ChaosBook.org chapter

world in a mirror

20 June 2011, version 13.4

symmetries are beautiful
our hymn to symmetry is a symphony in two movements:

1. symmetry induced interrelations amongst individual orbits
   - symmetry equates multiplets of equivalent orbits.
   - group operations that relate state space tiles do double duty as letters of a symbolic dynamics alphabet
2. symmetries affect global densities of trajectories, the factorization of spectral determinants

coordinate transformations transformation

$$x \to M\,x$$

maps the vector $x \in \mathcal{M}$ into $M\,x \in \mathcal{M}$. Transformation

$$f(x) \to M^{-1} f(x)$$

changes the coordinate system with respect to which the map $f(x) \in \mathcal{M}$ is measured

together, they yield

map in the transformed coordinates

$$\hat{f}(x) = M^{-1} f(M\,x)$$

symmetry of dynamics dynamical system $(\mathcal{M}, f)$ is invariant under transformation g if

the “law of motion" is invariant:

$$f(x) = g^{-1} f(gx)\,,$$

dynamics retains its form in the transformed coordinate frame for any state $x \in \mathcal{M}$

symmetry groups

finite group consists of a set of elements

$$G = \{e, g_2, \ldots, g_n\}$$

and a group multiplication rule $g_j \circ g_i$ satisfying

- closure: if $g_i, g_j \in G$, then $g_j \circ g_i \in G$
- associativity: $g_k \circ (g_j \circ g_i) = (g_k \circ g_j) \circ g_i$
- identity e: $g \circ e = e \circ g = g$ for all $g \in G$
- inverse g^{-1}: for every $g \in G$, there exists a unique element $h = g^{-1} \in G$ such that $h \circ g = g \circ h = e$.

$|G| = n$, the number of elements, is the order of the group symmetry of a dynamical system group is a symmetry of a dynamics, or, dynamical system $(\mathcal{M}, f)$ is invariant / G-equivariant under a symmetry group G if

- for solution $f(x) \in \mathcal{M}$, $y = gf(x)$ is also a solution
- the "equations of motion" $f : \mathcal{M} \to \mathcal{M}$ (a discrete time map f, or the continuous flow f^t) commute with all actions of G,

$$f(gx) = gf(x)$$

- the "law of motion" retains its form in symmetry-transformed coordinate frame

$$f(x) = g^{-1} f(gx)$$

for any state $x \in \mathcal{M}$ and any finite non-singular $[d \times d]$ matrix representation g of element $g \in G$

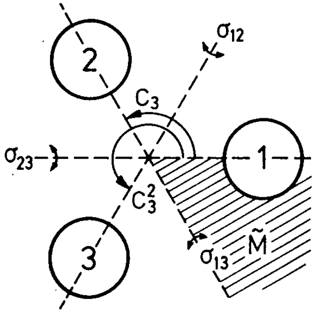
why "equivariant"? scalar function $h(x)$ is

G-invariant:

$$\text{if } h(x) = h(gx) \text{ for all } g \in G$$

map $f : \mathcal{M} \to \mathcal{M}$ maps vector into a different vector, hence a slightly different invariance condition $f(x) = g^{-1} f(gx)$

it is obvious from the context, but for verbal emphasis mathematicians like to distinguish the two cases by in/equi-variant example : 3-disk pinball symmetry is D_3



6 symmetries of three disks on an equilateral triangle. The fundamental domain is indicated by the shaded wedge

example : discrete symmetries of 3-dimensional flows
3-dimensional flows three types of discrete symmetry groups of order 2 can arise:

$$\begin{aligned} \text{reflections: } \sigma(x, y, z) &= (x, y, -z) \\ \text{rotations: } R(x, y, z) &= (-x, -y, z) \\ \text{inversions: } P(x, y, z) &= (-x, -y, -z) \end{aligned}$$

example : 1-d map with D_1 symmetry 2-element symmetry group $\mathrm{D}_1 = C_2 = \{e, R\}$ generated by a single reflection

$$f(-x) = -f(x)$$

Symmetry: if $\{x_n\}$ is a trajectory, than also $\{Rx_n\}$ is a trajectory because

$$Rx_{n+1} = Rf(x_n) = f(Rx_n)$$

group orbit

group orbit of $x \in \mathcal{M}$:

the set of points $g\,x$ generated by all actions $g \in G$

If G is a symmetry, intrinsic properties of an equilibrium (such as Floquet exponents) or a cycle p (period, Floquet multipliers) and its image under a symmetry transformation $g \in G$ are equal

a symmetry thus reduces the number of dynamically distinct solutions $\mathcal{M}_{x_0}$ of the system. So we also need to determine the symmetry of a solution, as opposed to the symmetry of the system generic orbit is complicated... solutions of an equivariant system can satisfy all of the system's symmetries, a subgroup of them, or have no symmetry at all

for a generic ergodic orbit $f^t(x)$ the trajectory and any of its images under action of $g \in G$ are distinct with probability one, $f^t(x) \cap gf^{t'}(x) = \emptyset$ for all t, t'. For compact invariant sets, such as fixed points and periodic orbits, especially the short ones, the situation is very different symmetry of a solution

definition: symmetry of a solution

let $p = \mathcal{M}_p \subset \mathcal{M}$ be an orbit of the system. The set of group actions $G_p \subseteq G$ which maps the orbit into itself,

$$G_p = \{g \in G : g\mathcal{M}_p = \mathcal{M}_p\} ,$$

is called the symmetry of the orbit $\mathcal{M}_p$

we shall denote by subgroup $G_p \subset G$ the maximal symmetry group of $\mathcal{M}_p$. For a discrete subgroup

$$G_p = \{e, b_2, b_3, \ldots, b_h\} \subseteq G ,$$

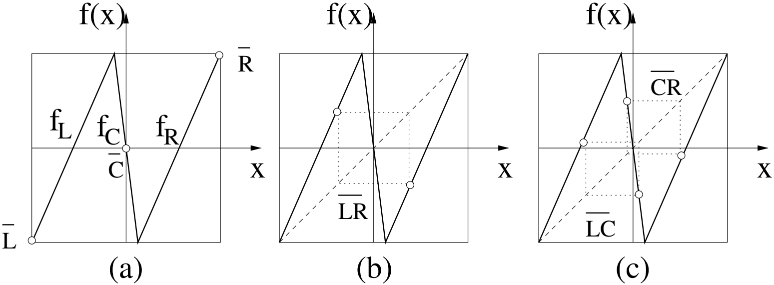
of order $h = |G_p|$, group elements map orbit points into orbit points reached at different times types of solution symmetries

(i) fully asymmetric a

(ii) G_p set-wise invariant cycles s built by repeats of relative cycle segments $\tilde{s}$

(iii) subgroup G_{EQ}-invariant equilibria or point-wise G_p-fixed cycles b

example : symmetries of 1-d sawtooth map orbits reflection symmetry group $\mathrm{D}_1 = C_2 = \{e, R\}$, $\qquad f(-x) = -f(x)$



(a) boundary fixed point $\overline{C}$, asymmetric fixed points pair $\{\overline{L}, \overline{R}\}$
(b) symmetric 2-cycle $\overline{LR}$
(c) asymmetric 2-cycles pair $\{\overline{LC}, \overline{CR}\}$

Asymmetric cycles: R generates a reflection of the orbit with the same number of points and the same stability properties

Symmetric cycles: A cycle s is symmetric (or self-dual) if operating with R on the set of cycle points reproduces the set. The period of a symmetric cycle is even ($n_s = 2n_{\tilde{s}}$), and the mirror image of the x_s cycle point is reached by traversing the irreducible segment $\tilde{s}$ (relative periodic orbit) of length $n_{\tilde{s}}$, $f^{n_{\tilde{s}}}(x_s) = Rx_s$

Boundary cycles: In the example at hand there is only one cycle which is neither symmetric nor antisymmetric, but lies on the boundary Fix(G): the fixed point $\overline{C}$ at the origin.

cosets Let $H = \{e, b_2, b_3, \ldots, b_h\} \subseteq G$ be a subgroup of order $h = |H|$. The set of h elements $\{c, cb_2, cb_3, \ldots, cb_h\}$, $c \in G$ but not in H, is called left coset cH. For a given subgroup H the group elements are partitioned into H and $m - 1$ cosets, where $m = |G|/|H|$

multiplicity of a solution If G_p is the symmetry group of orbit $\mathcal{M}_p$, elements of the coset space $g \in G/G_p$ generate the $m-1$ distinct copies of $\mathcal{M}_p$, so for discrete groups the multiplicity of an equilibrium or a cycle p is $m_p = |G|/|G_p|$.

classes $b \in G$ is conjugate to a if $b = c\,a\,c^{-1}$ where c is some other group element. If b and c are both conjugate to a, they are conjugate to each other

classes

application of all conjugations separates the set of group elements into mutually not-conjugate subsets called classes

identity e is always in the class $\{e\}$ of its own. This is the only class which is a subgroup, all other classes lack the identity element classes importance of classes follows from the way coordinate transformations act on mappings: action of elements of a class (say reflections, or discrete rotations) is equivalent up to a redefinition of the coordinate frame

we saw above that splitting of a group G into an solution symmetry group G_p and $m-1$ cosets cG_p relates a solution $\mathcal{M}_p$ to $m-1$ other distinct solutions $c\mathcal{M}_p$

all of them have equivalent symmetries: the symmetry of orbit $c\,p$ is conjugate to the p symmetry subgroup, $G_{c\,p} = c\,G_p\,c^{-1}$

next step is the key step:

if a set of solutions is equivalent by symmetry (a circle of equilibria, let's say), we would like to represent it by a single solution (pick a representative point on the circle).

definition: Invariant subgroup

A subgroup $H \subseteq G$ is an invariant subgroup or normal divisor if it consists of complete classes. Class is complete if no conjugation takes an element of the class out of H.

H divides G into H and $m-1$ cosets, each of order $|H|$. Think of action of H within each subset as identifying its $|H|$ elements as equivalent. This leads to the notion of G/H as the factor group or quotient group G/H of G, with respect to the normal divisor (or invariant subgroup) H. Its order is $m = |G|/|H|$, and its multiplication table can be worked out from the G multiplication table class by class, with the subgroup H playing the role of identity. G/H is homeomorphic to G, with $|H|$ elements in a class of G represented by a single element in G/H

so far we have discussed the structure of a group as an abstract entity. Now we switch gears to what we really need this for: describe the action of the group on the state space of a dynamical system of interest

definition: fixed-point subspace

fixed-point subspace of a given subgroup $H \subset G$, G a symmetry of dynamics, is the set state space points left point-wise invariant under any subgroup action

$$\mathrm{Fix}(H) = \{x \in \mathcal{M} : h\,x = x \text{ for all } h \in H\}\,.$$

typical point in $\mathrm{Fix}(H)$ moves with time, but remains within $f(\mathrm{Fix}(H)) \subseteq \mathrm{Fix}(H)$ for all times. This suggests a systematic approach to seeking compact invariant solutions. The larger the symmetry subgroup, the smaller $\mathrm{Fix}(H)$, easing the numerical searches, so start with the largest subgroups H first

definition: invariant subspace

$\mathcal{M}_\alpha \subset \mathcal{M}$ is an invariant subspace if

$$\{\mathcal{M}_\alpha : gx \in \mathcal{M}_\alpha \text{ for all } g \in G \text{ and } x \in \mathcal{M}_\alpha\} .$$

$\{0\}$ and $\mathcal{M}$ are always invariant subspaces. So is any Fix(H) which is point-wise invariant under action of G.

We can often decompose the state space into smaller invariant subspaces, with group acting within each "chunk" separately:

definition: irreducible subspace

a space $\mathcal{M}_\alpha$ whose only invariant subspaces are $\{0\}$ and $\mathcal{M}_\alpha$ is called irreducible.

as a first, coarse attempt at classification of orbits by their symmetries, we take note three types of equilibria or cycles: asymmetric a, symmetric equilibria or cycles s built by repeats of relative cycles $\tilde{s}$, and boundary equilibria classes **asymmetric cycles:** an equilibrium or periodic orbit is not symmetric if $\{x_a\} \cap \{gx_a\} = \emptyset$, where $\{x_a\}$ is the set of periodic points belonging to the cycle a. Thus $g \in G$ generate $|G|$ distinct orbits with the same number of points and the same stability properties.

symmetric cycles: a cycle s is symmetric (or self-dual) if it has a non-trivial symmetry subgroup, i.e., operating with $g \in G_p \subset G$ on the set of cycle points reproduces the set. $g \in G_p$ acts a shift in time, mapping the cycle point $x \in \mathcal{M}_p$ into $f^{T_p/|G_p|}(x)$

Boundary solutions: an equilibrium x_q or a larger compact invariant solution in a fixed-point subspace $\mathrm{Fix}(G)$, $gx_q = x_q$ for all $g \in G$ lies on the boundary of domains related by action of the symmetry group. A solution that is point-wise invariant under all group operations has multiplicity 1 example: group D_1 - a reflection symmetric 1d map Consider a 1d map f with reflection symmetry $f(-x) = -f(x)$. An example is the bimodal “sawtooth" map, piecewise-linear on the state space $\mathcal{M} = [-1, 1]$ split into three regions $\mathcal{M} = \{\mathcal{M}_L, \mathcal{M}_C, \mathcal{M}_R\}$ which we label with a 3-letter alphabet L(eft), C(enter), and R(ight). The symbolic dynamics is complete ternary dynamics, with any sequence of letters $\mathcal{A} = \{L, C, R\}$ corresponding to an admissible trajectory. Denote the reflection operation by $Rx = -x$

a symmetry reduces computation of periodic orbits to repeats of shorter, relative periodic orbit segments

equivariance of a flow under a symmetry means that the symmetric image of a cycle is again a cycle, with the same period and stability.

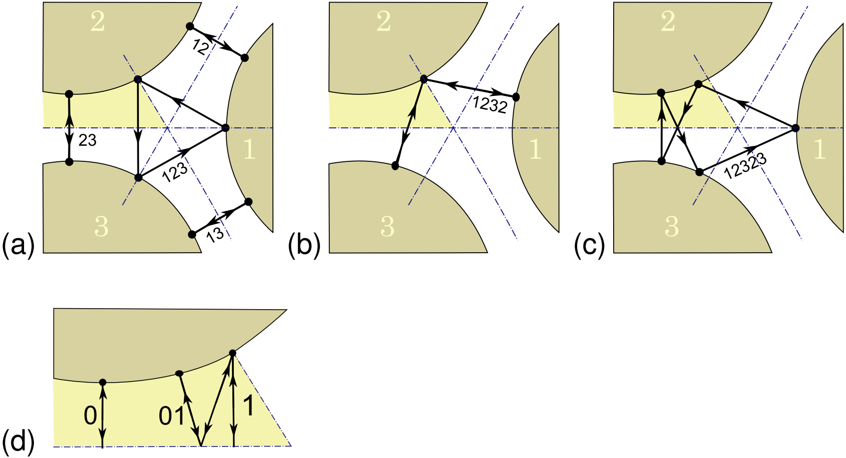
the new orbit may be topologically distinct (in which case it contributes to the multiplicity of the cycle) or it may be the same cycle relative periodic orbits a cycle is symmetric under symmetry operation g if g acts on it as a shift in time, advancing the starting point to the starting point of a symmetry related segment. A symmetric cycle p can thus be subdivided into m_p repeats of a irreducible segment, “prime" in the sense that the full state space cycle is a repeat of it. Thus in presence of a symmetry the notion of a periodic orbit is replaced by the notion of the shortest segment of the full state space cycle which tiles the cycle under the action of the group. In what follows we refer to this segment as a relative periodic orbit relative periodic orbits relative periodic orbits (or equvariant periodic orbits) are orbits $x(t)$ in state space $\mathcal{M}$ which exactly recur

$$x(t) = g\,x(t+T)$$

for a fixed relative period T and a fixed group action $g \in G$. This group action is referred to as a "phase," or a "shift." For a discrete group $g^m = e$ for some finite m, so the corresponding full state space orbit is periodic with period mT.

The period of the full orbit is given by the $m_p \times$ (period of the relative periodic orbit), and the ith Floquet multiplier $\Lambda_{p,i}$ is given by $\Lambda_{\tilde{p},i}^{m_p}$ of the relative periodic orbit. The elements of the quotient space $b \in G/G_p$ generate the copies bp, so the multiplicity of the full state space cycle p is $m_p = |G|/|G_p|$ symmetries of a 3-disk game of pinball

we illustrate these ideas with the



3-disk pinball cycles: (a) $\overline{12}$, $\overline{13}$, $\overline{23}$, $\overline{123}$. Cycle $\overline{132}$ turns clockwise. (b) Cycle $\overline{1232}$; the symmetry related $\overline{1213}$ and $\overline{1323}$ not drawn. (c) $\overline{12323}$; $\overline{12123}$, $\overline{12132}$, $\overline{12313}$, $\overline{13131}$ and $\overline{13232}$ not drawn. (d) The fundamental domain, i.e., the 1/6th wedge indicated in (a), consisting of a section of a disk, two segments of symmetry axes acting as straight mirror walls, and the escape gap to the left. The above 14 full-space cycles example: $C_{3v} = D_3$ invariance - 3-disk game of pinball As the three disks are equidistantly spaced, our game of pinball has a sixfold symmetry. The symmetry group of relabeling the 3 disks is the permutation group S_3; however, it is more instructive to think of this group geometrically, as C_{3v} (dihedral group D_3), the group of order $|G| = 6$ consisting of the identity element e, three reflections across axes $\{\sigma_{12}, \sigma_{23}, \sigma_{13}\}$, and two rotations by $2\pi/3$ and $4\pi/3$ denoted $\{C, C^2\}$. Applying an element (identity, rotation by $\pm 2\pi/3$, or one of the three possible reflections) of this symmetry group to a trajectory yields another trajectory. For instance, σ_{12}, the flip across the symmetry axis going through disk 1 interchanges the symbols 2 and 3; it maps the cycle $\overline{12123}$ into $\overline{13132}$. Cycles $\overline{12}$, $\overline{23}$, and $\overline{13}$ are related to each other by rotation by $\pm 2\pi/3$, or, equivalently, by a relabeling of the disks subgroups of D_3 are $D_1 = \{e, \sigma\}$, consisting of the identity and any one of the reflections, of order 2, and $C_3 = \{e, C, C^2\}$, of order 3, so possible cycle multiplicities are $|G|/|G_p| = 2$, 3 or 6

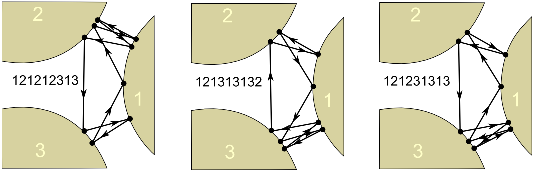
The C_3 subgroup $G_p = \{e, C, C^2\}$ invariance is exemplified by 2 cycles $\overline{123}$ and $\overline{132}$ which are invariant under rotations by $2\pi/3$ and $4\pi/3$, but are mapped into each other by any reflection, and the multiplicity is $|G|/|G_p| = 2$

the C_v type of a subgroup is exemplified by the invariances of $\hat{p} = 1213$. This cycle is invariant under reflection $\sigma_{23}\{\overline{1213}\} = \overline{1312} = \overline{1213}$, so the invariant subgroup is $G_{\hat{p}} = \{e, \sigma_{23}\}$, with multiplicity is $m_{\hat{p}} = |G|/|G_p| = 3$; the cycles in this class, $\overline{1213}$, $\overline{1232}$ and $\overline{1323}$, are related by $2\pi/3$ rotations a cycle of no symmetry, such as $\overline{12123}$, has $G_p = \{e\}$ and contributes in all six copies (the remaining cycles in the class are $\overline{12132}$, $\overline{12313}$, $\overline{12323}$, $\overline{13132}$ and $\overline{13232}$)

Besides the above discrete symmetries, for Hamiltonian systems cycles may be related by time reversal symmetry. An example are the cycles $\overline{121212313}$ and $\overline{121212323} = \overline{313212121}$ which have the same periods and stabilities, but are related by no space symmetry

So far we have used symmetry to effect a reduction in the number of independent cycles in cycle expansions. The next step achieves much more:

1. Discrete symmetries can be used to restrict all computations to a fundamental domain, the $\mathcal{M}/G$ quotiented subspace of the full state space $\mathcal{M}$.
2. Discrete symmetry tessellates the state space into copies of a fundamental domain, and thus induces a natural partition of state space. The state space is completely tiled by a fundamental domain and its symmetric images



cycle $\overline{121212313}$ has multiplicity 6; shown here is $\overline{121313132} = \sigma_{23}\overline{121212313}$. However, $\overline{121231313}$ which has the same stability and period is related to $\overline{121313132}$ by time reversal, but not by any C_{3v} symmetry

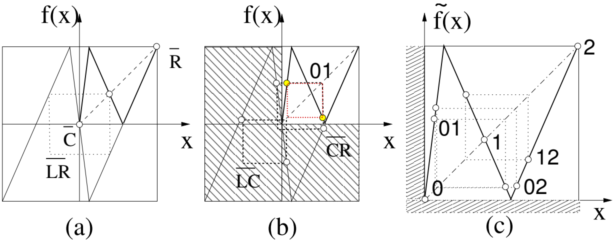
1. cycle multiplicities induced by the symmetry are removed by desymmetrization, reduction of the full dynamics to the dynamics on a fundamental domain. Each symmetry-related set of global cycles p corresponds to precisely one fundamental domain (or relative) cycle $\tilde{p}$. Conversely, each fundamental domain cycle $\tilde{p}$ traces out a segment of the global cycle p, with the end point of the cycle $\tilde{p}$ mapped into the irreducible segment of p with the group element $h_{\tilde{p}}$. The relative periodic orbits in the full space, folded back into the fundamental domain, are periodic orbits

1. The group elements $G = \{e, g_2, \dots, g_{|G|}\}$ which map the fundamental domain $\tilde{\mathcal{M}}$ into its copies $g\tilde{\mathcal{M}}$, serve also as letters of a symbolic dynamics alphabet.

if the dynamics is invariant under a discrete symmetry, the state space $\mathcal{M}$ can be completely tiled by the fundamental domain $\tilde{\mathcal{M}}$ and its images $\mathcal{M}_a = a\tilde{\mathcal{M}}$, $\mathcal{M}_b = b\tilde{\mathcal{M}}$, $\ldots$ under the action of the symmetry group $G = \{e, a, b, \ldots\}$,

$$\mathcal{M} = \tilde{\mathcal{M}} \cup \mathcal{M}_a \cup \mathcal{M}_b \cdots \cup \mathcal{M}_{|G|} = \tilde{\mathcal{M}} \cup a\tilde{\mathcal{M}} \cup b\tilde{\mathcal{M}} \cdots .$$

Now we can use the invariance condition to move the starting point x into the fundamental domain $x = a\tilde{x}$, and then use the relation $a^{-1}b = h^{-1}$ to also relate the endpoint y to its image in the fundamental domain. While the global trajectory runs over the full space $\mathcal{M}$, the restricted trajectory is brought back into the fundamental domain $\tilde{\mathcal{M}}$ any time it exits into an adjoining tile; the two trajectories are related by the symmetry operation h which maps the global endpoint into its fundamental domain image.



sawtooth map with the D_1 symmetry $f(-x) = -f(x)$ restricted to the fundamental domain. $f(x)$ is indicated by the thin line, and fundamental domain map $\tilde{f}(\tilde{x})$ by the thick line.
(a) Boundary fixed point $\overline{C}$ is the fixed point $\overline{0}$. The asymmetric fixed point pair $\{\overline{L},\overline{R}\}$ is reduced to the fixed point $\overline{2}$, and the full state space symmetric 2-cycle $\overline{LR}$ is reduced to the fixed point $\overline{2}$
(b) The asymmetric 2-cycle pair $\{\overline{LC},\overline{CR}\}$ is reduced to 2-cycle $\overline{02}$
(c) All fundamental domain fixed points and 2-cycles

example: group D_1 and reduction to the fundamental domain

Consider again the reflection-symmetric bimodal Ulam sawtooth map $f(-x) = -f(x)$, with symmetry group $D_1 = \{e, R\}$. The state space $\mathcal{M} = [-1, 1]$ can be tiled by half-line $\tilde{\mathcal{M}} = [0, 1]$, and $R\tilde{\mathcal{M}} = [-1, 0]$, its image under a reflection across $x = 0$ point. The dynamics can then be restricted to the fundamental domain $\tilde{x}_k \in \tilde{\mathcal{M}} = [0, 1]$; every time a trajectory leaves this interval, it is mapped back using R

the fundamental domain map $\tilde{f}(\tilde{x})$ is obtained by reflecting $x < 0$ segments of the global map $f(x)$ into the upper right quadrant. $\tilde{f}$ is also bimodal and piecewise-linear, with $\tilde{\mathcal{M}} = [0, 1]$ split into three regions $\tilde{\mathcal{M}} = \{\tilde{\mathcal{M}}_0, \tilde{\mathcal{M}}_1, \tilde{\mathcal{M}}_2\}$ which we label with a 3-letter alphabet $\tilde{\mathcal{A}} = \{0, 1, 2\}$. The symbolic dynamics is again complete ternary dynamics, with any sequence of letters $\{0, 1, 2\}$ admissible

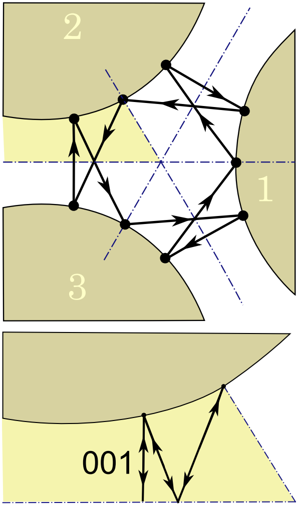
However, the interpretation of the “desymmetrized" dynamics is quite different - the multiplicity of every periodic orbit is now 1, and relative periodic orbits of the full state space dynamics are all periodic orbits in the fundamental domain

In (a) the boundary fixed point $\overline{C}$ is also the fixed point $\overline{0}$. In this case the set of points invariant under group action of D_1, $\tilde{\mathcal{M}} \cap R\tilde{\mathcal{M}}$, is just this fixed point $x = 0$, the reflection symmetry point.

The asymmetric fixed point pair $\{\overline{L},\overline{R}\}$ is reduced to the fixed point $\overline{2}$, and the full state space symmetric 2-cycle $\overline{LR}$ is reduced to the fixed point $\overline{1}$. The asymmetric 2-cycle pair $\{\overline{LC},\overline{CR}\}$ is reduced to the 2-cycle $\overline{01}$. Finally, the symmetric 4-cycle $\overline{LCRC}$ is reduced to the 2-cycle $\overline{02}$. This completes the conversion from the full state space for all fundamental domain fixed points and 2-cycles

(a)

(b)



(a) The pair of full-space 9-cycles, the counter-clockwise $\overline{121232313}$ and the clockwise $\overline{131323212}$ correspond to

(b) one fundamental domain 3-cycle $\overline{001}$.

example: 3-disk game of pinball in the fundamental domain

If the dynamics is symmetric under interchanges of disks, the absolute disk labels $\epsilon_i = 1, 2, \cdots, N$ can be replaced by the symmetry-invariant relative disk$\rightarrow$disk increments g_i, where g_i is the discrete group element that maps disk $i-1$ into disk i. For 3-disk system g_i is either reflection σ back to initial disk (symbol '0') or rotation by C to the next disk (symbol '1'). An immediate gain arising from symmetry invariant relabeling is that N-disk symbolic dynamics becomes $(N-1)$-nary, with no restrictions on the admissible sequences an irreducible segment corresponds to a periodic orbit in the fundamental domain, a one-sixth slice of the full 3-disk system, with the symmetry axes acting as reflecting mirrors. A set of orbits related in the full space by discrete symmetries maps onto a single fundamental domain orbit. The reduction to the fundamental domain desymmetrizes the dynamics and removes all global discrete symmetry-induced degeneracies: rotationally symmetric global orbits (such as the 3-cycles $\overline{123}$ and $\overline{132}$) have multiplicity 2, reflection symmetric ones (such as the 2-cycles $\overline{12}$, $\overline{13}$ and $\overline{23}$) have multiplicity 3, and global orbits with no symmetry are 6-fold degenerate.

Peculiar effects arise for orbits that run on a symmetry lines that border a fundamental domain. The state space transformation $\mathbf{h} \neq \mathbf{e}$ leaves invariant sets of boundary points; for example, under reflection σ across a symmetry axis, the axis itself remains invariant. Some care need to be exercised in treating the invariant “boundary" set $\mathcal{M} = \tilde{\mathcal{M}} \cap \mathcal{M}_a \cap \mathcal{M}_b \cdots \cap \mathcal{M}_{|G|}$. The properties of boundary periodic orbits that belong to such pointwise invariant sets will require a bit of thinking.

in our 3-disk example, no such orbits are possible, but they exist in other systems, such as in the bounded region of the Hénon-Heiles potential and in 1*d* maps. For the symmetrical 4-disk billiard, there are in principle two kinds of such orbits, one kind bouncing back and forth between two diagonally opposed disks and the other kind moving along the other axis of reflection symmetry; the latter exists for bounded systems only. While for low-dimensional state spaces there are typically relatively few boundary orbits, they tend to be among the shortest orbits, and they play a key role in dynamics while such boundary orbits are invariant under some symmetry operations, their neighborhoods are not. This affects the Jacobian matrix M_p of the orbit and its Floquet multipliers

here we have used a particularly simple direct product structure of a global symmetry that commutes with the flow to reduce the dynamics to a symmetry reduced $(d-1-N)$-dimensional state space $\mathcal{M}/G$.

desymmetrization of Lorenz flow

Lorenz equation is invariant under $G = \{e, R\}$, where R is $[x, y]$-plane rotation by π about the z-axis:

$$(x, y, z) \to R(x, y, z) = (-x, -y, z)\,.$$

$R^2 = 1$ condition decomposes the state space into $\mathcal{M} = \mathcal{M}^+ \oplus \mathcal{M}^-$, the z-axis $\mathcal{M}^+$ and the $[x, y]$ plane $\mathcal{M}^-$.

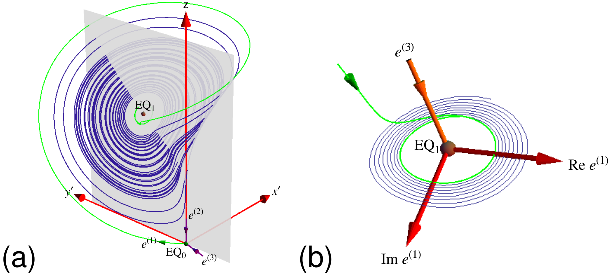
the 1-dimensional $\mathcal{M}^+$ subspace is the fixed-point subspace of D_1, with the z-axis points left point-wise invariant under the group action

$$\mathrm{Fix}(D_1) = \{x \in \mathcal{M}^+ : g\,x = x \text{ for } g \in \{e, R\}\}\,.$$

A apoint $x(t)$ in $\mathrm{Fix}(G)$ remains within $x(t) \subseteq \mathrm{Fix}(G)$ for all times; the subspace $\mathcal{M}^+ = \mathrm{Fix}(G)$ is flow invariant. The Lorenz equation is reduced to the exponential contraction to the EQ_0 equilibrium,

$$\dot{z} = -b\,z\,.$$

in higher-dimensional state spaces the flow-invariant $\mathcal{M}^+$ subspace can itself be high-dimensional, with interesting dynamics of its own. This subspace is a topological obstruction: the number of winds of a trajectory around $\rightarrow$ a natural symbolic dynamics the state space is divided into a half-space fundamental domain $\tilde{\mathcal{M}} = \mathcal{M}/D_1$ and its 180^o rotation $R\tilde{\mathcal{M}}$. Take the fundamental domain $\tilde{\mathcal{M}}$ to be the half-space between the viewer and $\mathcal{P}$. Then the full Lorenz flow is captured by re-injecting back into $\tilde{\mathcal{M}}$ any trajectory that exits it, by a rotation of π around the z axis



(a) Lorenz attractor plotted in $[x', y', z]$, the doubled-polar angle coordinates, with points related by π-rotation in the $[x, y]$ plane identified. Stable eigenvectors of EQ_0: $\mathbf{e}^{(3)}$ and $\mathbf{e}^{(2)}$, along the z axis. Unstable manifold orbit $W^u(EQ_0)$ (green) is a continuation of the unstable $\mathbf{e}^{(1)}$ of EQ_0.

(b) Blow-up of the region near EQ_1: The unstable eigenplane of EQ_1 is defined by $\mathrm{Re}\,\mathbf{e}^{(2)}$ and $\mathrm{Im}\,\mathbf{e}^{(2)}$, the stable eigenvector $\mathbf{e}^{(3)}$. The descent of the EQ_0 unstable manifold (green) defines the innermost edge of the strange attractor. As it is clear from (a), it also defines its outermost edge a state space redefinition that maps a pair of points related by symmetry into a single point accomplished by expressing (x, y) in polar coordinates $(x, y) = (r \cos\theta, r \sin\theta)$, and then plotting the flow in the "doubled-polar angle representation:"

$$
\begin{aligned}
(x', y') &= (r \cos 2\theta, r \sin 2\theta) \\
&= ((x^2 - y^2)/r, 2xy/r) \, ,
\end{aligned}
$$

fundamental domain
if invariant under a set of discrete symmetries G, the state space $\mathcal{M}$ is tiled by a set of symmetry-related tiles, and the dynamics can be reduced to dynamics within one such tile, the fundamental domain $\mathcal{M}/G$

if the symmetry is continuous the dynamics is reduced to a lower-dimensional desymmetrized system $\mathcal{M}/G$

families of symmetry-related full state space cycles are replaced by fewer, shorter “relative" cycles

notion of a prime periodic orbit is replaced by the notion of a relative periodic orbit, the shortest segment of the full state space cycle which tiles the cycle under the action of the group terra incognita we have some inklings of the “terra incognita:” for example, symplectic symmetry induces existence of KAM-tori, and in general dynamical settings we are encountering more and more examples of partially hyperbolic invariant tori, isolated tori that are consequences of dynamics, not of a global symmetry, and which cannot be represented by a single relative periodic orbit, but require a numerical computation of full $(N+1)$-dimensional compact invariant sets and their infinite-dimensional linearized Jacobian matrices, marginal in $(N+1)$ dimensions, and hyperbolic in the rest the main result of this chapter can be stated as follows: If a dynamical system $(\mathcal{M}, f)$ has a symmetry G, the symmetry should be deployed to "quotient" the state space $\mathcal{M}/G$, i.e., identify all $x \in \mathcal{M}$ related by the symmetry

in presence of a discrete symmetry G, associated with each full state space cycle p is a maximal symmetry subgroup $G_p \subseteq G$ of order $1 \leq |G_p| \leq |G|$, whose elements leave p invariant. The symmetry subgroup G_p acts on p as time shift, tiling it with $|G_p|$ copies of its shortest invariant segment, the relative periodic orbit $\tilde{p}$. The elements of the coset $b \in G/G_p$ generate $m_p = |G|/|G_p|$ distinct copies of p

this reduction to the fundamental domain $\tilde{\mathcal{M}} = \mathcal{M}/G$ simplifies symbolic dynamics and eliminates symmetry-induced degeneracies. For the short orbits the labor saving is dramatic. For example, for the 3-disk game of pinball there are 256 periodic points of length 8, but reduction to the fundamental domain non-degenerate prime cycles reduces the number of the distinct cycles of length 8 to 30.

there are no periodic orbits Amusingly, in this extension of "periodic orbit" theory from unstable 1-dimensional closed orbits to unstable $(N+1)$-dimensional compact manifolds $\mathcal{M}_p$ invariant under continuous symmetries, there are either no or proportionally few periodic orbits. Likelihood of finding a periodic orbit is zero. One expects some only if in addition to a continuous symmetry one has a discrete symmetry, or the particular invariant compact manifold $\mathcal{M}_p$ is invariant under a discrete subgroup of the continuous symmetry. Relative periodic orbits are almost never eventually periodic, i.e., they almost never lie on periodic trajectories in the full state space, unless forced to do so by a discrete symmetry, so looking for periodic orbits in systems with continuous symmetries is a fool's errand atypical as they are (no chaotic solution will be confined to these discrete subspaces) they are important for periodic orbit theory, as there the shortest orbits dominate.

we feel your pain, but trust us: once you grasp the relation between the full state space $\mathcal{M}$ and the desymmetrized G-quotiented $\mathcal{M}/G$, you will find the life as a fundamentalist so much simpler that you will never return to your full state space confused ways of yesteryear.

Résumé initially we made a lame attempt to classify “all possible motions:” (1) equilibria, (2) periodic orbits, (3) everything else

now one can discern in the fog of dynamics outline of a more serious classification - long time dynamics takes place on the closure of a set of all invariant compact sets preserved by the dynamics, and those are:

(1) 0-dimensional equilibria $\mathcal{M}_q$

(2) 1-dimensional periodic orbits $\mathcal{M}_p$, (3) global symmetry induced N-dimensional relative equilibria $\mathcal{M}_{tw}$

(3) (N+1)-dimensional relative periodic orbits $\mathcal{M}_p$

(5) terra incognita

what if the “law of motion" retains its form in a family of coordinate frames $f(x) = g^{-1} f(gx)$ related by a group of continuous symmetries? The notion of “fundamental domain” is of no use here. Instead, (read the next chapter, “Relativity of cyclists”) continuous symmetries reduce dynamics to a desymmetrized system of lower dimensionality, by elimination of “ignorable" coordinates.