ChaosBook.org chapter

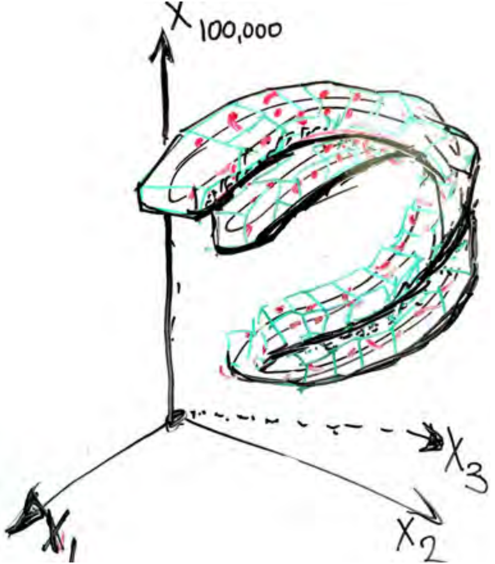
dimension of turbulence

25 March 2020, version 16.4

overview

1. what this chapter is about
2. why are we here
3. inertial manifold
4. state space
5. dimension of the inertial manifold

inertial manifold



strange attractor stuffed into a finite-dimensional body bag

Q: what is the physical dimension of a turbulent flow ?

question

does an attractor of a dissipative flow have a "dimension" ?

Foias *et al*[1]

mathematician's answer

dimension of 'inertial manifold' is finite

[1] C. Foias et al., C. R. Acad. Sci., Paris **301**, 285–288 (1985).

Q: what is the physical dimension of a turbulent flow ?

question

does an attractor of a dissipative flow have a “dimension” ?

Ginelli, Chaté, Radons, *et al*[1,2,3,4]

physicist’s answer

‘Lyapunov covariant vectors’ split into

(a) finite number of ‘physical,’ entangled directions, in the tangent space of the attractor

(b) infinitely many hyperbolically decaying directions that are isolated and do not mix and

[1] A. Politi et al., Physica D **224**, 90–101 (2006).

[2] F. Ginelli et al., Phys. Rev. Lett. **99**, 130601 (2007).

[3] H.-l. Yang et al., Phys. Rev. Lett. **102**, 074102 (2009).

[4] K. A. Takeuchi et al., Phys. Rev. Lett. **103**, 154103 (2009).

Q: what is the physical dimension of a turbulent flow ?

question

does an attractor of a dissipative flow have a “dimension” ?

Ding, Cvitanović *et al*[1]

physicist’s answer

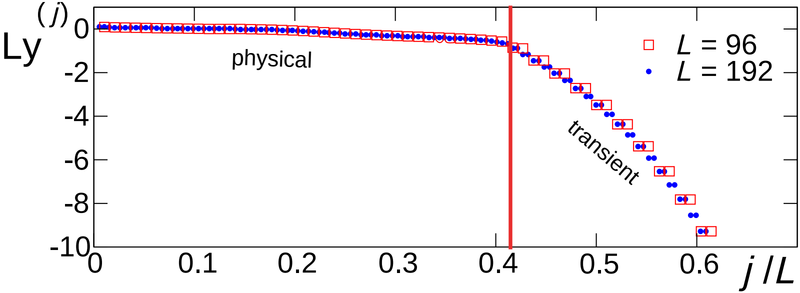
Floquet vectors of unstable periodic orbits identify the *local* number of degrees of freedom that captures the physics of a ‘turbulent’ PDE on a compact spatial domain

that number is proportional to the size L^D of the D-dimensional PDE system

[1] X. Ding et al., Phys. Rev. Lett. **117**, 024101 (2016).

the killer plot : to be explained here

Kuramoto-Sivashinsky Lyapunov spectrum
cells $L = 22, 96, 192$: it scales!



Now double # computational elements, fixed L :
all new ones go to the transient spectrum[2] !

[2] H.-l. Yang et al., Phys. Rev. Lett. **102**, 074102 (2009).

part 1

1. why are we here
2. inertial manifold
3. state space
4. dimension of the inertial manifold

Navier-Stokes equations (1822)

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = \frac{1}{R}\nabla^2 \mathbf{v} - \nabla p + \mathbf{f}, \qquad \nabla \cdot \mathbf{v} = 0,$$

velocity field $\mathbf{v} \in \mathbb{R}^3$; pressure field p ; driving force $\mathbf{f}$

goal : describe turbulence

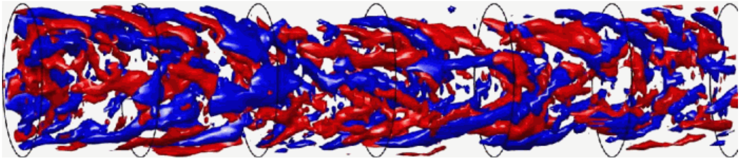
starting from the equations (no statistical assumptions)

describe :

pipe experiment data point

a state of turbulent pipe flow at instant in time

Stereoscopic Particle Image Velocimetry $\rightarrow$ 3-*d* velocity field over the entire pipe[3]



[3] Casimir W.H. van Doorne (PhD thesis, Delft 2004)

equations are known

Navier-Stokes equation

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \frac{1}{\mathrm{Re}} \nabla^2 \mathbf{u}$$

requires at least $\approx 100,000$-dimensional DNS
(direct numerical simulation)

the ‘physical’ dimension is still unknown: at least 100?

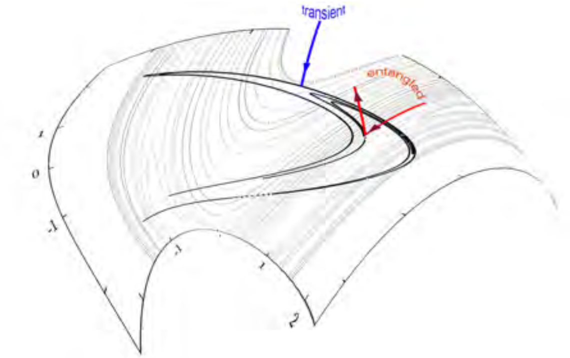
part 2

1. why are we here
2. inertial manifold
3. state space
4. dimension of the inertial manifold

what this chapter is about:

the attracting set of a dissipative flow

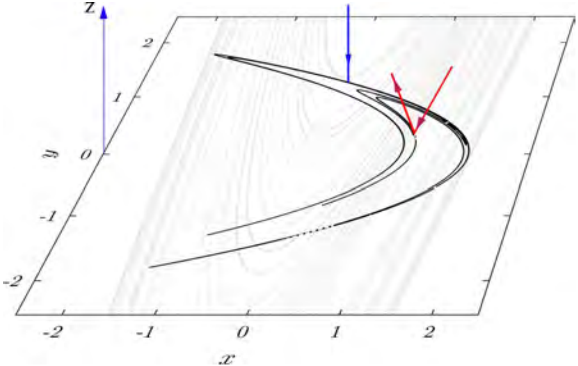
is embedded with the (curvilinear) inertial manifold embedded into ∞-dimensional state space



but try to draw THAT :)

what this chapter is about:

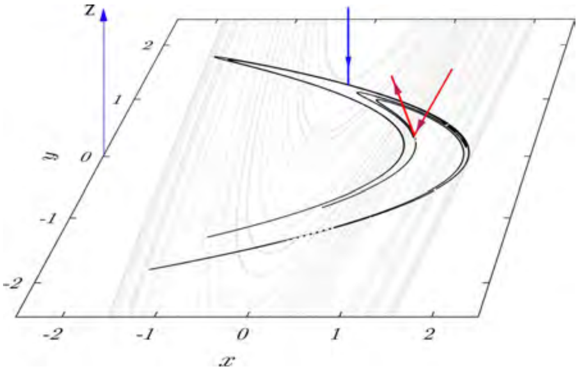
it is believed that the attracting set of a dissipative flow



- is confined to :
  a finite-dimensional smooth *inertial manifold*
- "*z*" directions :
  the remaining ∞ of *transient dimensions*

what this chapter is about:

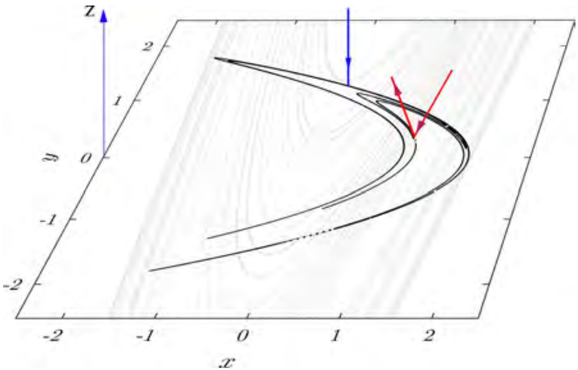
state space of dissipative flow is split into



- inertial manifold : spanned locally by entangled covariant vectors, tangent to unstable / stable manifolds
- the rest : spanned by the remaining ∞ of the contracting, decoupled, transient covariant vectors

what this chapter is about:

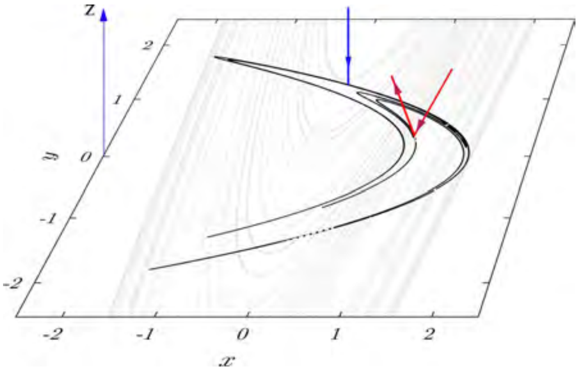
inertial manifold



- dynamics of the vectors that span the inertial manifold is entangled, with small angles and frequent tangencies
- any transient covariant vector : isolated, nearly orthogonal to all other covariant vectors

what this chapter is about:

goal : construct inertial manifold for a turbulent flow



- tile it with a finite collection of bricks centered on recurrent states, each brick $\approx 10 - 100$ dimensions
- span of ∞ of transient covariant vectors : no intersection with the entangled modes

if all this works out, it is kinda amazing

computation of turbulent solutions

requires at least
$\rightarrow$ integration of 10^4-10^6 coupled ordinary differential equations

inertial manifold, tiled

50 linear tiles cover the (nonlinear, curved) inertial manifold

each tile 100 dimensional (fingers crossed :)

part 3

1. why are we here
2. inertial manifold
3. **state space**
4. dimension of the inertial manifold

1 spatial dimension “Navier-Stokes”

computationally not ready yet to explore
the inertial manifold of $3D$ turbulence - we start with

Kuramoto-Sivashinsky equation

$$u_t + u\nabla u = -\nabla^2 u - \nabla^4 u \,, \qquad x \in [-L/2, L/2] \,,$$

describes spatially extended systems such as

- flame fronts in combustion
- reaction-diffusion systems
- ...

1-dimensional “Navier-Stokes”

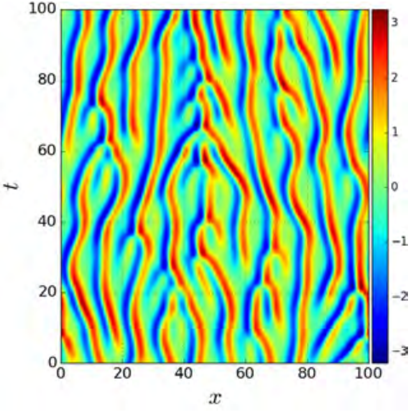
Kuramoto-Sivashinsky equation

$$u_t + u\, u_x + u_{xx} + \nu\, u_{xxxx} = 0$$

- “inertial” term $u\partial_x u$; nonlinear
- energy-in “anti-diffusive” term $-\partial_x^2 u$,
- “hyper-viscosity” $\partial_x^4 u$ - suppresses fast variations

only parameter: dimensionless length $\tilde{L} = \frac{L}{2\pi}$

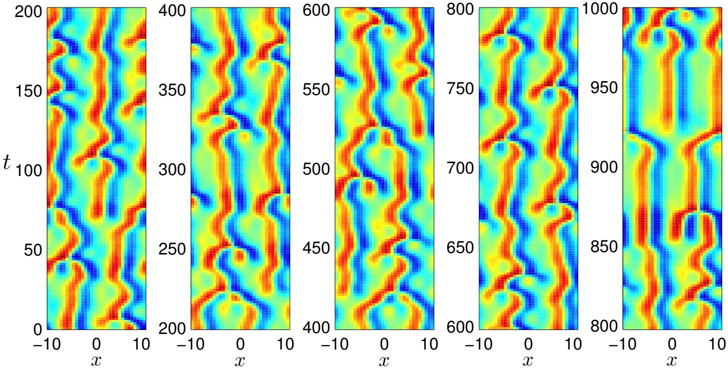
Kuramoto-Sivashinsky on a large spacetime domain



[horizontal] space $x \in [0, 100]$ [up] time evolution

- turbulent behavior
- simpler physical, mathematical and computational setting than Navier-Stokes

evolution of Kuramoto-Sivashinsky on small periodic domain

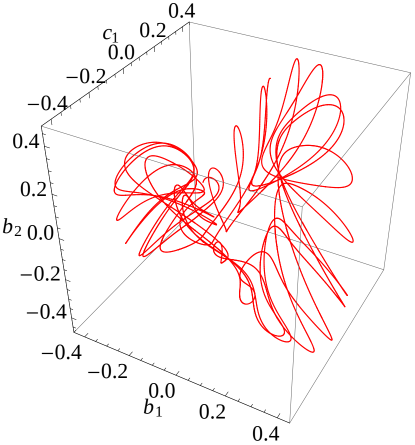


[horizontal] space $x \in [-11, 11]$ [up] time evolution

color: magnitude of $u(x, t)$

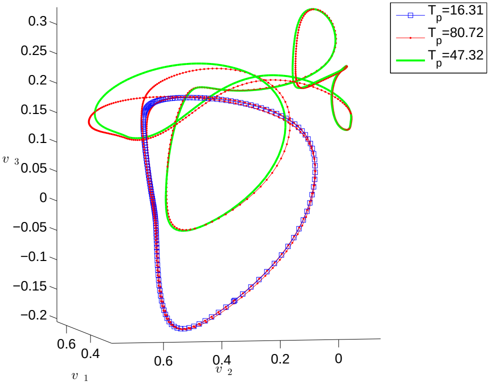
a relative periodic orbit

full state space : many periods



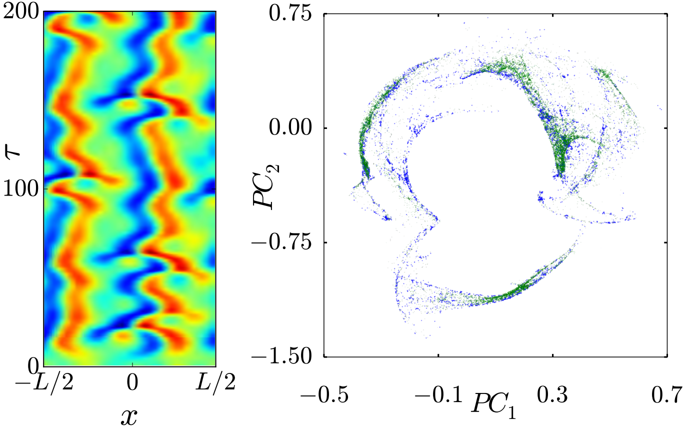
have computed: 60 000 periodic orbits

can explore shadowing



(impossible without symmetry reduction)

(relative) periodic orbits are dense in the attractor



- [left] turbulent trajectory segment in [space×time]
- Poincaré section, turbulent trajectory (natural measure)
- periodic points, from 479 periodic orbits[4]

[4] N. B. Budanur, "Exact Coherent Structures in Spatiotemporal Chaos: From Qualitative Description to Quantitative Predictions", PhD thesis (School of Physics, Georgia Inst. of Technology, Atlanta, 2015).

part 4

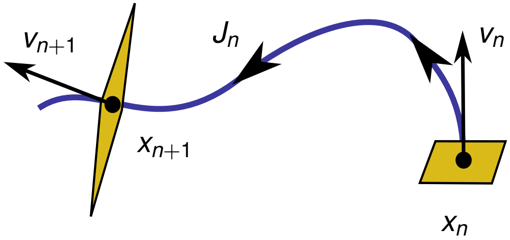
1. why are we here
2. inertial manifold
3. state space
4. **dimension of the inertial manifold**

what is the dimension of the inertial manifold?

we determine it in 6 independent ways

- Lyapunov exponents (diagnostic only, previous work)
- Lyapunov vectors (sharp, previous work)
- four periodic orbits determinations (presented here)

linearized deterministic flow



$$x_{n+1} + z_{n+1} = f(x_n) + J_n\, z_n\,, \quad J_{ij} = \partial f_i / \partial x_j$$

in one time step a linearized neighborhood of x_n is

(1) advected by the flow

(2) transported by the Jacobian matrix J_n into a neighborhood given by the J eigenvalues and eigenvectors[5]

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[5] rechristianed by (F. Ginelli et al., Phys. Rev. Lett. **99**, 130601 [2007]) to "covariant Lyapunov vectors"

method (0) : algorithmic advance

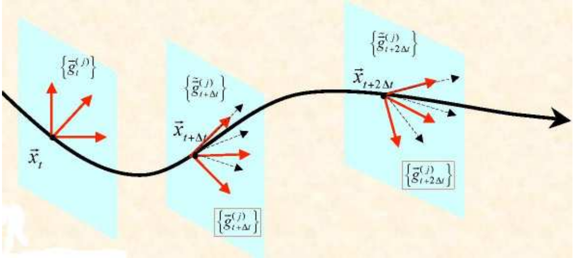
F. Ginelli, H. Chaté, G. Radons, A. Politi, P. Poggi, A. Turchi, R. M. Samelson, C. L. Wolfe:

computation of covariant “Lyapunov” vectors

Phys. Rev. Lett. 99, 130601 (2007); Tellus A 59, 355 (2007);

J. Phys. A 46, 254005 (2013)

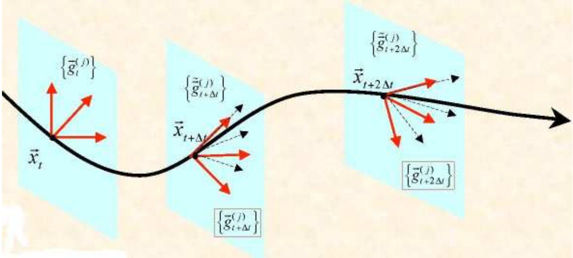
covariant vectors are non-normal



(references are hyperlinked)

method (0)

global ergodic trajectory, $t \in [-\infty, \infty]$



Jacobian matrix : orthogonal frame $\rightarrow$ non-orthogonal frame

$\rightarrow$

QR decomposed into an R-matrix + Gram-Schmidt frame

$\rightarrow$

next Jacobian matrix, and so on

beautiful insight of

F. Ginelli, H. Chaté, G. Radons, A. Politi, P. Poggi, A. Turchi, H.-l. Yang, K. A. Takeuchi

physical dynamics is hyperbolically separated from the infinity of transient modes :

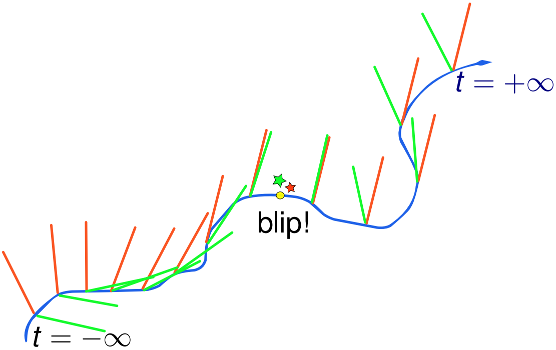
physical dimension of an inertial manifold

Phys. Rev. Lett. 102, 074102 (2009); Phys. Rev. E 84, 046214 (2011); Phys. Rev. Lett. 117, 024101 (2016)

- Kuramoto-Sivashinsky? OK!
- complex Ginzburg-Landau? OK!
- Navier-Stokes? dunno...

(references are hyperlinked)

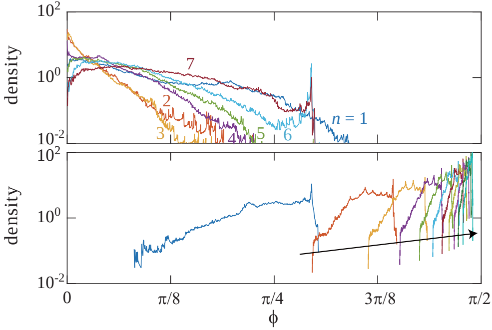
eigenvector span the “physical” manifold



a pair of “entangled” eigenvectors
distinct Lyapunov exponents
dance along t from $-\infty$ to $-\infty$ orbit

at the instant “blip!” they are (nearly) collinear

(0) distribution of angles between eigenvectors



histogram of angles between *n*th leading covariant vector and the next, accumulated over many long orbits :

- (top) For $n = 1 \cdots 7$ (eigenvector within the entangled manifold) the angles can be arbitrarily small
- (bottom) For the remaining, transient eigenvectors, $n = 8, 11, 12, \cdots$: angles are bounded away from zero

(2) machine learning

Linot & Graham[6]

Deep learning to discover and predict dynamics on an inertial manifold, arXiv:2001.04263

[6] A. J. Linot and M. D. Graham, *Deep learning to discover and predict dynamics on an inertial manifold*, 2020.

(3) approximate inertial manifold

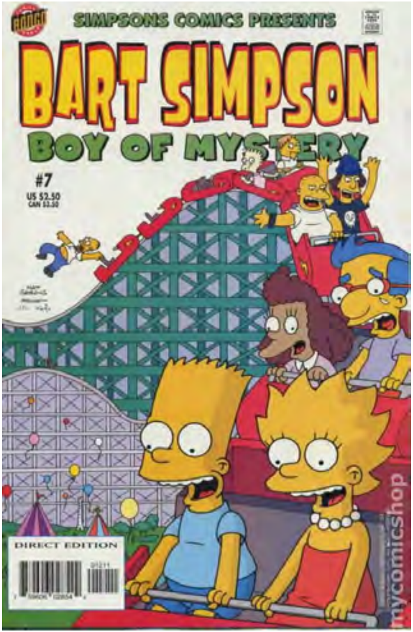
Akram, Hassanaly & Raman[7]

A priori analysis of reduced description of dynamical systems using approximate inertial manifolds

[7] M. Akram et al., J. Comput. Phys. **409**, 109344 (2020).

OK, so the frame
is
locally flat

but where the (blip) are we in the state space?



we are here

next : cartography of a roller coaster ride

part 5

1. why are we here
2. inertial manifold
3. state space
4. dimension of the inertial manifold
5. new : cartography of the inertial manifold

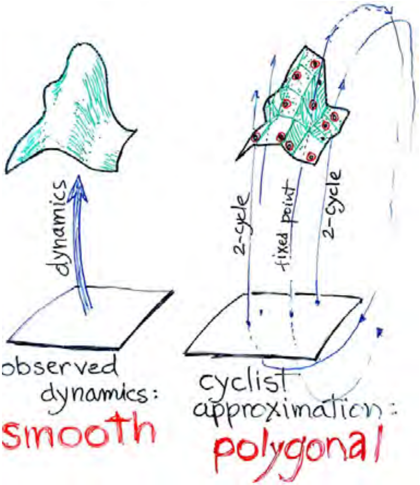
cartography for fluid dynamicists

cover the inertial manifold with a set of flat charts

we can do this with

finite-dimensional bricks embedded in $10^{100\,000}$ dimensions!

tile the inertial manifold by **recurrent flows**

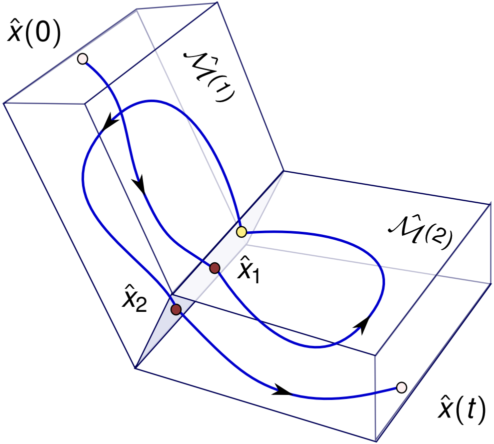


a fixed point

a 2-cycle, etc.

smooth dynamics (left frame)
tesselated by the skeleton of recurrent flows,
together with (right frame) their linearized neighborhoods

charting the inertial manifold

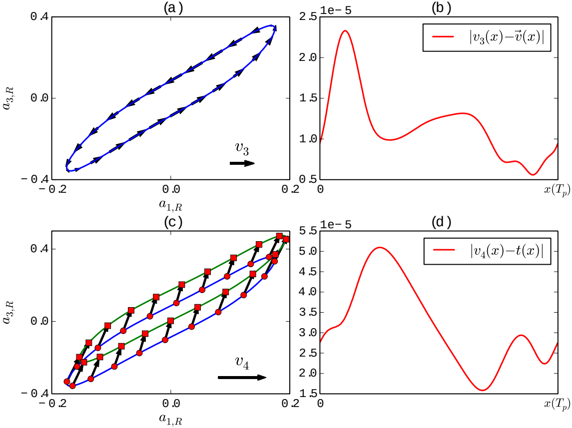


two tangent “entangled” tiles = finite-dimensional bricks

shaded plane :
when integrating your equations, switch the local chart

method (1) : local relative periodic orbit, one period

two marginal eigenvectors



[right panels]
all eigenvectors computed to the machine precision[8]

[8] X. Ding and P. Cvitanović, SIAM J. Appl. Dyn. Syst. **15**, 1434–1454 (2016).

(1) algorithmic breakthrough : all Floquet exponents to machine precision

	$\mu^{(i)}$	$e^{iT_p\omega^{(i)}}$
1=2	0.0331970261043278	-0.42330856002164 + i 0.905985575499084
3=4	(2 marginal)	
5	-0.216338085869672	1
6=7	-0.265233812289065	-0.867175421594352 + i 0.49800279937231
...	...	...
29	-316.19797864063	1
30	-320.666664811713	-1

Floquet exponents for the shortest pre-periodic orbit :

$\mu^{(i)}$ = real part of the exponent.
either the multiplier sign for a real exponent, or
$\omega^{(i)} \to$ the multiplier phase for a complex Floquet exponent

(1) algorithmic breakthrough : all Floquet exponents to machine precision

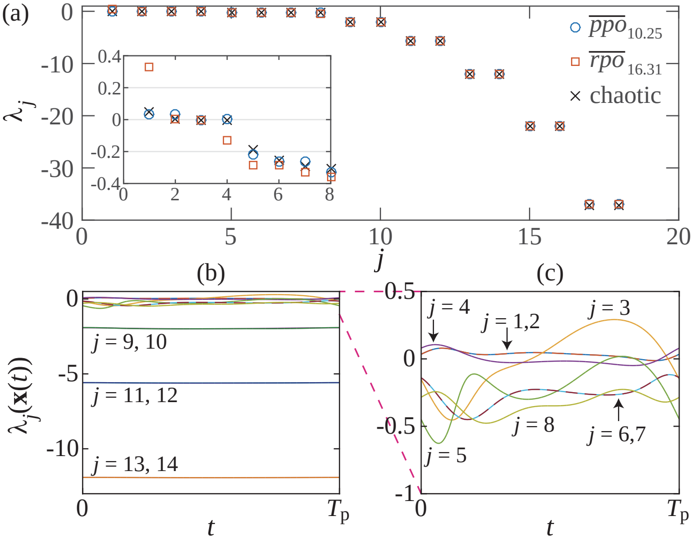
why is this a big deal?

periodic Schur decomposition : resolves Floquet multipliers differing by thousands of orders of magnitude

here the smallest Floquet multiplier for the shortest periodic orbit is

$$|\Lambda_{62}| \simeq e^{-6080.4 \times 10.25} \approx 10^{-27069}$$

(1) Floquet and Lyapunov exponents, $L = 22$ small cell



8 entangled modes, rest transient :

inertial manifold is 8 dimensional!

(1) dimension of the inertial manifold from an individual orbit (??)

Floquet exponents separate into entangled vs. transient

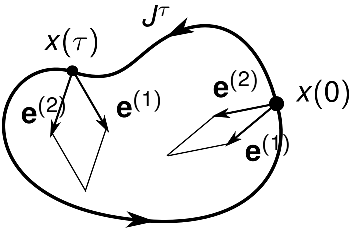
for every single periodic orbit! (checked 500 orbits)

if true for Navier-Stokes, that would make life easy!

(2) dimension of the inertial manifold from ensemble of orbits

- principal angles between hyperplanes spanned by Floquet vectors

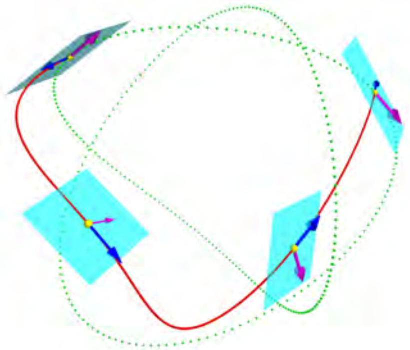
(2) Floquet vectors



a parallelepiped spanned by a pair of Floquet eigenvectors ('covariant vectors') transported along the orbit

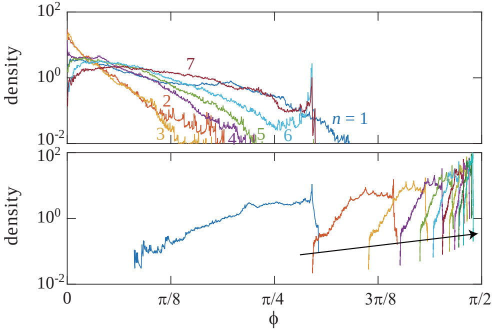
- Jacobian matrix not self-adjoint : the eigenvectors are not orthogonal, the eigenframe is 'non-normal'
- Measure the angle between eigenvectors $\mathbf{e}^{(i)}(x(t))$ and $\mathbf{e}^{(j)}(x(t))$

(2) example : Kuramoto-Sivashinsky relative periodic orbit



dotted green : a group orbit
solid red : a relative periodic orbit
planes : a tangent space spanned and transported by 2 Floquet vectors

(2) distribution of principal angles between Floquet subspaces



histogram of angles between S_n (n leading Floquet vectors) and $\bar{S}_n$ (the rest), accumulated over the 400 orbits :

- (top) For $n = 1 \cdots 7$ (S_n within the entangled manifold) the angles can be arbitrarily small
- (bottom) For the $\bar{S}_n$ spanned by transient modes, $n = 8, 10, 12, \cdots, 28$: angles bounded away from unity

.

(3), (4) dimension of the inertial manifold from a chaotic trajectory shadowing a given orbit

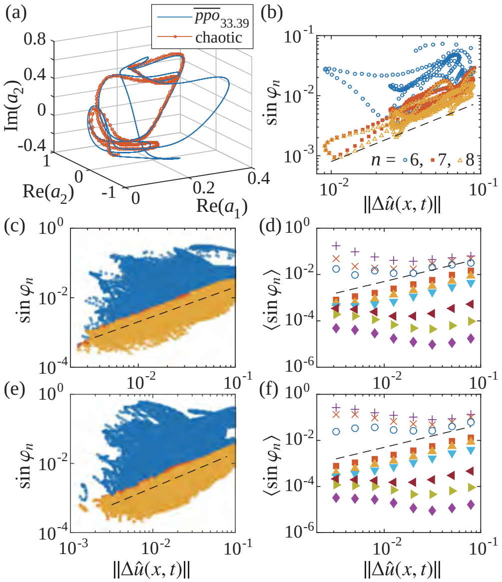
two independent measurements

(3) shadowing separation vector lies within the orbit's Floquet entangled manifold

(4) shadowing separation vector lies within the chaotic trajectories covariant vectors' entangled manifold

'separation vector' = difference vector between the chaotic orbit point and periodic orbit point at their (locally) closest passage

accumulate 1000's of near recurrences

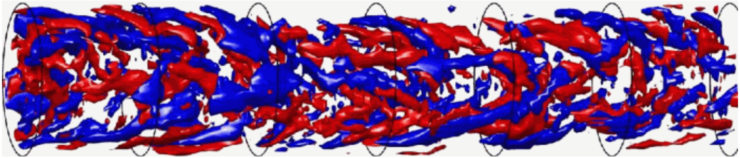


(3)

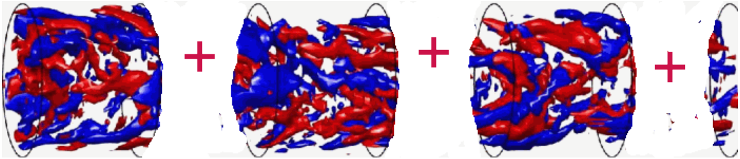
chaotic trajectory shadows periodic orbits within the entangled subspace

what about large or ∞ domains ?

spatiotemporal chaos

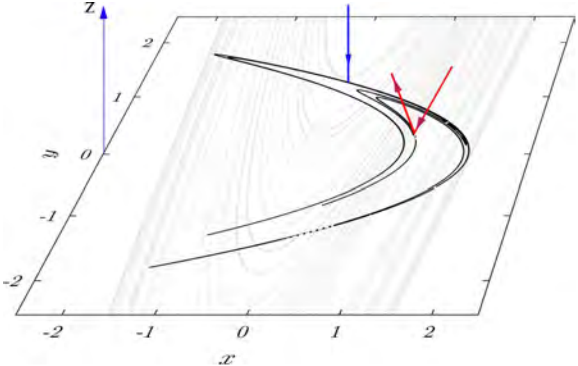


spatiotemporal chaos is extensive



summary for the impatient

state space of dissipative flow is split into



- inertial manifold : spanned locally by entangled covariant vectors, tangent to unstable / stable manifolds
- the rest : spanned by the remaining ∞ of the contracting, decoupled, transient covariant vectors

Benettin *et al.*[9] Lyapunov exponents algorithm relies on construction of orthogonal sets of Gram-Schmidt vectors. They are not covariant, i.e., the Gram-Schmidt vectors at a given state space point are not mapped by the linearized dynamics into the Gram-Schmidt vectors of the forward images of this point

in contrast, the Jacobian matrix J_n eigenvectors $\mathbf{e}^{(i)}$, which span the d-dimensional tangent space, and are covariantly transported by the flow, are generically not normal

[9] G. Benettin et al., Meccanica **15**, 9–20 (1980).

strongly contracting $\Lambda^{(j)}$ multiplier forward in time, becomes the leading $1/\Lambda^{(j)}$ multiplier backward in time. Matrix power method then pulls out this eigenvalue as the leading one within the subspace $T\mathcal{M}^{[j]}$

increase the dimension of the subspace by one, and you get the next $\Lambda^{(j+1)}$

repeat, and you get all eigenvalues and eigenvectors, even those insanely contacting ones, like $\Lambda^{(j)} \approx 10^{-137}$

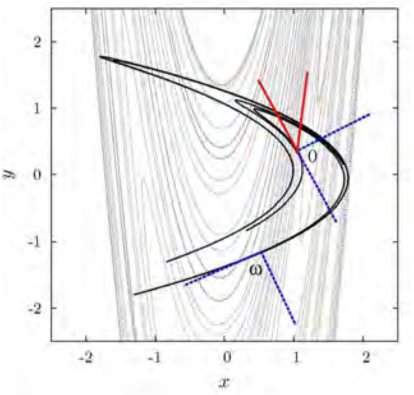
‘physical’ tangent space is transported across the whole curved strange manifold ergodically, and nonhyperbolicity of the attractor is used as a test that trajectory initiated along a given direction stays within the attractor

probability density of angles between adjacent physical eigenvectors computed over long time is flat, not peaked at 90^o

‘trivial,’ hyperbolically decaying eigen-directions that are isolated exhibit no such small inter-angles anywhere along the ergodic trajectory

angle between stable / unstable eigendirections

Hénon attractor



non-hyperbolicity

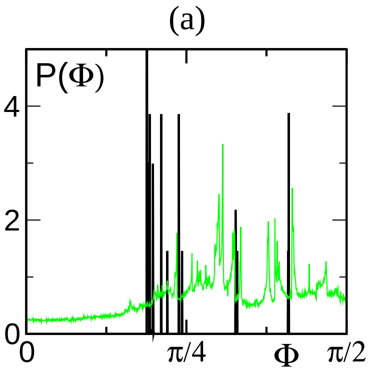
attractor (black line) and a finite-length approximation of its stable manifold (dotted line)

red vectors: the covariant vectors at the state space point

blue vectors: Gram-Schmidt vectors

separate the women from girls : hyperbolicity

Hénon attractor



Hénon map (green)

$x_{n+1} = 1 - 1.4\, x_n^2 + 0.3 x_{n-1}$

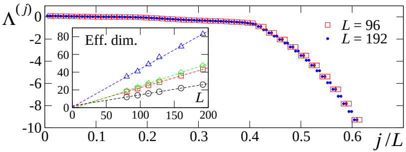
Lozi map (black)

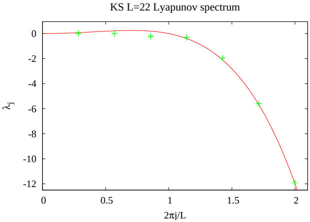
$x_{n+1} = 1 - 1.4\, |x_n| + 0.3 x_{n-1}$

probability distribution of the angle between stable and unstable eigenvector the leading Lyapunov vectors are tangent to the attractor. Perturbations that are on the attractor can be found in the subspace of the leading Lyapunov vectors

the main advance of using Lyapunov vectors instead of eigenvalues alone is that the approximate orthogonality of the ‘isolated’ ones provides a clear threshold between the ‘physical’ and the rest

example : Kuramoto-Sivashinsky flow



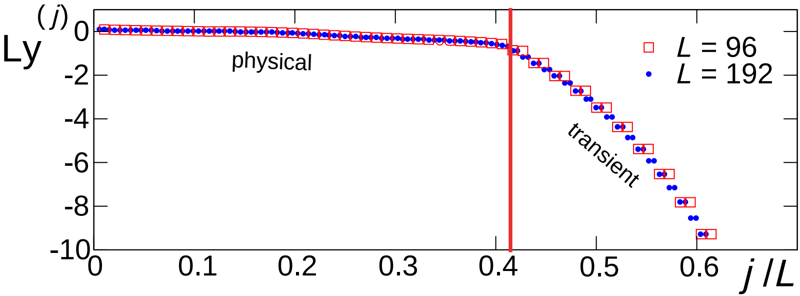


(left) Inset: number of non-negative exponents (circles), Kaplan-Yorke dimension (squares), metric entropy (diamonds, multiplied by 50), and number of physical Lyapunov vectors (triangles).

(right) First 14 Lyapunov exponents λ_j for the full state space, periodic b.c. KS for $L = 22$, from a 62 real Fourier modes long-time simulation

Full line corresponds to the stability eigenvalues of the $u(x,t) = 0$ stationary solution $(j/\tilde{L})^2 - (j/\tilde{L})^4$, for arbitrary system size L.

Kuramoto-Sivashinsky physical dimension grows linearly with the domain size!



Now double # Fourier modes : all new ones go to the transient spectrum [10]

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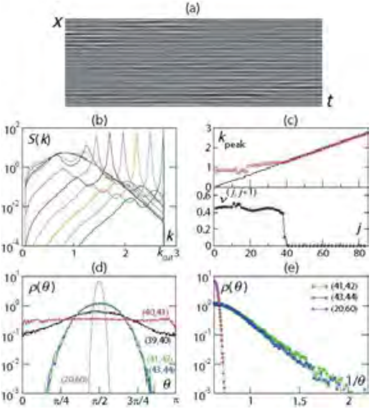
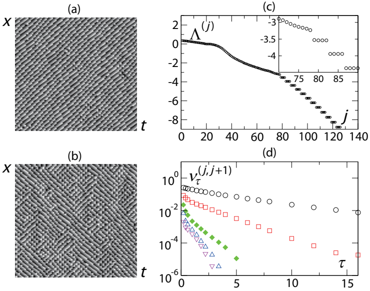
[10] Yang et al (Phys. Rev. Lett. 2009)

example : Kuramoto-Sivashinsky flow

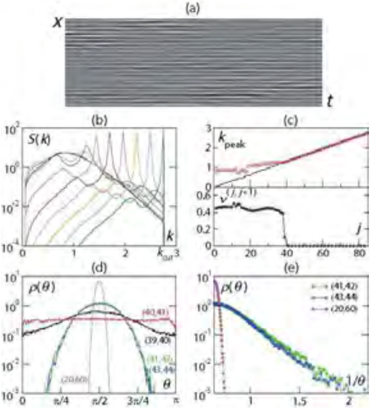
dimension of the Kuramoto-Sivashinsky flow is roughly four times the number of positive/marginal Floquet (or Lyapunov) exponents, and twice its Kaplan-Yorke estimate one might be unimpressed by the KS example, as the $-k^4$ hyper-diffusion term kills all higher Fourier modes very effectively

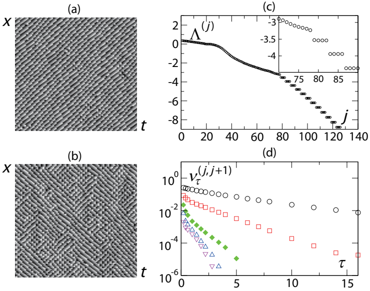
complex Ginzburg-Landau equation is persuasive; the nonlinearity is of $u(x)^3$ variety (instead of $u\partial u$ of Navier-Stokes and KS), but there is only a $-k^2$ diffusive term, and nevertheless there is a clear threshold for the 'isolated' Lyapunov vectors



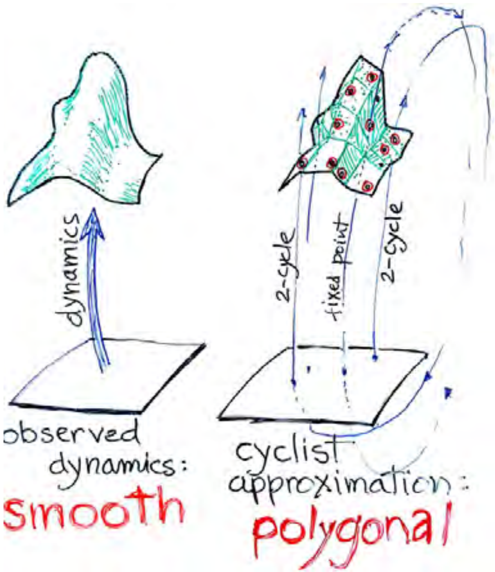


(left panel) KS system, $L = 96$. (a) Spatiotemporal plot of a typical vector ($j = 46$) in the isolated region. (b) Spatial power spectra of vectors of indices $j = 1, 16, 32, 38, 44, 52, 60, 68, 76, 84$. (c) Top panel: peak wavenumber in the power spectra (red circles) and $k = [j/2] \cdot 2\pi/L$ (black line). Bottom panel: DOS violation fraction $\nu_\tau^{(j,j+1)}$, neighboring vectors. (d) (e) Angle distributions between pairs of vectors.





(right panel) Complex Ginzburg-Landau, amplitude turbulence regime, $L = 64$. (a,b) Spatiotemporal plots of the phase component of a typical vector $j = 91$ in the isolated region (c) Lyapunov spectrum; inset: close-up around threshold. (d) Time fraction $\nu_\tau^{(j,j+1)}$ of DOS violation, as a function of τ ($j = 78$, 82, 86, 90, 94, from top to bottom).



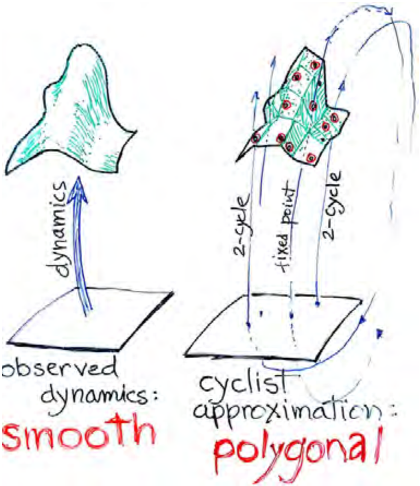
this is the periodic-orbit implementation of the idea of state space tessellation follow an ant as it traces out a trajectory $\hat{x}^{(1)}(\tau)$, confined to the Poincaré section $\mathcal{P}^{(1)}$ The moment $\langle(\hat{x}^{(1)}(\tau) - \hat{x}'^{(2)})|v^{(2)}\rangle$ changes sign, the ant has crossed the ridge and continues its merry stroll within the $\mathcal{P}^{(2)}$ Poincaré section we propose to construct a global atlas by deploying a set of linear Poincaré sections in neighborhoods of the most important equilibria and/or periodic orbits as local charts

physical task is to, for a given dynamical flow, pick a set of qualitatively distinct templates whose Poincaré sections are locally tangent to the strange attractor

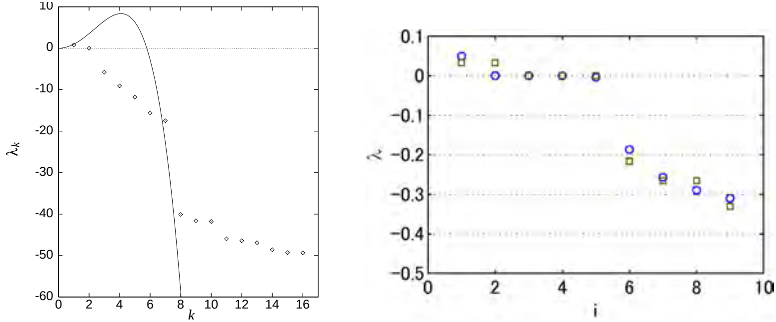
state space tessellation by periodic orbits

two 1-cycles

a 2-cycle that alternates between the neighborhoods of the two 1-cycles, shadowing first one of the two 1-cycles, and then the other



smooth dynamics (left frame) tesselated by the skeleton of periodic points, together with their linearized neighborhoods, (right frame)



(left) Lyapunov exponents λ_k versus k for the periodic orbit $\overline{1}$ compared with the stability eigenvalues of the $u(x, t) = 0$ stationary solution $k^2 - \nu k^4$. λ_k for $k \geq 8$ fall below the numerical accuracy of integration and are not meaningful

(right) Lyapunov exponents λ_j for the full state space, periodic b.c. KS for $L = 22$, from a 124 real Fourier modes (blue circles) long-time simulation overlayed on the $T_p = 10.2534$ periodic orbit (green squares)

periodic orbits

the idea is to coarsely cover the *continuous-symmetry reduced* nonlinear strange attractor with a set of tangent hyperplanes

any adjacent pair intersects in a ‘ridge’ hyperplane of one less dimension

the task:

for a given strange attractor, pick a set of Poincaré section-fixing points, such that each local section is approximately tangent to the strange attractor

résumé

if a physical flow is confined to a lower-dimensional manifold, one should use this fact to implement a dimensionality reduction

we have described dimensionality reduction by the method linear Poincaré sections, a linear procedure particularly simple and practical to implement

We propose instead to construct a global atlas by deploying sets of linear Poincaré sections as charts of neighborhoods of the most important (relative) equilibria and/or (relative) periodic orbits

résumé
6 ways to determine the dimension of the inertial manifold

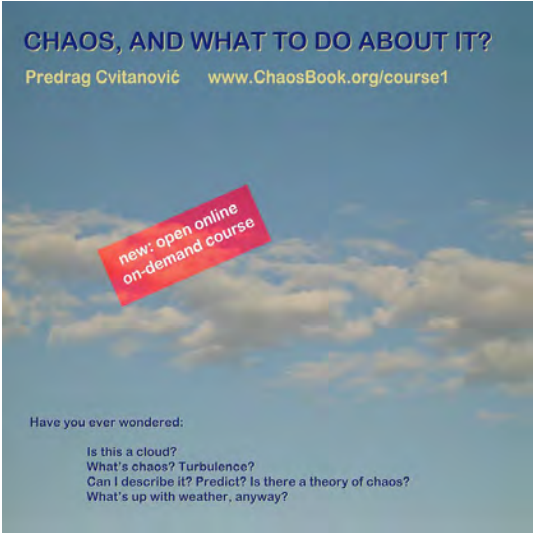
Tangent spaces separate into entangled vs. transient

1. Lyapunov exponents (plausible, previous work)
2. Lyapunov vectors (sharp, previous work)
3. for each individual orbit Floquet exponents separate into entangled vs. transient (new)
4. for an ensemble of orbits principal angles between hyperplanes spanned by Floquet vectors separate into entangled vs. transient (new)
5. for a chaotic trajectory shadowing a given orbit the separation vector lies within the orbit's Floquet entangled manifold (new)
6. for a chaotic trajectory shadowing a given orbit the separation vector lies within the chaotic trajectories covariant vectors' entangled manifold (new)

homework - bonus points:

do it for Navier-Stokes!

what next? take the course!



student raves :
...10^6 times harder than any other online course...