got symmetry?
here is how you slice it

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dynamical description of turbulent flows

state space

a manifold $\mathcal{M} \in \mathbb{R}^d$: d numbers determine the state of the system

representative point

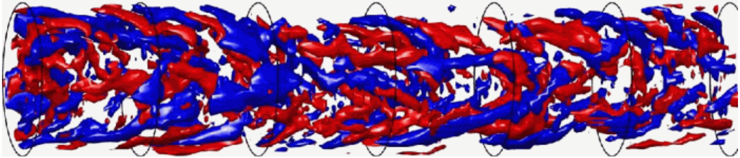
$x(t) \in \mathcal{M}$
a state of physical system at instant in time

today's experiments

example of a representative point

$x(t) \in \mathcal{M}$, $d = \infty$
a state of turbulent pipe flow at instant in time

Stereoscopic Particle Image Velocimetry $\rightarrow$ 3-*d* velocity field over the entire pipe[1]

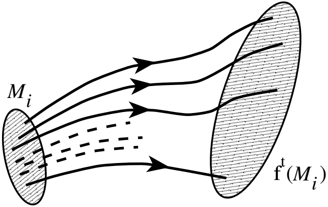


[1] Casimir W.H. van Doorne (PhD thesis, Delft 2004)

deterministic dynamics

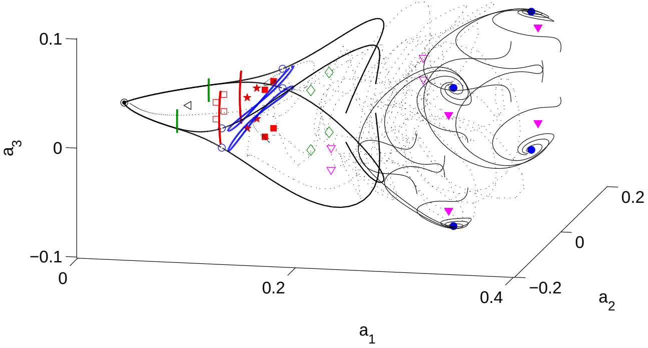
map $f^t(x_0)$ = representative point time t later

evolution



f^t maps a region $\mathcal{M}_i$ of the state space into the region $f^t(\mathcal{M}_i)$.

have : chart over 61,506 dimensional state space of turbulent flow



equilibria of turbulent plane Couette flow, their unstable manifolds, and a turbulent video mapped out as one happy family

for movies, please click through ChaosBook.org/tutorials today's talk's focus:

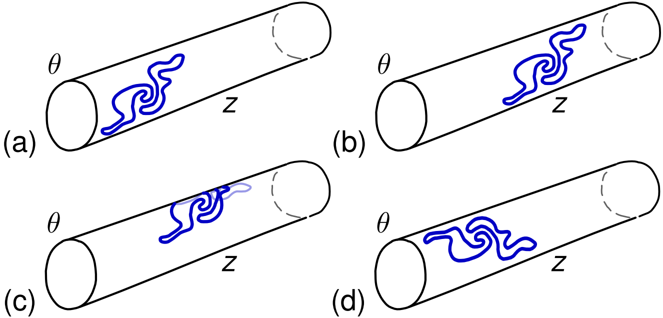
nature loves symmetry

symmetry of a dynamical system

a group G is a symmetry of the dynamics if

for every solution $f(x) \in \mathcal{M}$ and $g \in G$, $gf(x)$ is also a solution

example: $\mathrm{SO}(2)_z \times \mathrm{O}(2)_\theta$ symmetry of pipe flow



a solution, shifted by a stream-wise translation, azimuthal rotation g_p is also a solution

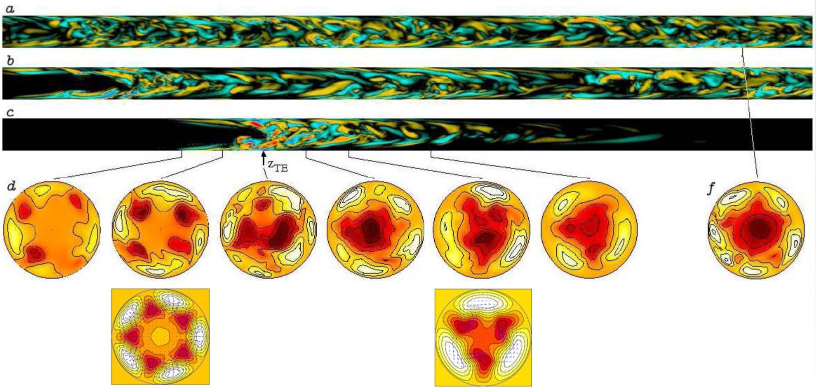
b) stream-wise
c) stream-wise, azimuthal
d) azimuthal flip

Das Problem

> mathematicians like symmetry more than Nature
>
> Rich Kerswell

turbulence in pipe flows

pipe flows : amazing data! amazing numerics!



Nature, she don't care : turbulence breaks all symmetries

Die Faulheit

drifting is energetically cheap

flows are lazy, rather than doing work, solutions drift along non-shape-changing symmetry directions

Das Problem

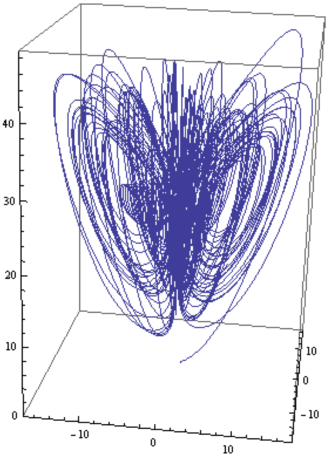
complex Lorenz equations

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{y}_1 \\ \dot{y}_2 \\ \dot{z} \end{bmatrix} = \begin{bmatrix} -\sigma x_1 + \sigma y_1 \\ -\sigma x_2 + \sigma y_2 \\ (\rho_1 - z)x_1 - \rho_2 x_2 - y_1 - e y_2 \\ \rho_2 x_1 + (\rho_1 - z)x_2 + e y_1 - y_2 \\ -bz + x_1 y_1 + x_2 y_2 \end{bmatrix}$$

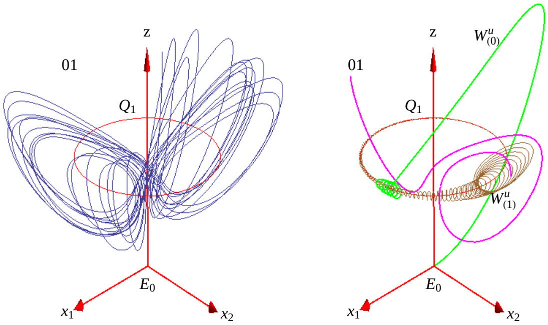
$\rho_1 = 28, \rho_2 = 0, b = 8/3, \sigma = 10, e = 1/10$

- A typical $\{x_1, x_2, z\}$ trajectory
- superimposed: a trajectory whose initial point is close to the relative equilibrium Q_1

attractor



continuous symmetry induces drifts



- generic chaotic trajectory (blue)
- E_0 equilibrium
- E_0 unstable manifold - a cone of such (green)
- Q_1 relative equilibrium (red)
- Q_1 unstable manifold, one for each point on Q_1 (brown)
- relative periodic orbit $\overline{01}$ (purple)

Das Durcheinander

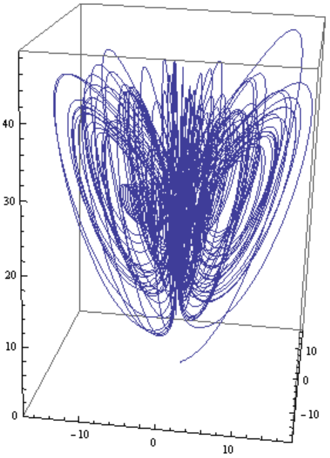
what to do?

it's a mess

the goal

reduce this messy strange attractor to something simple

attractor



Die Lösung

what to do?

it's a mess

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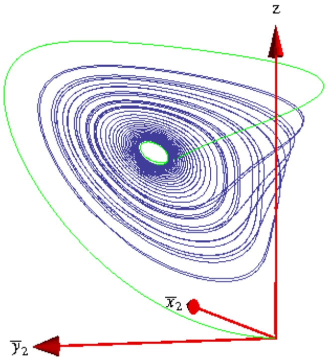
reduce this messy strange attractor to something simple

symmetry reduced state space



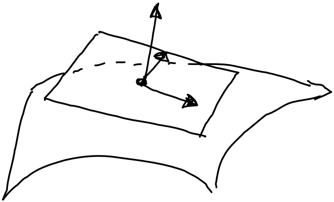
amazing!

Das Gebot

what I teach you now you must do

symmetries of dynamics

time vs. shifts



$v(x)$: tangent along the time flow

$t^{(1)}(x)$, $t^{(2)}(x)$: two group tangents along infinitesimal symmetry shifts

a flow $\dot{x} = v(x)$ is G-equivariant if

$$v(x) = g^{-1}\, v(g\, x)\,, \qquad \text{for all } g \in G\,.$$

equations of motion of the same form in all frames

example: SO(2) invariance

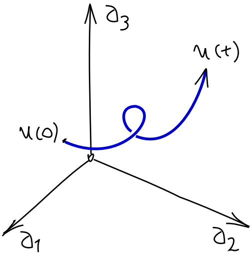
complex Lorenz equations

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{y}_1 \\ \dot{y}_2 \\ \dot{z} \end{bmatrix} = \begin{bmatrix} -\sigma x_1 + \sigma y_1 \\ -\sigma x_2 + \sigma y_2 \\ (\rho_1 - z)x_1 - \rho_2 x_2 - y_1 - e y_2 \\ \rho_2 x_1 + (\rho_1 - z)x_2 + e y_1 - y_2 \\ -bz + x_1 y_1 + x_2 y_2 \end{bmatrix}$$

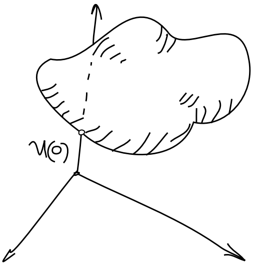
invariant under a SO(2) rotation by finite angle ϕ:

$$g(\phi) = \begin{pmatrix} \cos\phi & \sin\phi & 0 & 0 & 0 \\ -\sin\phi & \cos\phi & 0 & 0 & 0 \\ 0 & 0 & \cos\phi & \sin\phi & 0 \\ 0 & 0 & -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

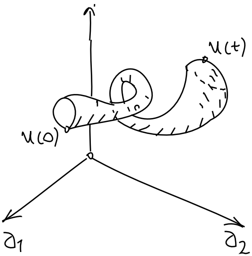
trajectories, orbits



trajectory $x(t)$



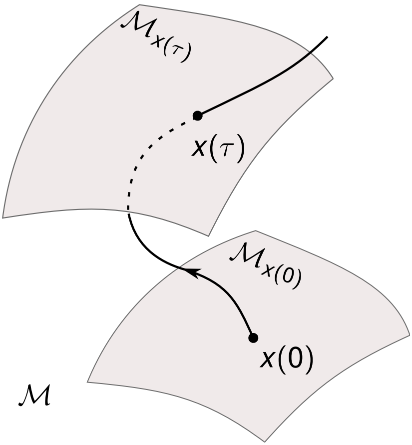
group orbit $g\,x(0)$



wurst $g\,x(t)$

stratification by group orbits

group orbits

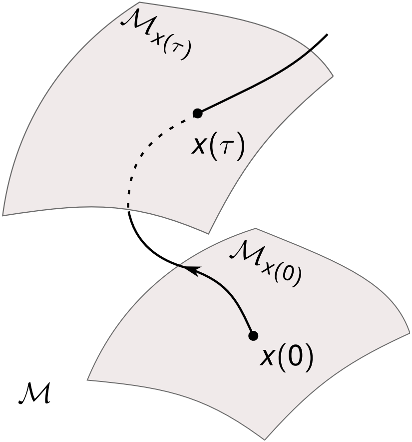


group orbit $\mathcal{M}_x$ of x is the set of all group actions

$$\mathcal{M}_x = \{g\,x \mid g \in G\}$$

stratification by group orbits

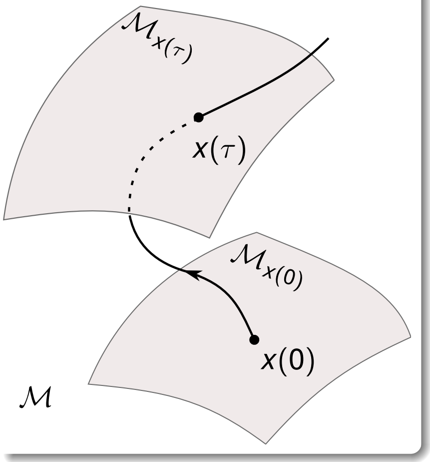
group orbits



any point on the manifold $\mathcal{M}_{x(t)}$ is equivalent to any other

stratification by group orbits

group orbits



action of a symmetry group stratifies the state space into a union of group orbits

each group orbit an equivalence class

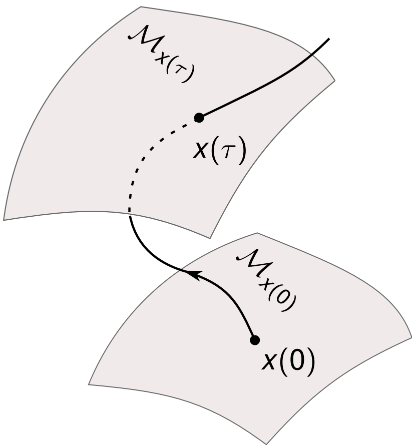
the goal

replace each group orbit by a unique point in a lower-dimensional

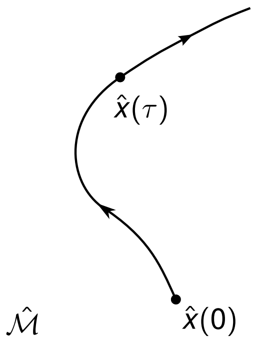
symmetry reduced state space $\mathcal{M}/G$

symmetry reduction

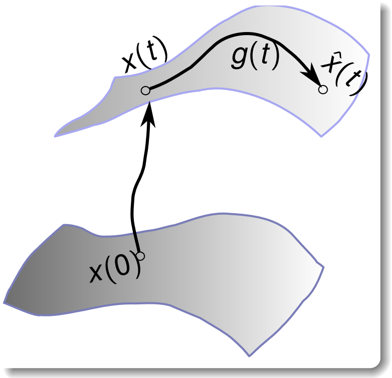
full state space



reduced state space



moving frame



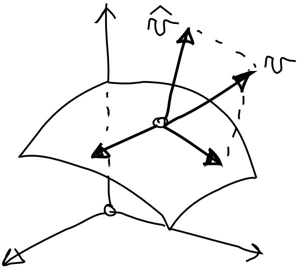
Cartan : can move wherever

free to redefine the flow to any time-dependent frame moving along symmetry directions

how relativists do it : connections or 'gauge fixing'

2-continuous parameter symmetry :
each state space point x owns 3 tangent vectors

local tangent space



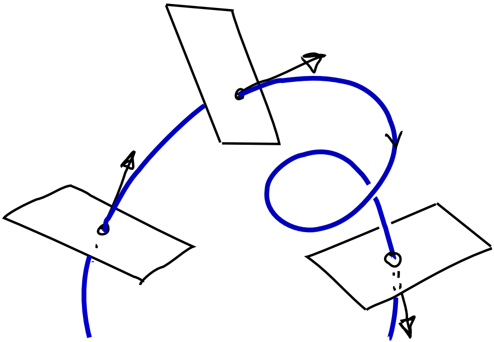
$v(x)$ along the time flow

$t^{(1)}(x)$, $t^{(2)}(x)$ along infinitesimal symmetry shifts

Kim Jong Il gauge

follow flow $\hat{v}(x)$ normal to group tangent directions

method of “connections”



never stray along the group directions, always move orthogonally to the group orbit

North Korean gauge :
slacking along non-shape-changing directions is forbidden

sophisticates do it : Faddeev-Popov gauge fixing

the equivalence principle

integrate over classes of gauge equivalent fields
instead of all fields A_μ^a

the representative in the class of equivalent fields is fixed by a gauge condition,

$$\partial_\mu A_\mu^a = 0\,,$$

a plane intersected by the gauge orbits

$$A_\mu = A_\mu^a t_a \to A_\mu^\Omega = \Omega A_\mu \Omega^{-1} + \partial_\mu \Omega \Omega^{-1}$$

- abelian orbits intersect the plane at the same angle
- non-abelian intersection angle depends on the field

Zutiefst Nutzlos

elegant, deep and useless : no symmetry reduction

method of slices

cut group orbits by a hypersurface (not a Poincaré section), each group orbit of symmetry-equivalent points represented by the single point

cut how?

inspiration: pattern recognition

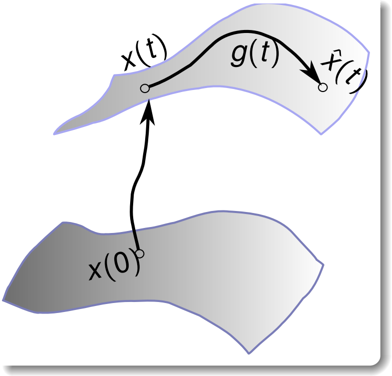
you are observing turbulence in a pipe flow, or your defibrillator has a mesh of sensors measuring electrical currents that cross your heart, and

you have a precomputed pattern, and are sifting through the data set of observed patterns for something like it

here you see a pattern, and there you see a pattern that seems much like the first one

how ‘much like the first one?’

moving frame



move until distance minimized

take the first pattern

'template' or 'reference state'

a point $\hat{x}'$ in the state space $\mathcal{M}$

and use the symmetries of the flow to

slide and rotate the 'template'

act with elements of the symmetry group G on $\hat{x}' \to g(\phi)\,\hat{x}'$

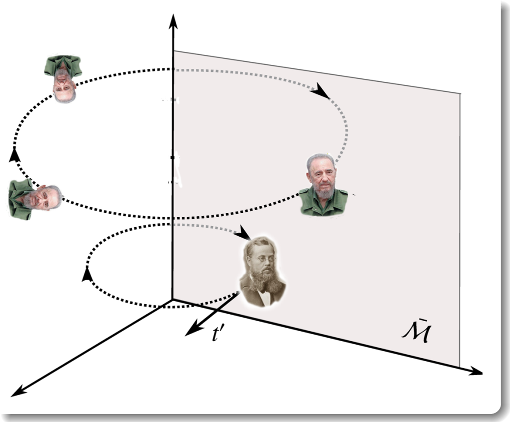
until it overlies the second pattern (a point x in the state space)

distance between the two patterns

$$|x - g(\phi)\,\hat{x}'| = |\hat{x} - \hat{x}'|$$

is minimized

idea: the closest match



template: Sophus Lie

(1) rotate bearded guy x traces out the group orbit $\mathcal{M}_x$

(2) replace the group orbit by the closest match $\hat{x}$ to the template pattern $\hat{x}'$

the closest matches $\hat{x}$ lie in the $(d-N)$ symmetry reduced state space $\hat{\mathcal{M}}$

distance

assume that G is a subgroup of the group of orthogonal transformations $\mathrm{O}(d)$, and measure distance $|x|^2 = \langle x|x\rangle$ in terms of the Euclidean inner product

numerical fluids: PDE discretization independent L2 distance is

energy norm

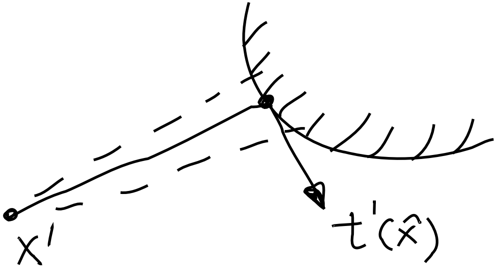
$$\|\mathbf{u}-\mathbf{v}\|^2 = \langle \mathbf{u}-\mathbf{v}|\mathbf{u}-\mathbf{v}\rangle = \frac{1}{V}\int_\Omega d\mathbf{x}\,(\mathbf{u}-\mathbf{v})\cdot(\mathbf{u}-\mathbf{v})$$

experimental fluid:

image discretization independent distance

is Hamming distance, or ???

idea: the closest match



extremal condition for nearest distance

minimal distance

is a solution to the extremum conditions

$$\frac{\partial}{\partial \phi_a} |x - g(\phi)\, \hat{x}'|^2$$

but what is

$$\frac{\partial}{\partial \phi_a} g(\phi)\,?$$

infinitesimal transformations

$$g \simeq 1 + \phi \cdot \mathbf{T}\,, \qquad |\delta\phi| \ll 1$$

- T_a are generators of infinitesimal transformations
- here T_a are $[d \times d]$ antisymmetric matrices

example: $\mathrm{SO}(2)$ invariance of complex Lorenz equations

complex Lorenz equations equations are invariant under $\mathrm{SO}(2)$ rotation by finite angle ϕ:

$$g(\phi) = \begin{pmatrix} \cos\phi & \sin\phi & 0 & 0 & 0 \\ -\sin\phi & \cos\phi & 0 & 0 & 0 \\ 0 & 0 & \cos\phi & \sin\phi & 0 \\ 0 & 0 & -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$\mathrm{SO}(2)$ has one generator of infinitesimal rotations

$$\mathbf{T} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

now have the ‘slice condition’

group tangent fields

flow field at the state space point x induced by the action of the group is given by the set of N *tangent fields*

$$t_a(x)_i = (\mathbf{T}_a)_{ij} x_j$$

slice condition

$$\frac{\partial}{\partial \phi_a} |x - g(\phi)\, \hat{x}'|^2 = 2\, \langle \hat{x} - \hat{x}' | t'_a \rangle = 0\,, \qquad t'_a = \mathbf{T}_a \hat{x}'$$

flow within the slice

slice fixed by $\hat{x}'$

reduced state space $\hat{\mathcal{M}}$ flow $\hat{v}(\hat{x})$

$$
\begin{aligned}
\hat{v}(\hat{x}) &= v(\hat{x}) - \dot{\phi}(\hat{x}) \cdot t(\hat{x}) \,, \qquad\qquad \hat{x} \in \hat{\mathcal{M}} \\
\dot{\phi}_a(\hat{x}) &= (v(\hat{x})^T t'_a)/(t(\hat{x})^T \cdot t') \,.
\end{aligned}
$$

- v : velocity, full space
- $\hat{v}$: velocity component in slice
- $\dot{\phi} \cdot t$: velocity component normal to slice
- $\dot{\phi}$: reconstruction equation for the group phases

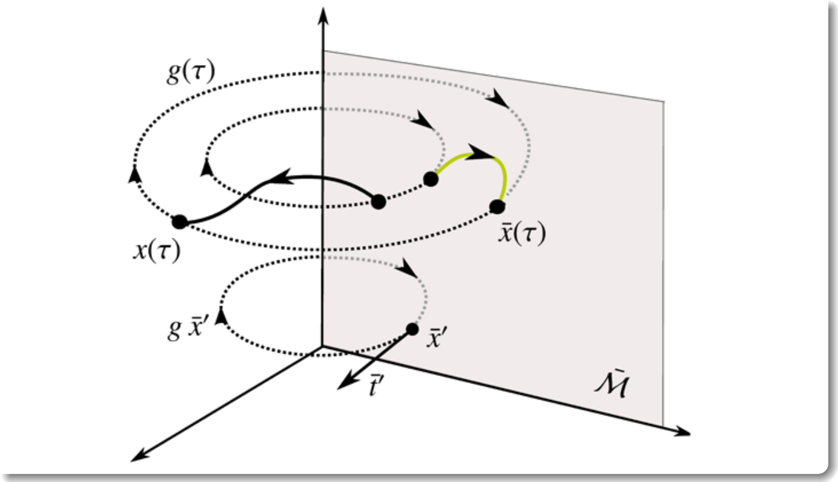
make Phil Morrison happy

call this

Cartan derivative

$$g^{-1}\dot{g}\,x = e^{-\phi\cdot\mathbf{T}}\,\frac{d}{d\,\tau}e^{\phi\cdot\mathbf{T}}x = \dot{\phi}\cdot t(x)$$

flow within the slice



full-space trajectory $x(\tau)$
rotated into the reduced state space $\hat{x}(\tau) = g(\phi)^{-1} x(\tau)$
by appropriate *moving frame* angles $\{\phi(\tau)\}$

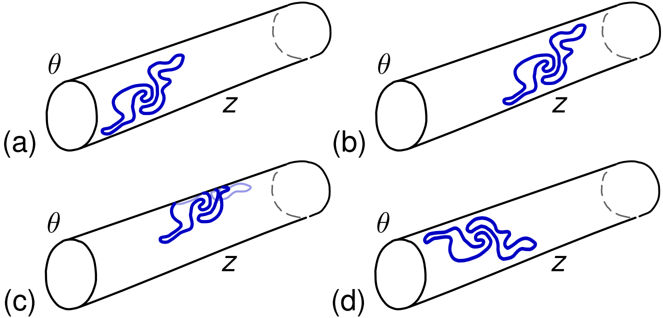
relative periodic orbit

a relative periodic orbit p is an orbit in state space $\mathcal{M}$ which exactly recurs

$$x_p(t) = g_p x_p(t + T_p)\,, \qquad x_p(t) \in \mathcal{M}_p$$

for a fixed relative period T_p and a fixed group action $g_p \in G$ that “rotates" the endpoint $x_p(T_p)$ back into the initial point $x_p(0)$.

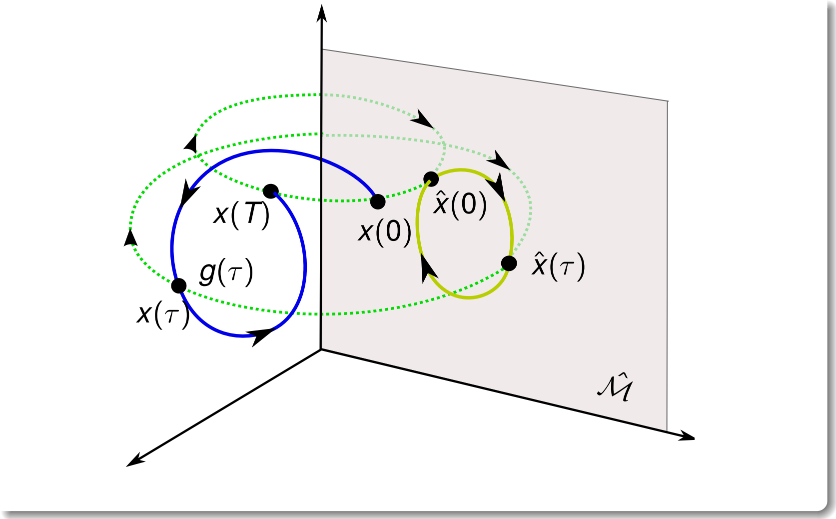
relative periodic orbits : $SO(2)_z \times O(2)_\theta$ symmetry of pipe flow



relative periodic orbit : recurs at time T_p, shifted by a streamwise translation, azimuthal rotation g_p

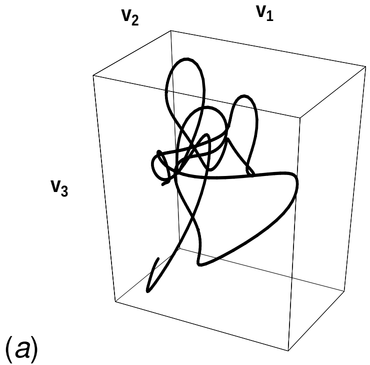
- **b)** stream-wise recurrent
- **c)** stream-wise, azimuthal recurrent
- **d)** azimuthal flip recurrent

relative periodic orbit→ periodic orbit



full state space relative periodic orbit $x(\tau)$
is rotated into the reduced state space periodic orbit

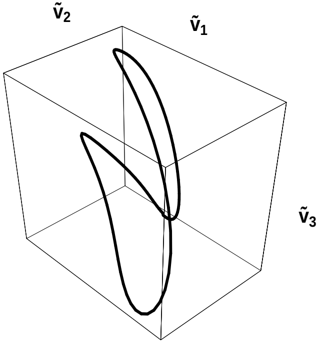
in full state space



a relative periodic orbit of the Kuramoto-Sivashinsky flow, 128d state space traced for four periods T_p, projected on

full state space coordinate frame $\{v_1, v_2, v_3\}$; a mess

in slice



(*b*)

a relative periodic orbit of the Kuramoto-Sivashinsky flow projected on

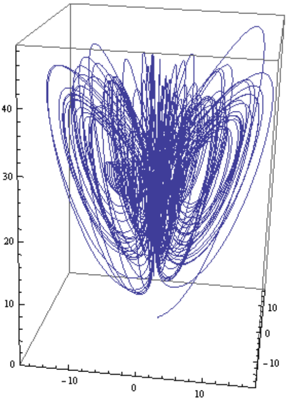
a slice $\{\tilde{v}_1, \tilde{v}_2, \tilde{v}_3\}$ frame

symmetry reduction achieved!

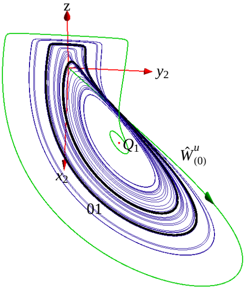
- all points equivalent by symmetries are represented by
  - a single point
- families of solutions are mapped to a single solution
  - relative equilibria become equilibria
  - relative periodic orbits become periodic orbits

die Lösung : complex Lorenz flow reduced

full state space



reduced state space



ergodic trajectory was a mess, now the topology is reveled
relative periodic orbit $\overline{01}$ now a periodic orbit

take-home message

rotation into a slice is not an average
over 3D pipe azimuthal angle

it is the full snapshot of the flow embedded in the

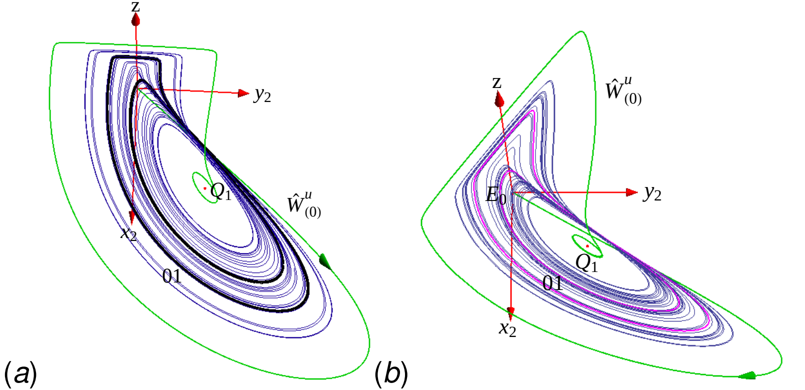
∞-dimensional state space

NO information is lost by symmetry reduction

- not modeling by a few degrees of freedom
- no dimensional reduction

slice trouble 1

portrait of complex Lorenz flow in reduced state space



any choices of the slice $\hat{x}'$ exhibit flow discontinuities

slice trouble 1

glitches!

group tangent of a generic trajectory orthogonal to the slice tangent at a sequence of instants τ_k

$$t(\tau_k)^T \cdot t' = 0$$

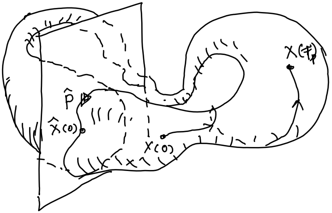
Nature couples many Fourier modes

group orbits of highly nonlinear states are highly contorted: many extrema, multiple sections by a slice

sliced wurst

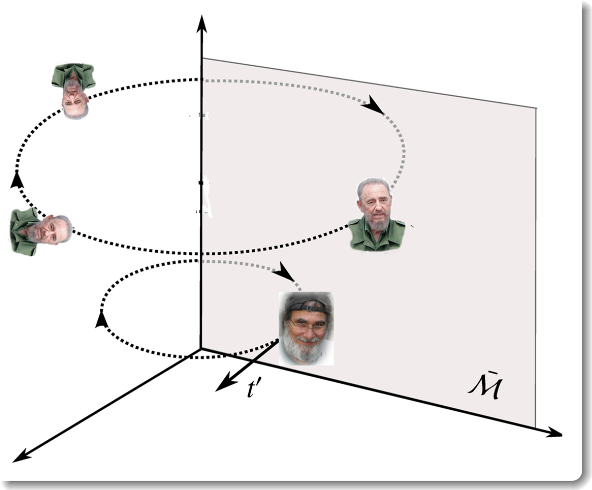
a slice hyperplane cuts every group orbit at least twice

slice



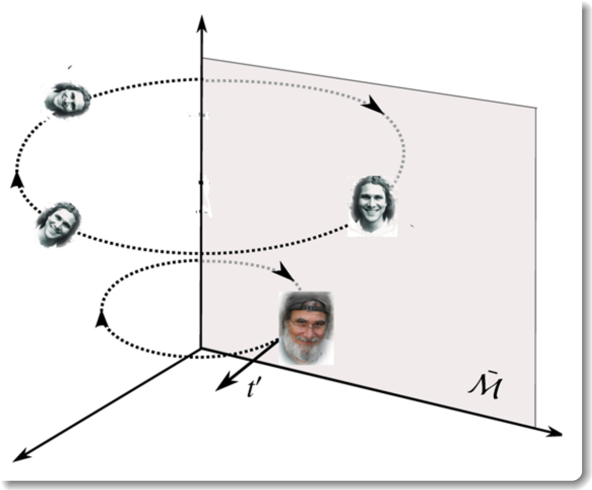
an $SO(2)$ relative periodic orbit is topologically a torus : the cuts are periodic orbit images of the same relative periodic orbit, the good close one, and the rest bad ones

trouble: slices cannot be global



representing a group orbit by the closest match to a good template $\hat{x}'$ (Phil Morrison)

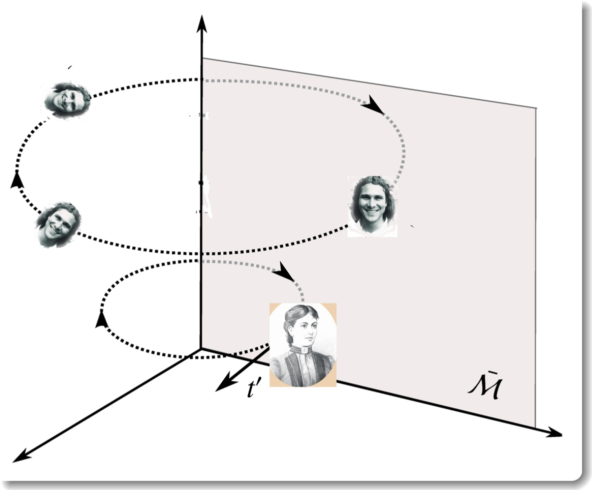
trouble: slices cannot be global



the 'closest match' to a bad template $\hat{x}'$ (young Phil Morrison) can be a mismatch

single template cannot be a good match globally

trouble: slices cannot be global

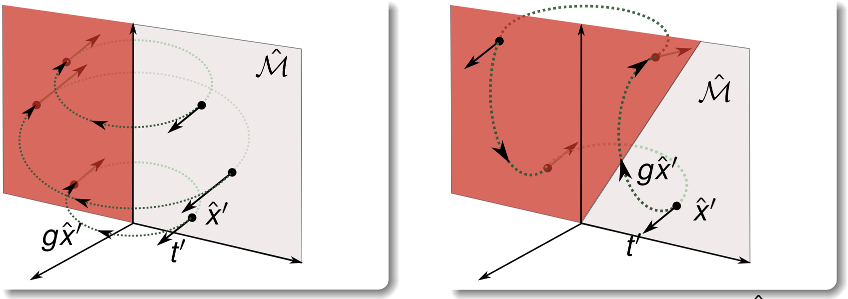


representing a group orbit by the closest match to a better template $\hat{x}'$ (Sonya Kovalewskaya)

to cover $\mathcal{M}/G$ globally, need:
a set of templates:

- 2 rolls
- 4 rolls
- ...

slice is good up to the chart border



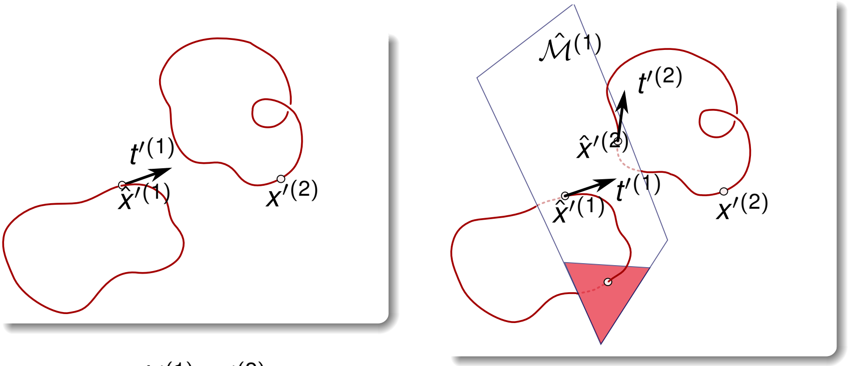
$\mathrm{SO}(2)$: two hyperplanes to a given template $\hat{x}'$; the slice $\hat{\mathcal{M}}$, and *chart border* $\hat{x}^* \in S$. Beyond :
group orbits pierce in the wrong direction
(a) a circle group orbit crosses the slice hyperplane twice.
(b) a group orbit for a combination of $m = 1$ and $m = 2$ Fourier modes resembles a baseball seam, and can be sliced 4 times, out of which only the point closest to the template is in the slice

charting the state space

for turbulent/chaotic systems a set of Poincaré sections is needed to capture the dynamics. The choice of sections should reflect the dynamically dominant patterns seen in the solutions of nonlinear PDEs

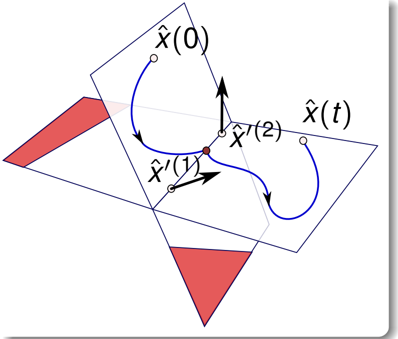
we propose to construct a global atlas of the dimensionally reduced state space $\hat{\mathcal{M}}$ by deploying linear Poincaré sections $\mathcal{P}^{(j)}$ across neighborhoods of the qualitatively most important patterns $\hat{x}'^{(j)}$

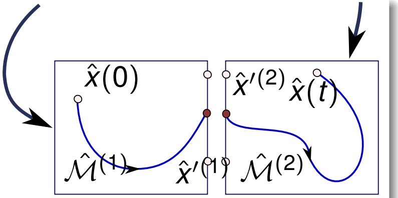
2-chart atlas



templates $\hat{x}'^{(1)}$, $x'^{(2)}$, with group orbits. Start with the template $\hat{x}'^{(1)}$. All group orbits traverse its $(d-1)$-dimensional slice hyperplane, including the group orbit of the second template $x'^{(2)}$. Replace the second template by its closest group-orbit point $\hat{x}'^{(2)}$, i.e., the point in slice $\hat{\mathcal{M}}^{(1)}$.

2-chart atlas



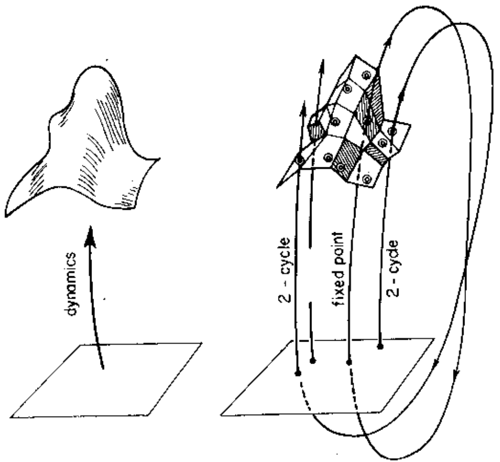


atlas of $(d-1)$-dimensional charts $\hat{\mathcal{M}}^{(1)}, \hat{\mathcal{M}}^{(2)}, \cdots$

two templates are the closest points viewed from either group orbit, they lie in both slices.

tangent vectors have different orientations, hence two distinct slice hyperplanes $\hat{\mathcal{M}}^{(1)}$ and $\hat{\mathcal{M}}^{(2)}$ which intersect in the *ridge*, a hyperplane of dimension $(d-2)$ (here drawn as a 'line') shared by the template pair.

the chart for the neighborhood of each template (a page of the atlas on the right side of the figure) extends only as far as this ridge. If the templates are sufficiently close, the chart border of



this is the periodic-orbit implementation of the idea of state space tessellation

summary

conclusion

- 'gauge fixing' - no insight into geometry of flows
- symmetry reduction by method of slices:
  efficient, allows exploration of high-dimensional flows hitherto unthinkable

to be done

- construct Poincaré sections
- use the information quantitatively (periodic orbit theory)

take-home message

if you have a symmetry

use it!

without symmetry reduction, no understanding of fluid flows, nonlinear field theories possible

amazing theory! amazing numerics! hope...

"Ask your doctor if taking a pill to solve all your problems is right for you."

triumph : all pipe flow solution in one happy family

first ’turbulent’ relative periodic orbits for pipe flows!