ChaosBook.org chapter

go with the flow

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dynamical systems

state space (often: phase space)

$\mathcal{M} \in \mathbb{R}^d$: d numbers determine the state of the system

$\mathcal{M}$ is a manifold - a torus, a cylinder, $\cdots$

representative point

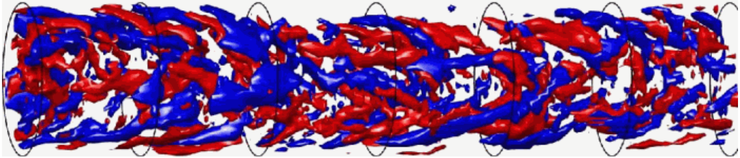
$x(t) \in \mathcal{M}$: a state of physical system at instant in time

today's experiments

example of a representative point

$x(t) \in \mathcal{M}$, $d = \infty$
a state of turbulent pipe flow at instant in time

Stereoscopic Particle Image Velocimetry $\rightarrow$ 3-*d* velocity field over the entire pipe[1]

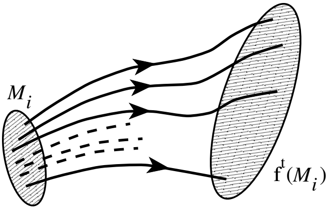


[1] Casimir W.H. van Doorne (PhD thesis, Delft 2004)

dynamics

map $f^t(x_0)$ = representative point time t later

evolution rule



f^t maps a region $\mathcal{M}_i$ of the state space into the region $f^t(\mathcal{M}_i)$.

smooth dynamical system

f^t can be differentiated as many times as needed

deterministic dynamics

evolution rule f maps a point into exactly one point at time t

dynamical system

the pair $(\mathcal{M}, f)$

dynamical systems

flow

evolution in continuous time $t \in \mathbb{R}$:

iteration of a map

$$x_{n+1} = f(x_n)$$

evolution advances in discrete time steps, integer time $n \in \mathbb{Z}$

flows

for infinitesimal times, flows can be defined by differential equations - a generalized vector field

$$v(x) = \dot{x}(t)$$

examples

Newton's laws for a mechanical system

general flows, mechanical or not, defined by a time-independent vector field $v(x)$

devil is in the details

fluid dynamics

have equations: can compute the trajectories

Navier-Stokes

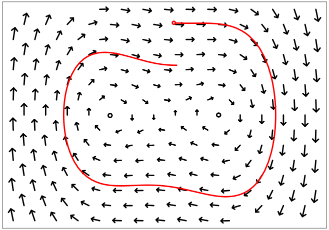
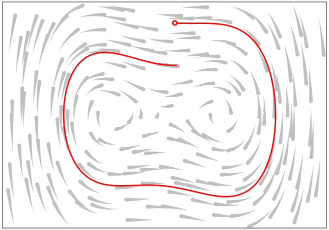
$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = \frac{1}{R}\nabla^2 \mathbf{v} - \nabla p + \mathbf{f}, \qquad \nabla \cdot \mathbf{v} = 0,$$

velocity field $\mathbf{v} \in \mathbb{R}^3$; pressure field p ; driving force $\mathbf{f}$

example: unforced Duffing system

a two-dimensional vector field $v(x)$

$$
\begin{aligned}
\dot{x}(t) &= y(t) \\
\dot{y}(t) &= -0.15\, y(t) + x(t) - x(t)^3
\end{aligned}
$$

(a)  (b) 

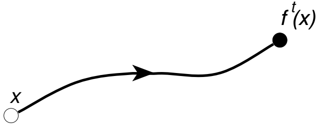
two visualization of the velocity vector field $v(x)$, superimposed over the configuration coordinates $(x(t), y(t)) \in \mathcal{M}$

velocity field belongs to a different space,

the tangent bundle $\mathbf{T}\mathcal{M}$

orbit vs. trajectory

trajectory



curve $x(t) = f^t(x)$ through the point x traced out by the evolution rule f^t

after a time t the point is at $f^t(x)$

orbit of x_0

subset in $\mathcal{M}$ of points reached by the (possibly infinite) trajectory of x_0

For a flow, an orbit is a continuous curve; for a map, it is a sequence of points

orbit is a dynamically invariant set

orbit, or a solution of x_0:

subset of points $\mathcal{M}_{x_0} \subset \mathcal{M}$ that belong to the infinite-time trajectory of a given point x_0

$\mathcal{M}_{x_0}$ is a dynamically invariant set, the totality of states that can be reached from x_0, with the full state space $\mathcal{M}$ foliated into a union of such orbits

we label a generic orbit $\mathcal{M}_{x_0}$ by a point belonging to it, for example $x_0 = x(0)$

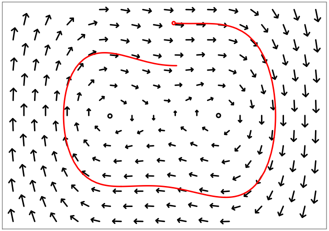
possible trajectories?

stationary:	$f^t(x) = x$	for all t
periodic:	$f^t(x) = f^{t+T_p}(x)$	for a given minimum period T_p
aperiodic:	$f^t(x) \neq f^{t'}(x)$	for all $t \neq t'$.

example : Duffing flow equilibria

x_q is an equilibrium point

$$\text{if} \qquad v(x_q) = 0$$



Duffing flow is bit of a bore: every trajectory ends up in one of the two attractive equilibrium points

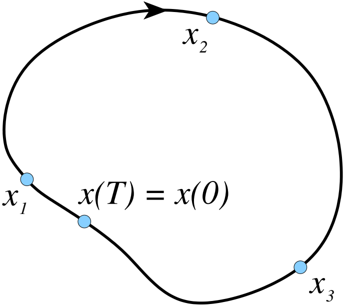
periodic orbits

periodic point

returns to the initial point after a finite time, $x = f^{T_p}(x)$

periodic orbit

p is the set of periodic points $p = \mathcal{M}_p = \{x_1, x_2, \cdots\}$ swept out by the trajectory of any one of them in the finite time T_p



periodic orbits - a very small subset of the state space, in the same sense that rational numbers are a set of zero measure on the unit interval generic orbit might be ergodic, unstable and essentially uncontrollable

ChaosBook strategy : "geometry of chaos"

populate the state space by a hierarchy of compact invariant sets (equilibria, periodic orbits, invariant tori, ...), each computable in a finite time

together with their invariant stable/unstable manifolds

orbits which are compact invariant sets we label by alphabet convenient in a particular application:

- $EQ = x_{EQ} = \mathcal{M}_{EQ}$ for an equilibrium
- $p = \mathcal{M}_p$ for a periodic orbit
- $\mathcal{M}_T$ for an invariant torus
- $\cdots$

close recurrences

for a generic dynamical system most motions are aperiodic, so give up on exact periodicity, consider instead close recurrences

non–wandering set

point x is recurrent or non-wandering if for any open neighborhood $\mathcal{M}_0$ of x there exists a later time t such that

$$f^t(x) \in \mathcal{M}_0$$

Ω : the non–wandering set of f, the union of all the non-wandering points of $\mathcal{M}$

Ω is the key to understanding the long-time behavior of a dynamical system

attractor

if there exists a connected state space volume that maps into itself under forward evolution, the flow is globally contracting onto a subset of $\mathcal{M}$, the attractor

there can coexist any number of distinct attracting sets, each with its own basin of attraction, the set of all points that fall into the attractor under forward evolution

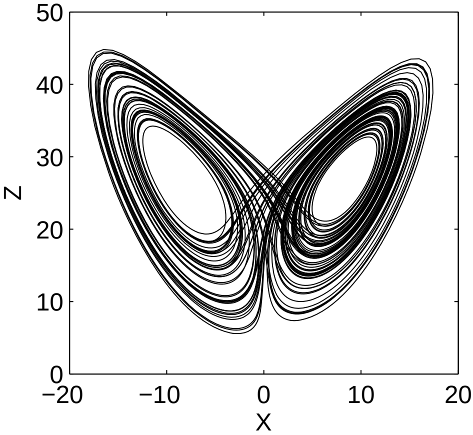
the attractor can be

- a fixed point
- a periodic orbit
- aperiodic
- or any combination of the above

strange attractor

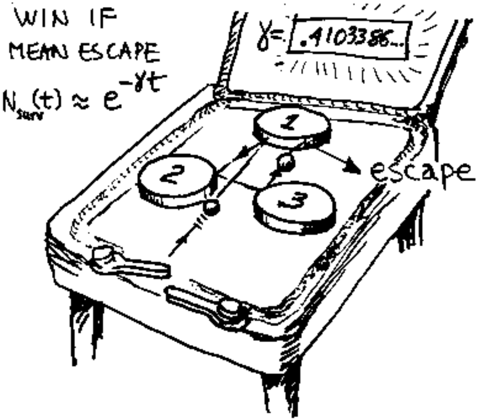
the most interesting case is that of an aperiodic recurrent attractor, to which we shall refer loosely as a strange attractor

example : Lorenz strange attractor



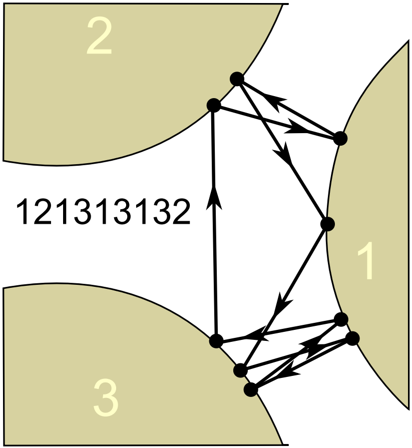
Edward Lorenz: the weather will remain unpredictable conversely, if we can enclose the non–wandering set Ω by a connected state space volume $\mathcal{M}_0$ and then show that almost all points within $\mathcal{M}_0$, but not in Ω, eventually exit $\mathcal{M}_0$, we refer to the non–wandering set Ω as a repeller

example : game of pinball



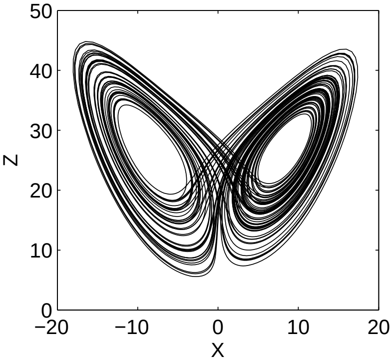
Pinball Wizard: no pinball bounces forever

periodic orbit: a perfect pinball shot



Pinball Wizard: cannot make it, it is an unstable solution

example : Lorenz "butterfly" strange attractor



$$\dot{x} = v(x) = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} \sigma(y - x) \\ \rho x - y - xz \\ xy - bz \end{bmatrix}$$

Lorenz fixed $\sigma = 10$, $b = 8/3$, varied the "Rayleigh number" ρ

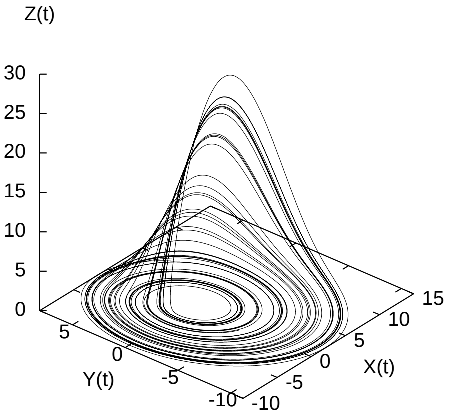
example : Rössler flow

Rössler flow

$$\begin{aligned} \dot{x} &= -y - z \\ \dot{y} &= x + ay \\ \dot{z} &= b + z(x - c)\,, \end{aligned}$$

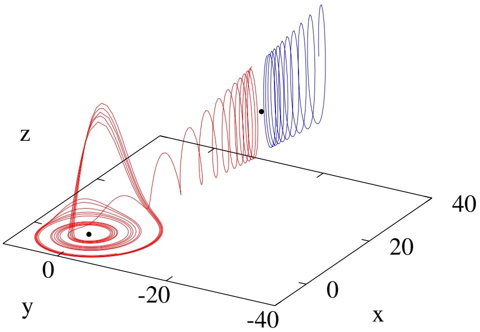
$$a = b = 0.2\,, \quad c = 5.7$$

typical numerically integrated long-time trajectory



equilibria of Rössler flow

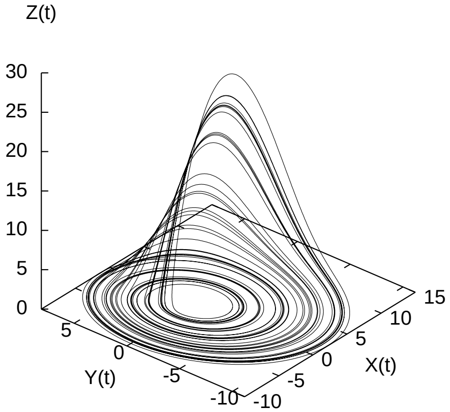
two trajectories of the Rössler flow initiated in the neighborhood of the "upper" equilibrium point



2 repelling equilibrium points (no dynamics there!)

$$(x^-, y^-, z^-) = (\,0.0070,\ -0.0351,\ 0.0351\,)$$
$$(x^+, y^+, z^+) = (\,5.6929,\ -28.464,\ 28.464\,)$$

a strange attractor?



a trajectory of the Rössler flow up to time $t = 250$

trajectories that start out sufficiently close to the origin seem to converge to a strange attractor

Charles Babbage:

“On two occasions I have been asked
[by members of Parliament],

’Pray, Mr. Babbage, if you put into the machine wrong figures,
will the right answers come out?’

I am not able rightly to apprehend the kind of confusion of ideas
that could provoke such a question.”

Résumé

a dynamical system – a flow, or an iterated map – is defined by specifying

a pair

$$(\mathcal{M}, f)$$

where $\mathcal{M}$ is the state space, and $f : \mathcal{M} \to \mathcal{M}$

the key concepts in exploration of the long time dynamics are the notions of

- recurrence and of
- non–wandering set of f, the union of all the non-wandering points of $\mathcal{M}$