE

BUT!:

"DETERMINISTIC" =

⇒ ASYMPTOTIC GEOMETRY RIGID, ORDERED

HENCE THE GOALS OF "DETERMINISTIC CHAOS" ARE

1. DESCRIBE GEOMETRY
2. EVALUATE MEASURABLE AVERAGES

$\langle \varphi \rangle$, $\langle \varphi(t)\varphi(0) \rangle$ CORRELATIONS, ...

¿ how quickly we lose control ?

¿ how many degrees of freedom are out of control ?

Lyapunov time

initial data accuracy $\delta x = |\delta \mathbf{x}(0)|$

system size L

predictable to a finite Lyapunov time

$$T_{\mathrm{Lyap}} \approx -\frac{1}{\lambda} \ln |\delta x / L| \,,$$

if Lyapunov time $\ll$ the observational time,

chaos matters

successes::

statistical mechanics, quantum mechanics, and questions of long term stability in celestial mechanics.

There are applications -- plumber's turbulent pipes, ... -- where the few important degrees of freedom can be isolated when

$$T_{Lyapunov} \ll T_{observation}$$

Chaos Rules

what to do ?

Lyapunov time for chaotic systems

$\mathbb{J}(t) \Rightarrow \Lambda(t)$ most unstable eigenvalue

$$= 10^{\lambda t}$$

Lyapunov time $= \frac{1}{\lambda} =$ # seconds to lose a digit of accuracy

examples:

Pluto	10^{15} sec
Obliquity of Mars	10^{14} sec
Chemical chaotic oscillator	10^{3} sec
Hydrodynamic chaotic flow	10 sec
cm^3 Argon at room temperature	10^{-10} sec
cm^3 Argon at triple point	10^{-15} sec

Chaos rules

beyond the Lyapunov time

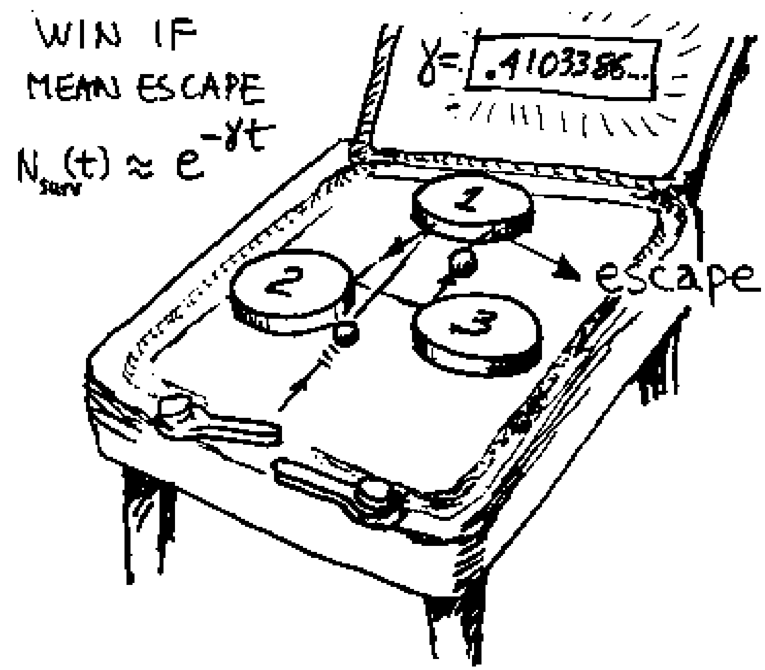
INSERT the overhead

Chaos: what is it good for?

TRANSPORT!

measurable prediction:

escape rate γ



each bounce a fraction lost

fraction of survivors $\sim e^{-\gamma}$.

$\gamma = ?$

ChaosBook: who needs it?

periodic orbit theory:

Take 3-disk radius 1, center-center separation 6, velocity 1

If you know that
$\gamma = 0.410338407769346489338461307 8192 \ldots$

you do not need ChaosBook. If clueless, hang on.

What is chaos?

if deterministic system

no structural stability

$\rightarrow$

``fractal'' transport coeefficients

$$\lambda - h = \gamma$$

[unmeasurable] [measurable]

``escape rate = Lyapunov − entropy'' measurable

.

tabletop experiment: measure macroscpic transport

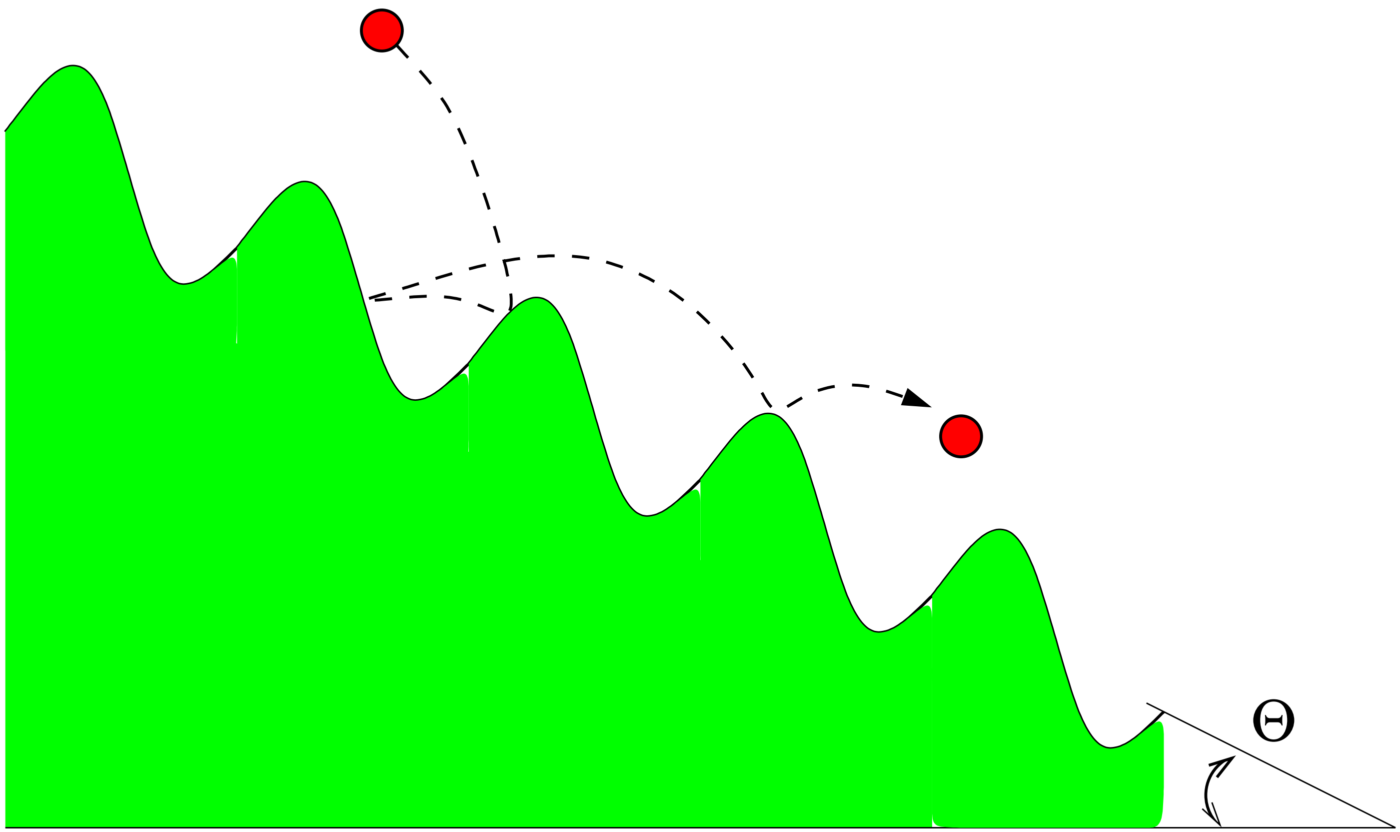
diffusion, conductance, drag

observe thus determinism on nanoscales

Chaos: what is it good for?

TRANSPORT!

measurable prediction: washboard mean velocity



artwork: Y. Lan

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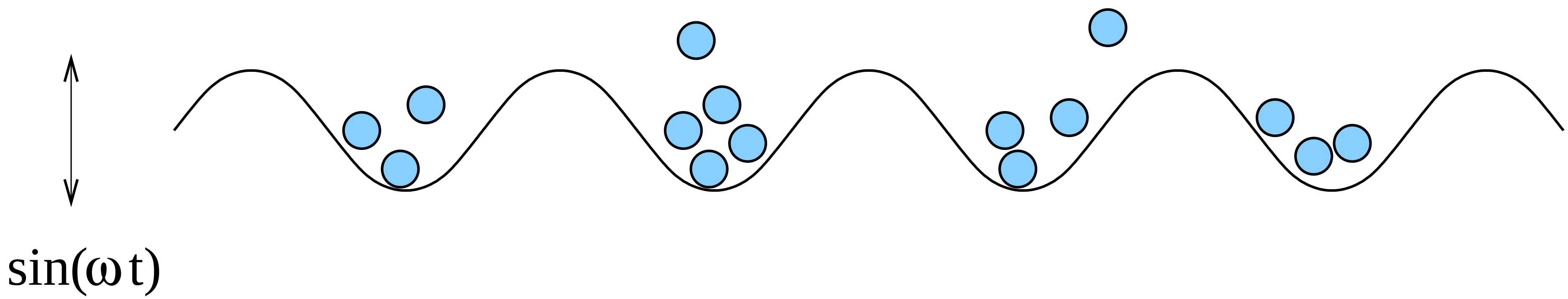
Cold atom lattice



artwork: Y. Lan

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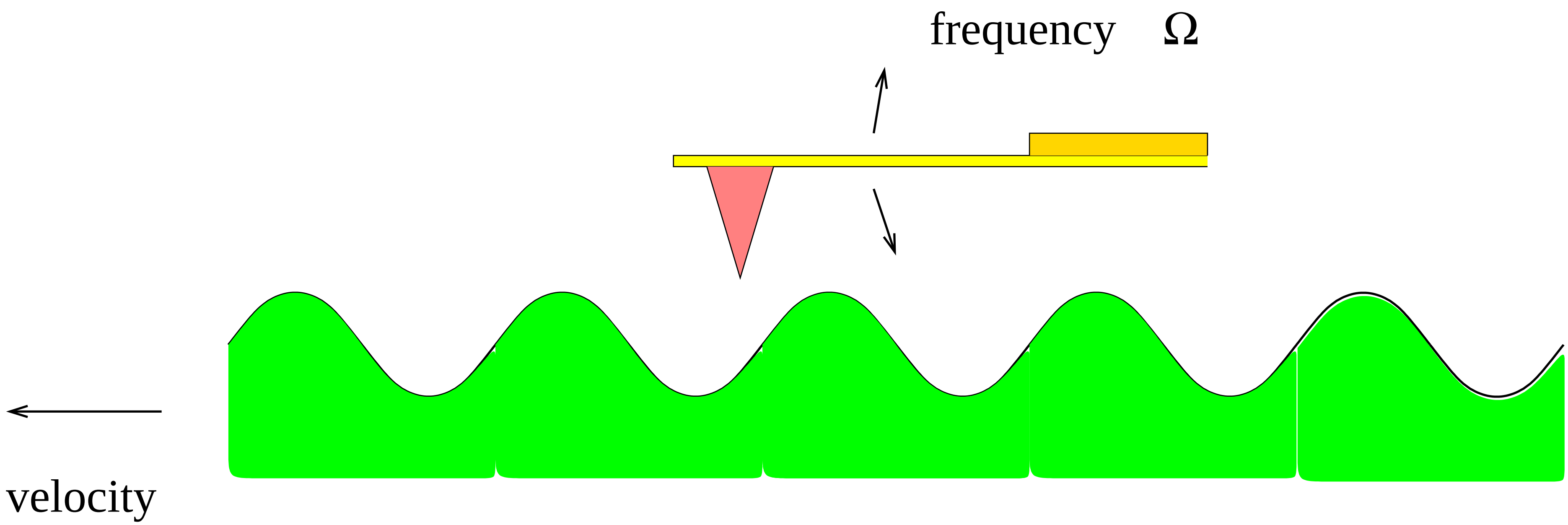
AFM tip drag force



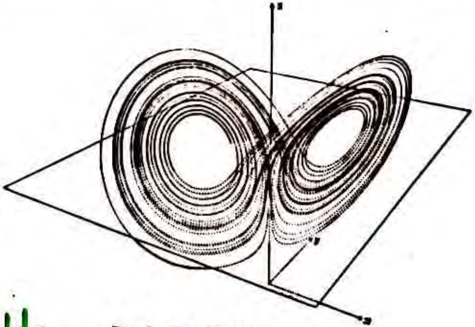
artwork: Y. Lan

H. Poincaré, describing in
‘‘Les méthodes nouvelles de la méchanique méleste’’ his discovery of homoclinic tangles:

The complexity of this figure will be striking, and I shall not even try to draw it.

LORENZ 1963

METEOROLOGY



"Raynolds no."

Рис. 2. $\dot{x} = -10x + 10y$
$\dot{y} = 28x - y - xz$
$\dot{z} = -\frac{8}{3}z + xy$

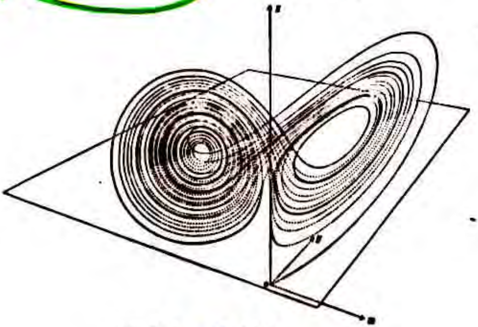
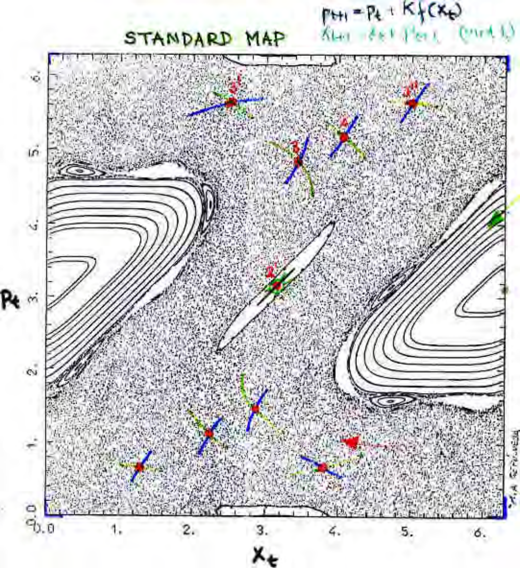


Рис. 3. $\dot{x} = -10x + 10y$
$\dot{y} = 40x - y - xz$
$\dot{z} = -\frac{8}{3}z + xy$

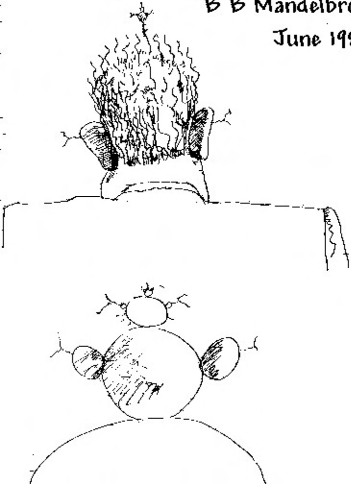
WEATHER: UNPREDICTABLE

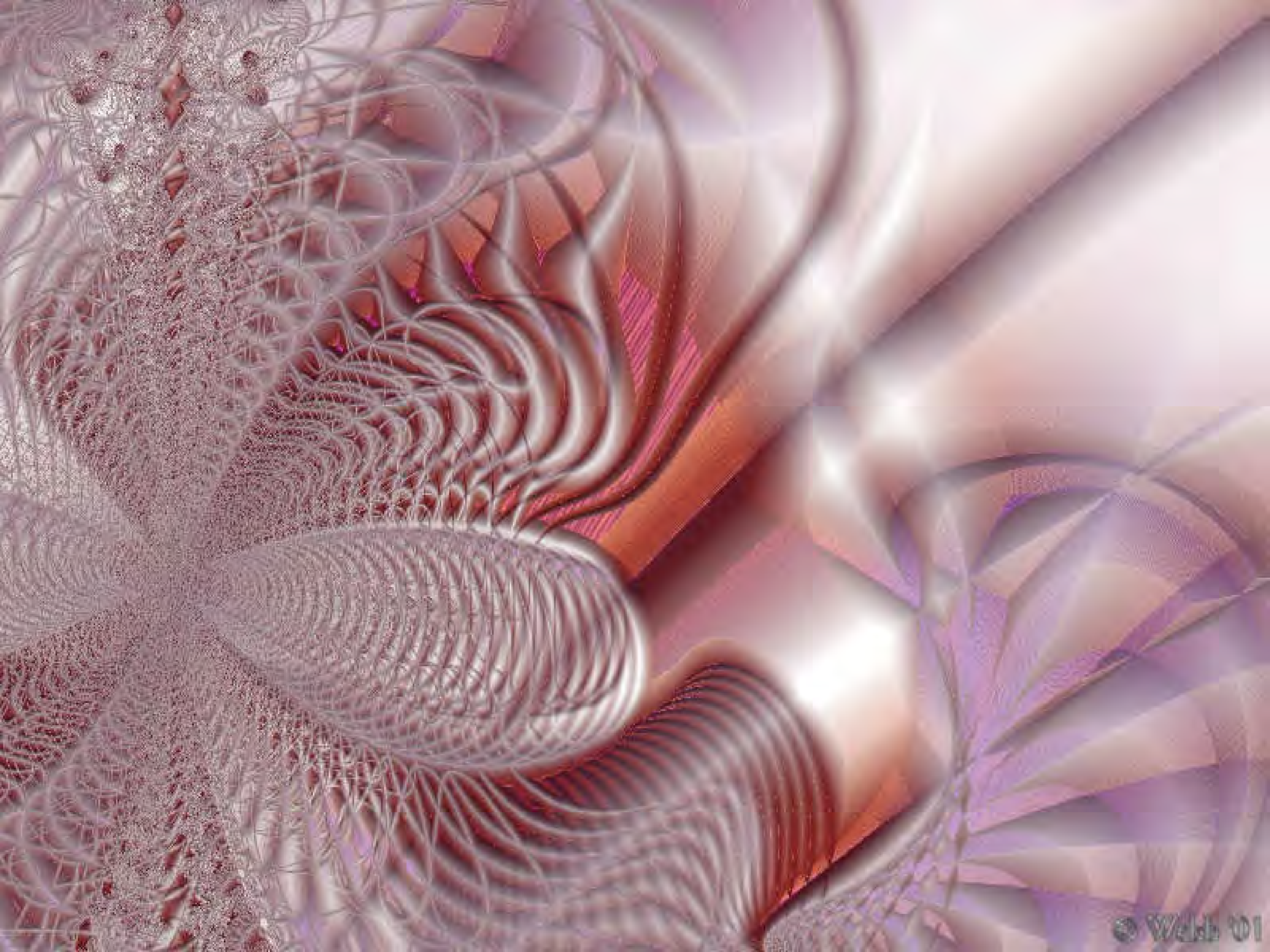


STANDARD MAP
$p_{t+1} = p_t + K f(x_t)$
$x_{t+1} = x_t + p_{t+1}$ (mod 1)
p_t
x_t
0.0
1.
2.
3.
4.
5.
6.


Google™
$z_{n+1} = z^2 + c$

B B Mandelbrot,
June 1984

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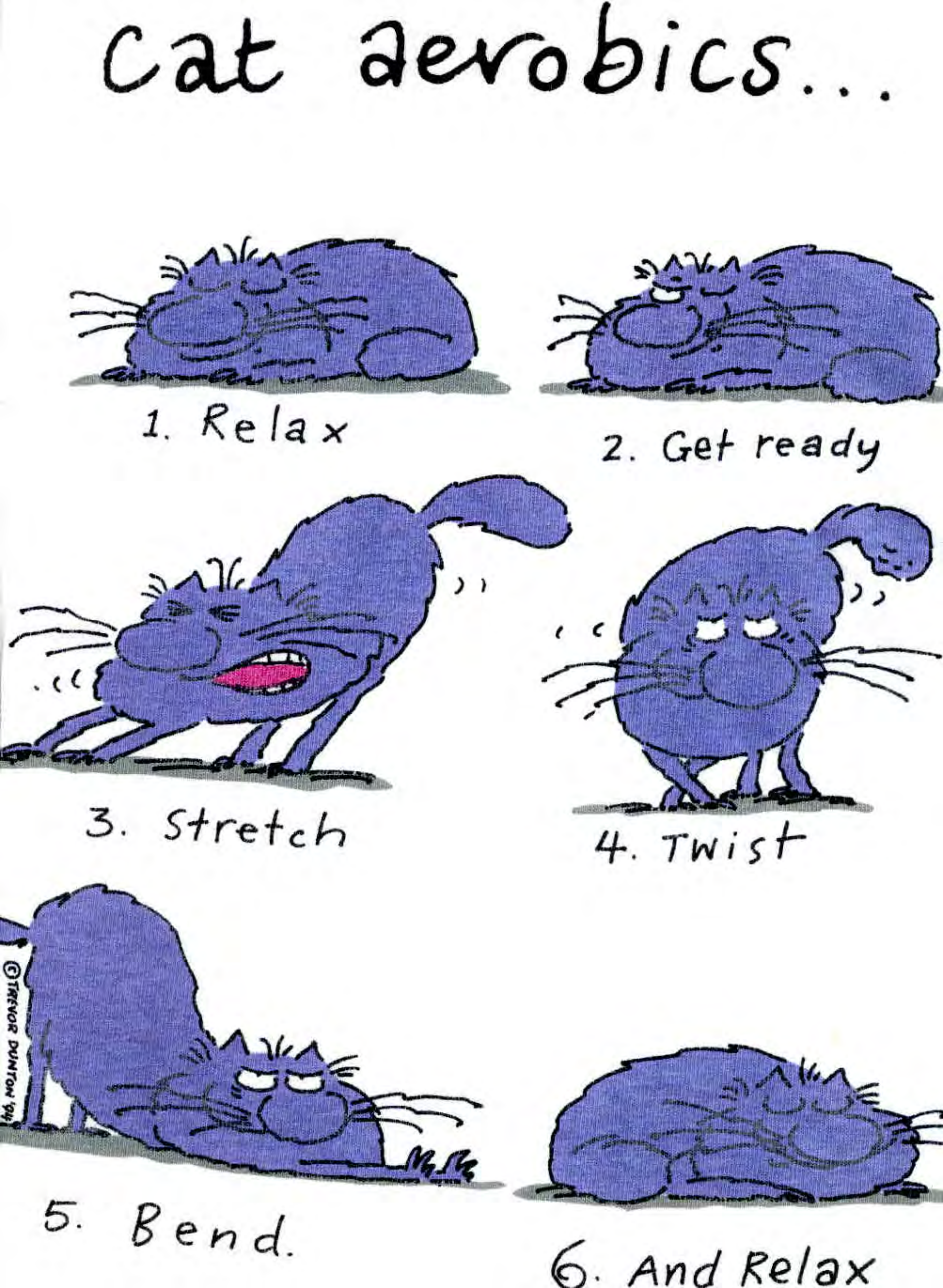
(Ruelle '71)

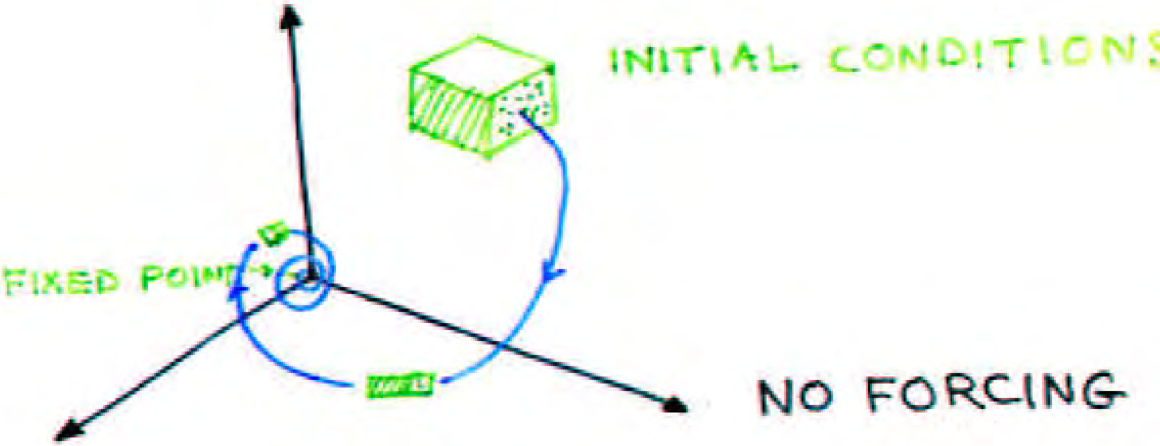
Strange attractors

3 degrees of freedom suffice for (weak) turbulence

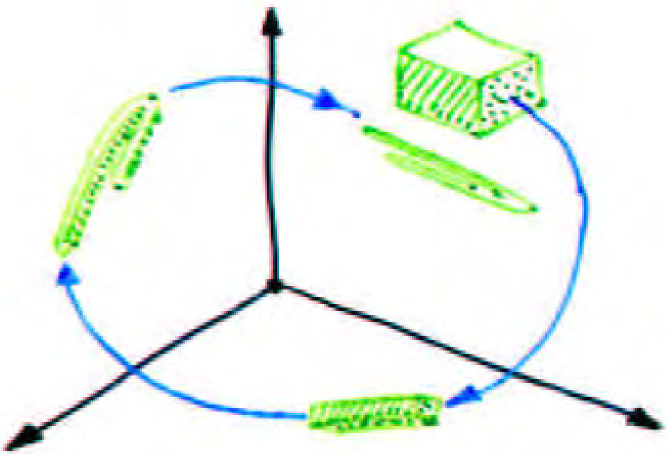
(do not need zillions of parts to make world complicated)


Cat aerobics...
1. Relax
2. Get ready
3. Stretch
4. Twist
5. Bend.
6. And Relax

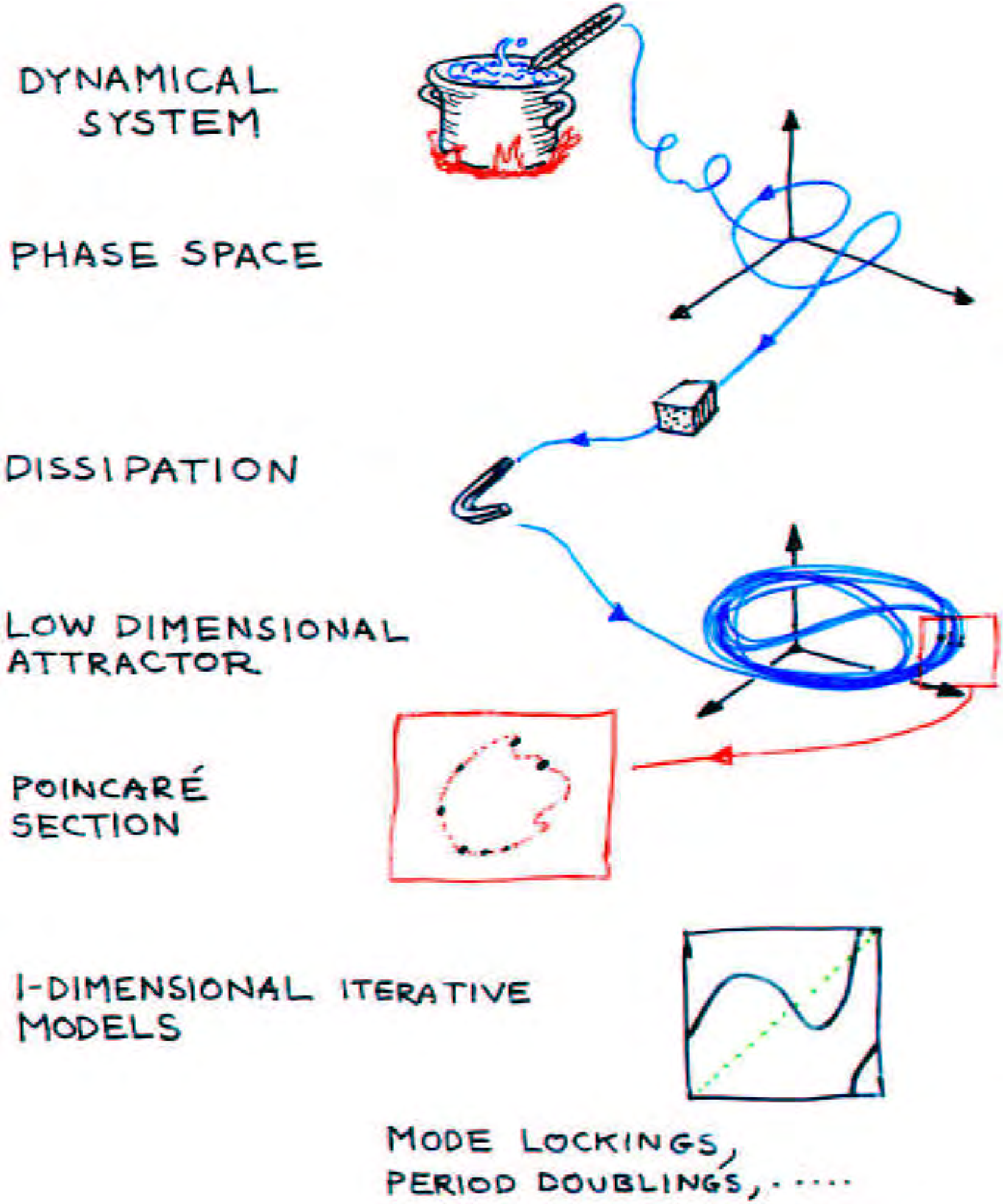
DISSIPATION
⇒ LOW DIMENSIONAL ATTRACTOR



NO FORCING



FORCING


DYNAMICAL SYSTEM
PHASE SPACE
DISSIPATION
LOW DIMENSIONAL ATTRACTOR
POINCARÉ SECTION
1-DIMENSIONAL ITERATIVE MODELS
MODE LOCKINGS, PERIOD DOUBLINGS, ...

Shmurblence

Shear Boundary Moderate Reynolds Turbulence

``turbulence:'':

refers to complex motions of fluids (∞-dimensional dynamical systems) which we understand poorly.

As soon as a phenomenon is understood better, it is renamed: ``a route to chaos'', ``spatio-temporal chaos'', $\cdots$

Is this a cloud?

Flame front flutter

Bunsen burner

1

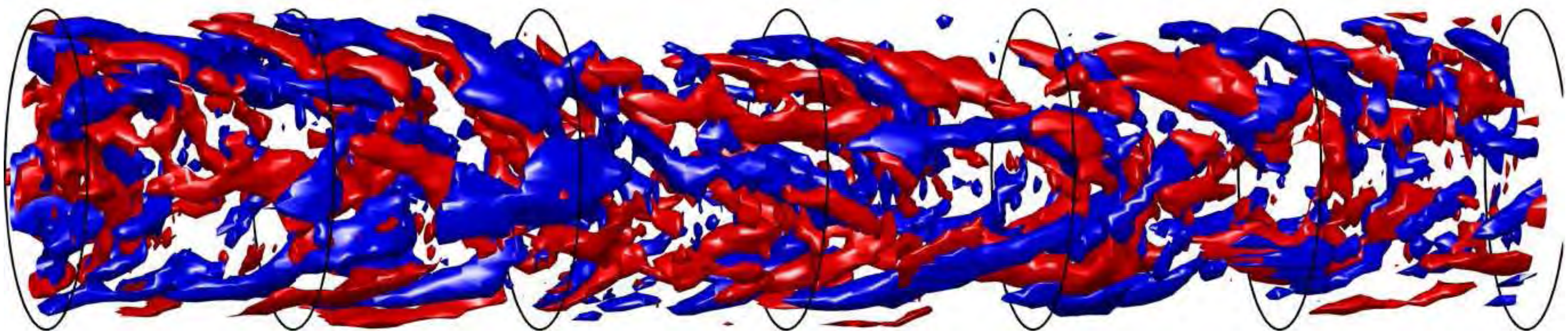
Q : 1-d turbulence –

flutter of a flame front?

[1] R.W. Bunsen (1811–1899), Doctorate U. Göttingen, age 19

Plumbing: Unstable Coherent Structures

Stereoscopic Particle Image Velocimetry → 3-d velocity field over the entire pipe[2]



Observed structures resemble numerically computed traveling waves

What lies beyond?

[2] Casimir W.H. van Doorne (PhD thesis, Delft 2004); Hof et al., Science (Sep 10, 2004)

Turbulence: A walk through a repertoire of unstable recurrent patterns?

As a turbulent flow evolves, every so often we catch a glimpse of a familiar pattern:

$\Longrightarrow$ other swirls $\Longrightarrow$

For any finite spatial resolution, the system follows approximately for a finite time a pattern belonging to a finite alphabet of admissible patterns. The long term dynamics = a walk through the space of such unstable patterns.

Résumé

deterministic chaos --

-- is defined by

long time dynamics are the notions of local instability (positive Lyapunov) and of recurrence (positive entropy)

WHY IS THIS DIFFERENT FROM THE TRADITIONAL PHYSICS?

GIVE UP ON:

INITIAL CONDITIONS + $\frac{d\vec{x}}{dt} = \vec{f}(x, ..)$

$\Longrightarrow$ DESCRIPTION OF THE SYSTEM

INSTEAD:

EXTREMELY NON-LINEAR PHYSICS DESCRIBED BY SELF-SIMILARITY EQUATIONS

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Future looks bright