Symmetry reduced averages over moderately turbulent flows

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Newton Institute - Sep 9, 2008

Turbulence: A walk through a repertoire of unstable recurrent patterns?

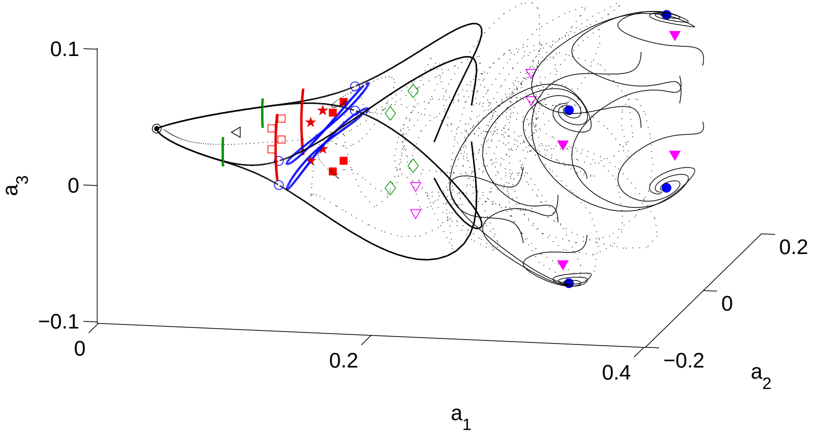
As a turbulent flow evolves, every so often we catch a glimpse of a familiar pattern:

 $\Longrightarrow$ other swirls $\Longrightarrow$ 

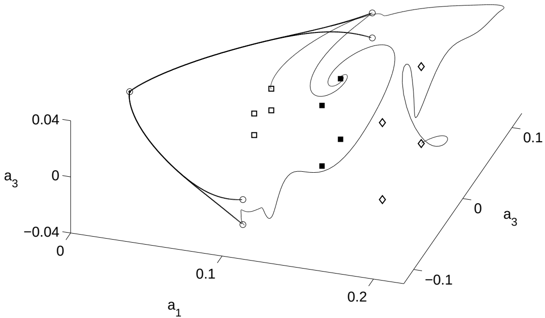
For any finite spatial resolution, the system follows approximately for a finite time a pattern belonging to a finite alphabet of admissible patterns. The long term dynamics =

a walk through the space of such unstable patterns.

Have: chart over state space



Have: state space heteroclinic connection cycles



EQ_3, EQ_4, EQ_5, $\to EQ_1$ heteroclinic connections: □,■, and ◊

EQ_0 ($\odot$) at the origin: laminar solution

○: EQ_2 + half shifts

Pipes and planes: what for?

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Want [an invitation to a discussion?]:

- long-time averages: dissipation/power input $\langle D \rangle , \cdots$

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- transport
    - turbulent Lagrangian mixing
    - turbulent heat (passive scalar) transport
    - wind driven ocean streams

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- 'image compression'

Trace formula for a deterministic flow

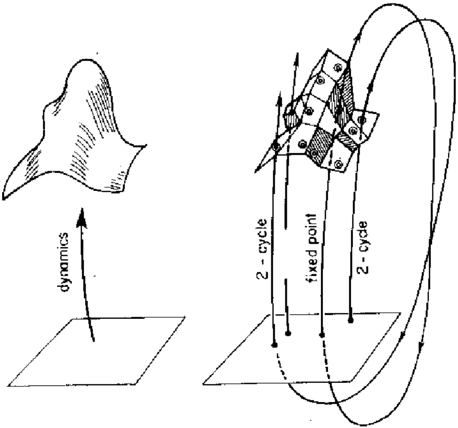
The long time averages on chaotic / turbulent / ergodic flows are given by the classical trace formula for flows:

$$\sum_{\alpha=0}^{\infty} \frac{1}{s - s_\alpha} = \sum_p T_p \sum_{r=1}^{\infty} \frac{e^{r(\beta \cdot A_p - s T_p)}}{\left| \det \left(\mathbf{1} - M_p^r \right) \right|} \,.$$

(no way to derive this in a slide)

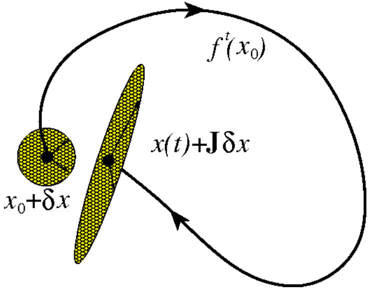
What does this mean?

Partitioning of state space by periodic orbits



How big is the neighborhood of a given cycle?

Fundamental matrix

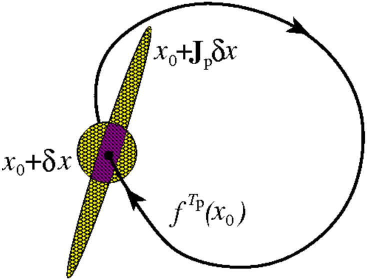


J_p maps spherical neighborhood of x_0
$\longrightarrow$
ellipsoidal neighborhood time t later

Neighbors

- separate along unstable directions
- approach each other along stable directions
- creep along the marginal directions

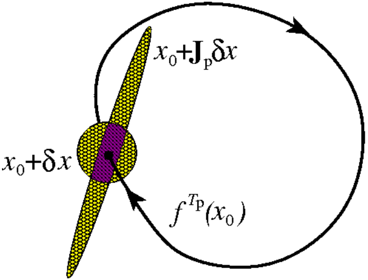
Cycle neighborhood



cycle p fundamental matrix J_p returns an infinitesimal spherical neighborhood of $x_0 \in p$ stretched into an ellipsoid

- overlap ratio along the expanding eigdirection $\mathbf{e}^{(i)}$ of $J_p(x)$ given by

Cycle neighborhood



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- the expanding eigenvalue $1/|\Lambda_{p,i}|$

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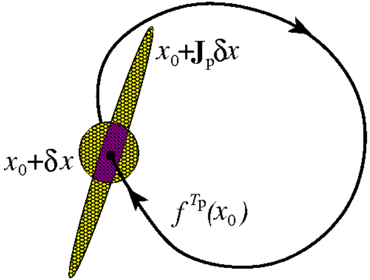

cycle p fundamental matrix J_p returns an infinitesimal spherical neighborhood of $x_0 \in p$ stretched into an ellipsoid

- overlap ratio along the expanding eigdirection $\mathbf{e}^{(i)}$ of $J_p(x)$ given by
- the expanding eigenvalue $1/|\Lambda_{p,i}|$
- $=$ fraction of trajectories still hanging out in the hood

Local trace

Fraction of trajectories that return to the neighborhood

$$t_p = \frac{1}{|\Lambda_\pi|} e^{-sT_p}$$

decreases exponentially

- with the cycle period

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- with the cycle period
- 'instability' $=$ product of expanding eigenvalues
  $\Lambda_p = \prod_e \Lambda_{p,e}$
- 1-few expanding directions, embedded in $10^4 - 10^6$-dimensional PDE discretization

Pseudocycles and shadowing

pseudocycles

$$t_\pi = (-1)^{k+1} t_{p_1} t_{p_2} \dots t_{p_k}$$

are sequences of shorter cycles that shadow a cycle with the symbol sequence $p_1 p_2 \dots p_k$ along segments p_1, $p_2, \dots,$ p_k.

Pseudocycle weight

$$t_\pi = (-1)^{k+1} \frac{1}{|\Lambda_\pi|} e^{-s T_\pi} \,.$$

falls off exponentially with the pseudocycle period and instability

$$T_\pi = T_{p_1} + \dots + T_{p_k} \,, \qquad \Lambda_\pi = \Lambda_{p_1} \Lambda_{p_2} \cdots \Lambda_{p_k} \,.$$

Cycle averaging formulas

Cycle averaging formulas for the expectation value of observable $a(x)$ (for example, the dissipation rate D of a given solution)

$$
\begin{aligned}
\langle a \rangle &= \langle A \rangle \,/\, \langle T \rangle \\
\langle A \rangle &= \sum{}' A_\pi t_\pi \\
\langle T \rangle &= \sum{}' T_\pi t_\pi
\end{aligned}
$$

A_π, T_π evaluated on pseudocycles

Example: binary symbolic dynamics

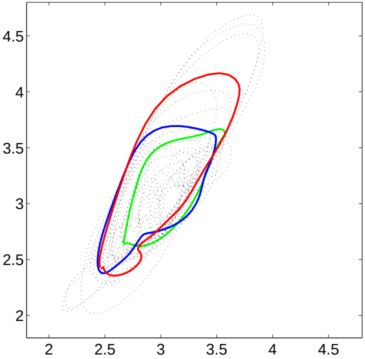
For the complete binary symbolic dynamics the mean cycle period $\langle T \rangle$ is given by

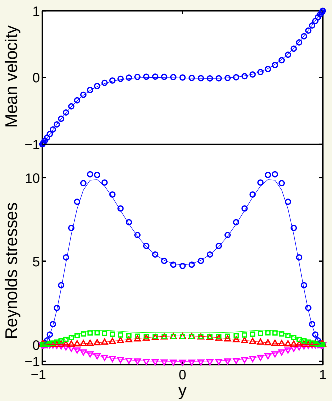
$$\begin{aligned}\langle T \rangle &= \frac{T_0}{|\Lambda_0|} + \frac{T_1}{|\Lambda_1|} + \left(\frac{T_{01}}{|\Lambda_{01}|} - \frac{T_0 + T_1}{|\Lambda_0 \Lambda_1|} \right) \\ &\quad + \left(\frac{T_{001}}{|\Lambda_{001}|} - \frac{T_{01} + T_0}{|\Lambda_{01} \Lambda_0|} \right) + \left(\frac{T_{011}}{|\Lambda_{011}|} - \frac{T_{01} + T_1}{|\Lambda_{01} \Lambda_1|} \right) + \dots .\end{aligned}$$

cycle expansions for averages are grouped into shadowing combinations, with nearby pseudocycles nearly cancelling each other.

A single periodic orbit: an example

Turbulent average (lines) versus periodic orbits (symbols)





blue cycle → core, boundary regions

Reynolds stresses[1]: $\langle u^2 \rangle$ $\langle uv \rangle$ $\langle v^2 \rangle$ $\langle w^2 \rangle$ in wall units

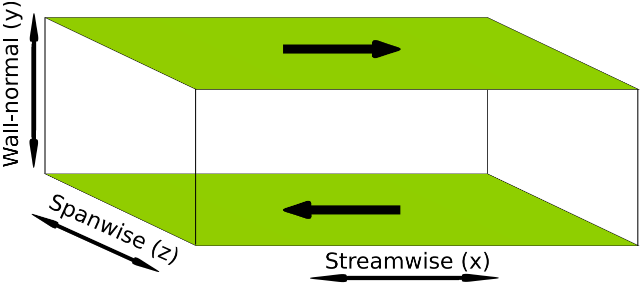
[1] Kawahara and Kida *JFM* (2002)

Pipes and planes: how to?

‘technical issues’

- Markov partition on state space
- symmetries
- ∞-aspect ratio cells
- periodic orbit averaging - ‘trace formulas’

Plane Couette flow



$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\,\mathbf{u} = -\boldsymbol{\nabla} p + \frac{1}{Re}\nabla^2 \mathbf{u}$$

$$\nabla \cdot \mathbf{u} = 0\,, \qquad \mathbf{u}(y = \pm 1) = \pm \hat{\mathbf{x}}$$

Plane Couette symmetries

Navier-Stokes equations are equivariant under

- $SO(2) \times SO(2)$ streamwise, spanwise translations $\tau(x, z)$

Symmetry group of plane Couette: $O(2) \times O(2)$

Plane Couette symmetries

Navier-Stokes equations are equivariant under

- $SO(2) \times SO(2)$ streamwise, spanwise translations $\tau(x, z)$
- flips and inversions $R = \{e, \sigma_x, \sigma_z, \sigma_{xz}\}$, where

$$\sigma_x[u, v, w](x, y, z) = [-u, -v, w](-x, -y, z)$$
$$\sigma_z[u, v, w](x, y, z) = [u, v, -w](x, y, -z)$$
$$\sigma_{xz}[u, v, w](x, y, z) = [-u, -v, -w](-x, -y, -z)$$

Symmetry group of plane Couette: $O(2) \times O(2)$

Plane Couette discrete translations

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- generated by

$$\tau_x\,[u, v, w](x, y, z) = [u, v, w](x + L_x/2, y, z)$$
$$\tau_z\,[u, v, w](x, y, z) = [u, v, w](x, y, z + L_z/2)$$

S-invariant solutions

- Physical arguments (Waleffe SSP and long simulations) suggest that $S = \{e, s_1, s_2, s_3\}$ is an important invariant subspace, with

$$
\begin{aligned}
s_1\,[u,v,w](x,y,z) &= [u,v,-w](x+L_x/2, y, -z) \\
s_2\,[u,v,w](x,y,z) &= [-u,-v,w](-x+L_x/2, -y, z+L_z/2) \\
s_3\,[u,v,w](x,y,z) &= [-u,-v,-w](-x,-y,-z+L_z/2)
\end{aligned}
$$

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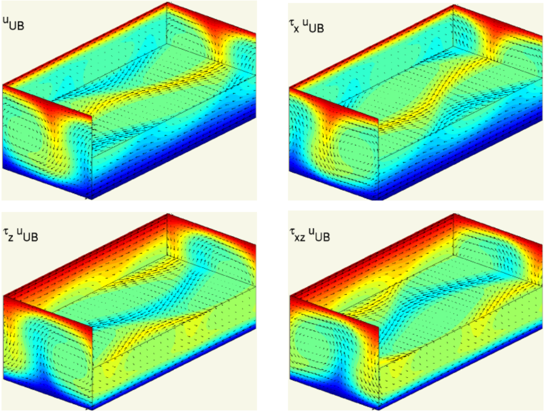
$$s_3\,[u, v, w](x, y, z) = [-u, -v, -w](-x, -y, -z + L_z/2)$$

- half-cell translations generate 4 symmetric copies of each state within the S-invariant subspace.

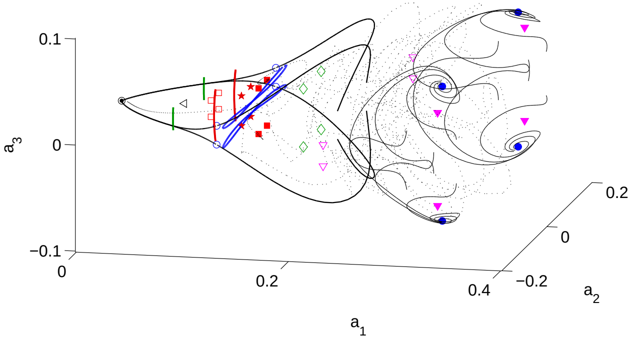
Discrete symmetry: 1/2-shifts equivariance



S-invariant subspace: continuous translations $\rightarrow$ discrete group

$$T = \{1, \tau_x, \tau_z, \tau_{xz}\}$$

1/2-shifts equivariance in state space

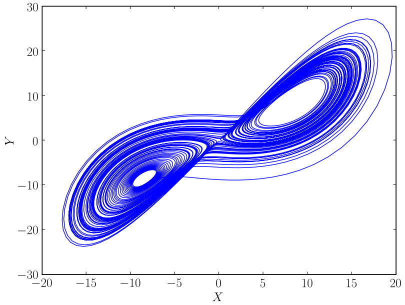


EQ_2 comes in 4 copies

Lorenz flow

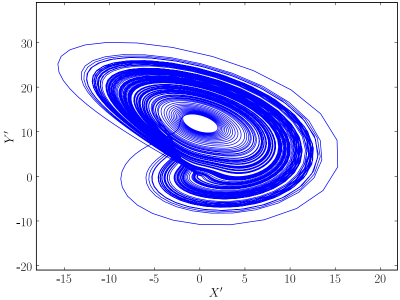
$$\dot{x} = \sigma(y - x)$$
$$\dot{y} = \rho x - y - xz$$
$$\dot{z} = xy - bz$$

EQ_1 comes in 2 copies



(Lorenz)/Z_2 quotient ('orbit,' 'image') space

- equivariant under $x, y \to -x, -y$



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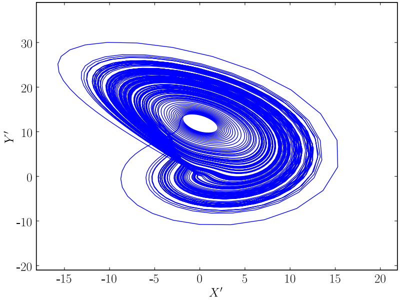
- identify equivalent points, so



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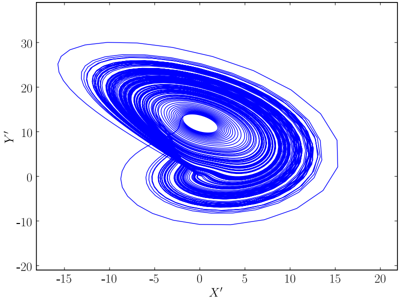

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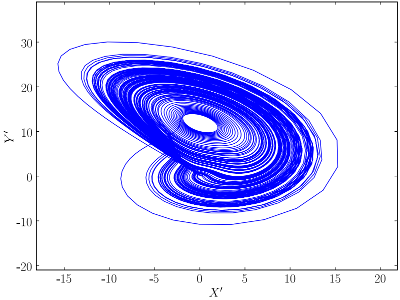
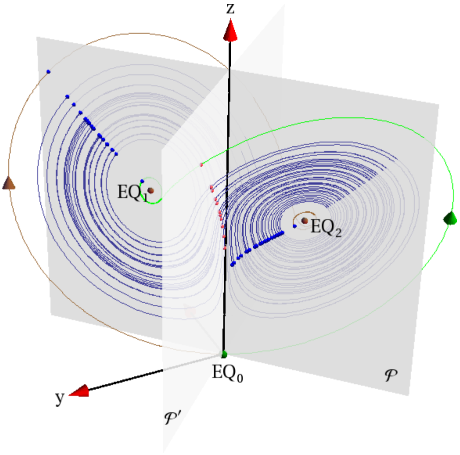
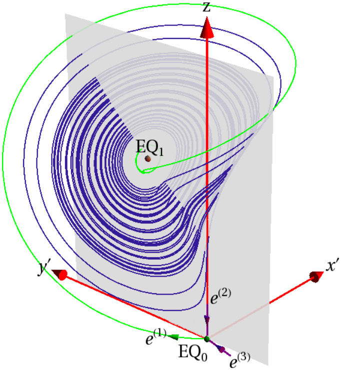


- identify equivalent points by plotting

$$[r, \theta, z] \to [r, 2\theta, z] = [(x^2 - y^2)/r, 2xy/r, z]$$

Poincaré section



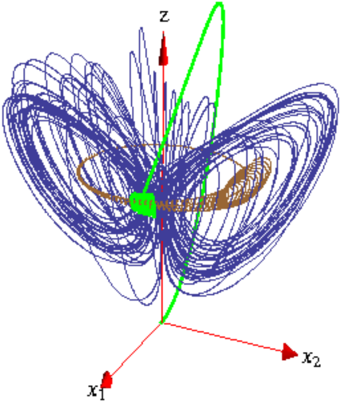


Equivariant flow with continuous symmetry

5-dimensional "Complex Lorenz" model of baroclinic instability in the atmosphere[2]

$$
\begin{aligned}
\dot{x}_1 &= -\sigma x_1 + \sigma y_1 \\
\dot{x}_2 &= -\sigma x_2 + \sigma y_2 \\
\dot{y}_1 &= (r_1 - z)x_1 - r_2 x_2 - y_1 - e y_2 \\
\dot{y}_2 &= r_2 x_1 + (r_1 - z)x_2 + e y_1 - y_2 \\
\dot{z} &= -bz + x_1 y_1 + x_2 y_2
\end{aligned}
$$

[dynamics for $r_1 = 28$, $b = 8/3$, $\sigma = 10$, $a = 1$, $e = 0.01$, $r_2 = 0$]



drifts in circles - rotation around z axis $\mathrm{SO}(2)$-equivariant.

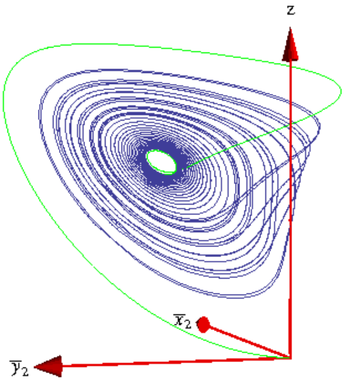
[2] Gibbon and McGuinness, *Physica D* **4** (1982)

Complex Lorenz/$\mathrm{SO}(2)$: invariant flow

Quotient $\mathrm{SO}(2)$ by plotting it in an invariant polynomial basis

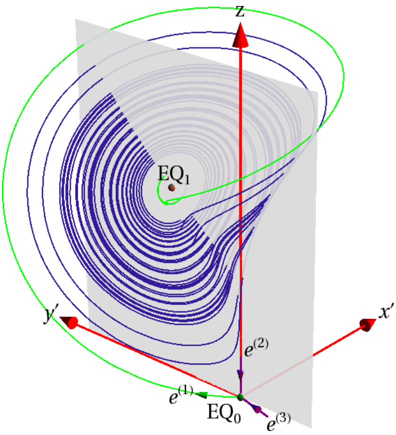
$$\bar{x}_1 = 0 \quad \text{(group section)}$$
$$\bar{x}_2 = (y_1^2 + y_2^2)/r$$
$$\bar{y}_1 = -(x_2 y_1 - x_1 y_2)/r$$
$$\bar{y}_2 = (x_1 y_1 + x_2 y_2)/r$$
$$\bar{z} = z\,, \quad r^2 = x_1^2 + x_2^2 + y_1^2 + y_2^2$$

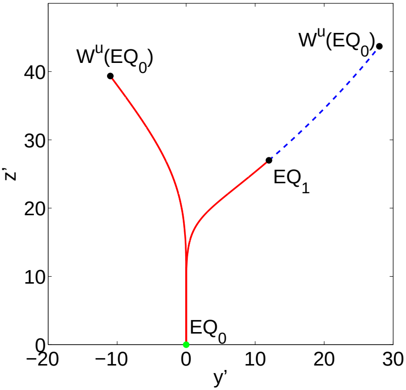
$4d$ flow, no $\mathrm{SO}(2)$-equivariance drift!



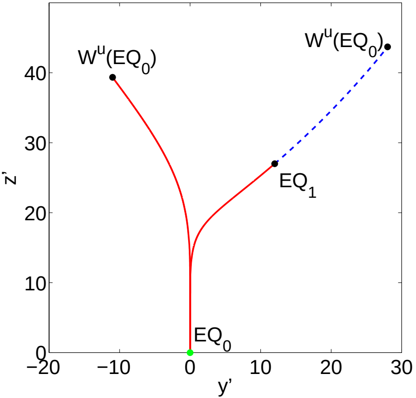
$\approx 1d$ return map; symbolic dynamics, periodic orbits

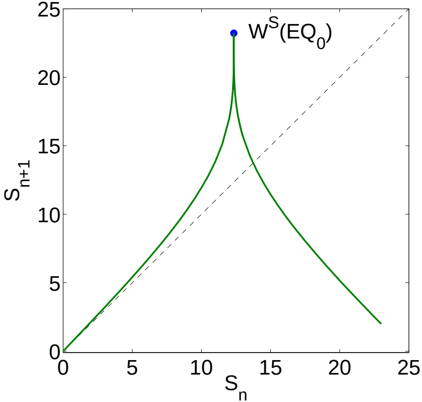
Poincaré section





Return map

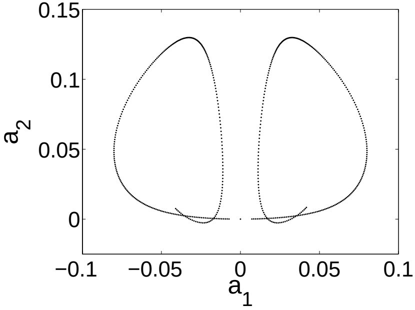




s = unstable manifold arclength along $EQ_1 \to W^u(EQ_0)$

Local unstable manifold return maps

Kuramoto-Sivashinsky: $\infty - d$ state space
but: each repelling Smale horseshoe has its approximate
local $1d$ return map $s \to f(s)$ onto the local unstable manifold:[3]



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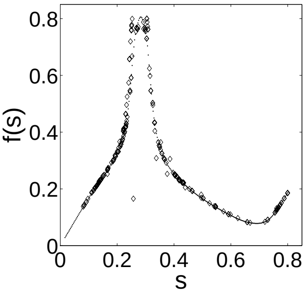

return map + cycles

symbolic dynamics, periodic orbits

[3] Lan and Cvitanović, *Phys. Rev. E* (2008)

“Fourier” analysis for compact groups

The new trace formula follows from Peter-Weyl Theorem

- representations of a compact group G are fully reducibile
- irreducible representations labeled by integers $m = (m_1, \cdots, m_N)$,
- character of irrep: $\chi_m(g)$
- projection operator onto the V_m irreducible subspace:

$$P_m = d_m \int_G [dg]\, \chi_m(g) D_m(g^{-1})\,.$$

Symmetry reduced trace formula

classical trace formula for flows:

$$\sum_{\alpha=0}^{\infty} \frac{1}{s - s_\alpha} = \sum_p T_p \sum_{r=1}^{\infty} \frac{e^{r(\beta \cdot A_p - s T_p)}}{\left|\det\left(\mathbf{1} - M_p^r\right)\right|} .$$

symmetry reduced trace formula for flows on irreducible subspace m:

$$\sum_{\beta=0}^{\infty} \frac{1}{s - s_{m,\beta}} = d_m \sum_p T_p \sum_{r=1}^{\infty} \chi_m(g_p^r) \frac{e^{r(\beta A_p - s T_p)}}{\left|\det\left(\mathbf{1} - \tilde{M}_{m,p}^r\right)\right|} .$$

sum over prime relative periodic orbits and their repeats

And thus our students became beautiful, in parts

Jonathan Halcrow → → Paris Hilton

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