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the laser fusion, laser isotope separation, and new laser development programs at LASL. The Division is heavily involved in fission reactor safety research, and to some extent in magnetic fusion research and other energy options. Separate sections in part II describe T-Division's role in the laser fusion, reactor safety, and other energy research efforts.

Part III, the latter half of the report, covers the major T-Division efforts by discipline. Under each discipline are reports from the appropriate group leaders outlining the research programs of their respective groups, as well as selected reports from individual staff members on recent research of special interest. In the interest of communicating to a wider audience, contributors have tried to present their work through verbal ideas and concepts rather than equations. We hope that these reports will generate interest and enthusiasm across disciplinary lines.

The Materials Science effort carried out by Group T-11, Statistical Physics and Materials Science, is described first. Group T-11 is performing the kind of complex function characteristic of many groups in T-Division—that is, undertaking high-quality research on problems of fundamental scientific interest, and thus maintaining strong credentials and ties with the academic community, and at the same time being responsive to the Laboratory needs by actively seeking out and solving practical problems with the most advanced theoretical tools.

Group T-12, Theoretical Molecular Physics, another new group, also performs a dual role as the theoretical arm of the laser isotope separation program and the major group at LASL devoted to theoretical molecular physics and theoretical chemistry.

The theoretical effort in nuclear and particle physics, comprising four separate groups, is perhaps widest in scope, ranging from the application of nuclear data in weapons and reactor problems to research on a fundamental quantum field theory of nuclei and the theory of elementary particles. This theoretical effort is closely tied to the experimental efforts in MP and P Divisions. Other efforts include Mathematics and Mathematical Physics, Theoretical Biology and Biophysics, Atomic and Molecular Physics, Fluid Dynamics, Transport Theory, and a small effort in Astrophysics. While the research in each area ranges from pure to applied, staff members from all groups are being encouraged to become even more responsive to the needs of the Laboratory.

The unusual working environment at LASL has produced a significant body of knowledge that is relatively unknown in the scientific community at large. Then too, there are many renowned scientists at LASL whose work is widely read but would benefit from a unified presentation. T Division has recently negotiated with University of California Press to publish these contributions in a series of monographs, "Los Alamos Series in Pure and Applied Sciences." Several manuscripts are presently under consideration for publication by University of California Press.

Indeed, communication with the scientific community outside the Laboratory is essential to the vitality of all LASL efforts. T Division therefore maintains a very active visitor and consultant program whose participants include many of the best scientists in the country. This past year, T Division, and in particular, members of T-7, have helped to organize two very successful conferences. The first, entitled "International Research Conference on the History of Computing," was intended, according to N. Metropolis and J. Worlton, to "record 'living history' in the context of discussions among the pioneers in the origins of electronic computing." A report on the conference by participant J. Howlett is reprinted at the end of this section by permission from the Association of Computing Machinery, Inc.

The other conference entitled "The Second Los Alamos workshop on Mathematics in the Natural Sciences" was held August 26—September 1, 1976; Rota (MIT and Los Alamos Fellow), Orszag (MIT and Visiting Staff Member), and Metropolis were the organizers. Four principal speakers each gave three lectures:

F. Dyson (Institute for Advanced Study), "The Inverse Scattering Problem,"

M. Kac (The Rockefeller Institute), "Nonlinear Differential Equations and Inverse Problems,"

J. Schwartz (Courant Institute of Math. Sciences), "Semantic Mechanisms for Controlling Parallel, Competing, and Mutually Suspicious Processes," and

A. P. Calderon (University of Chicago), "Singular Integrals and Pseudo-Differential Operators."

A lecture was given by each of twelve speakers, six by professors whose research could be related to

Laboratory projects, and six from the Laboratory's T Division who described some of the mathematical challenges arising in their work. The conference was very well attended and provoked a lively interaction among the participants.

One topic that enjoyed widespread interest and entered the discussion surprisingly frequently was soliton solutions to nonlinear differential equations. An article on solitons written by one of the speakers at the Conference, David Campbell of T-8, appears in the Nuclear and Particle Physics section. Another point of general interest was the successful use of computers as an empirical tool to help find new approaches to very complex mathematical problems. The work of M. Feigenbaum, T-DOT and conference speaker, is one such example. It represents a breakthrough in the understanding of one-dimensional discrete recursive systems which are used to model, for example, population dynamics. This work is described in Feigenbaum's article, "Universality in Complex Discrete Dynamics," in the section on "Mathematics and Mathematical Physics."

The pictures in Fig. I.2 were taken during the conference.

THEORETICAL DIVISION ORGANIZATION

MARCH 1977

DIVISION LEADER
Peter A. Carruthers

Associate Division Ldr.
Charles Critchfield

Alternate Division Ldr.
George I. Bell

Associate Division Ldr.
John F. Barnes

Associate Division Ldr.
L. M. Simmons, Jr.

Assistant for
Finance & Admin.
Betty J. Simes

GROUP T-1 – Transport & Reactor Theory
Group Ldr. – Warren F. Miller, Jr.
Alt. Grp. Ldr. – Donald J. Dudziak

GROUP T-2 – Nuclear Data
Group Ldr. – Phillip G. Young
Alt. Grp. Ldr. Raphael J. LaBauve
Assoc. Grp. Ldr. – Robert MacFarlane

GROUP T-3 – Hydrodynamics
Group Ldr. – Cyril W. Hirt
Alt. Grp. Ldr. – Thomas D. Butler
Assoc. Grp. Ldr. – Richard A. Gentry
Asst. Grp. Ldr. – Anthony A. Amsden

GROUP T-4 – Equation of State and Opacity
Grp. Ldr. – Walter F. Huebner
Assoc. Grp. Ldrs. – G. Kerley
A. Merts

GROUP T-5 – Medium Energy Physics Theory
Group Ldr. – Richard R. Silbar

GROUP T-6 – Laser Theory
Group Ldr. – Dale B. Henderson
Alt. Grp. Ldr. – Joseph M. Kindel

GROUP T-7 – Mathematic Modeling & Analysis
Group Ldr. – Burton Wendroff
Alt. Grp. Ldr. – Nicholas C. Metropolis

GROUP T-8 – Elementary Particles & Field Theory
Group Ldr. – Geoffrey B. West

GROUP T-9 – Nuclear Physics Theory
Group Ldr. – Charles L. Critchfield

GROUP T-10 – Theoretical Biology & Biophysics
Group Ldr. – George I. Bell

GROUP T-11 – Statistical Physics & Materials Theory
Group Ldr. – Donald F. DuBois
Assoc. Grp. Ldrs. – George A. Baker, Jr.
William Visscher

GROUP T-12 – Theoretical Molecular Physics
Group Ldr. – David C. Cartwright
Assoc. Grp. Ldr. – Cyrus Cantrell

GROUP T-14 – Detonation Theory & Application
Group Ldr. – C. L. Mader

Fig. I.1.

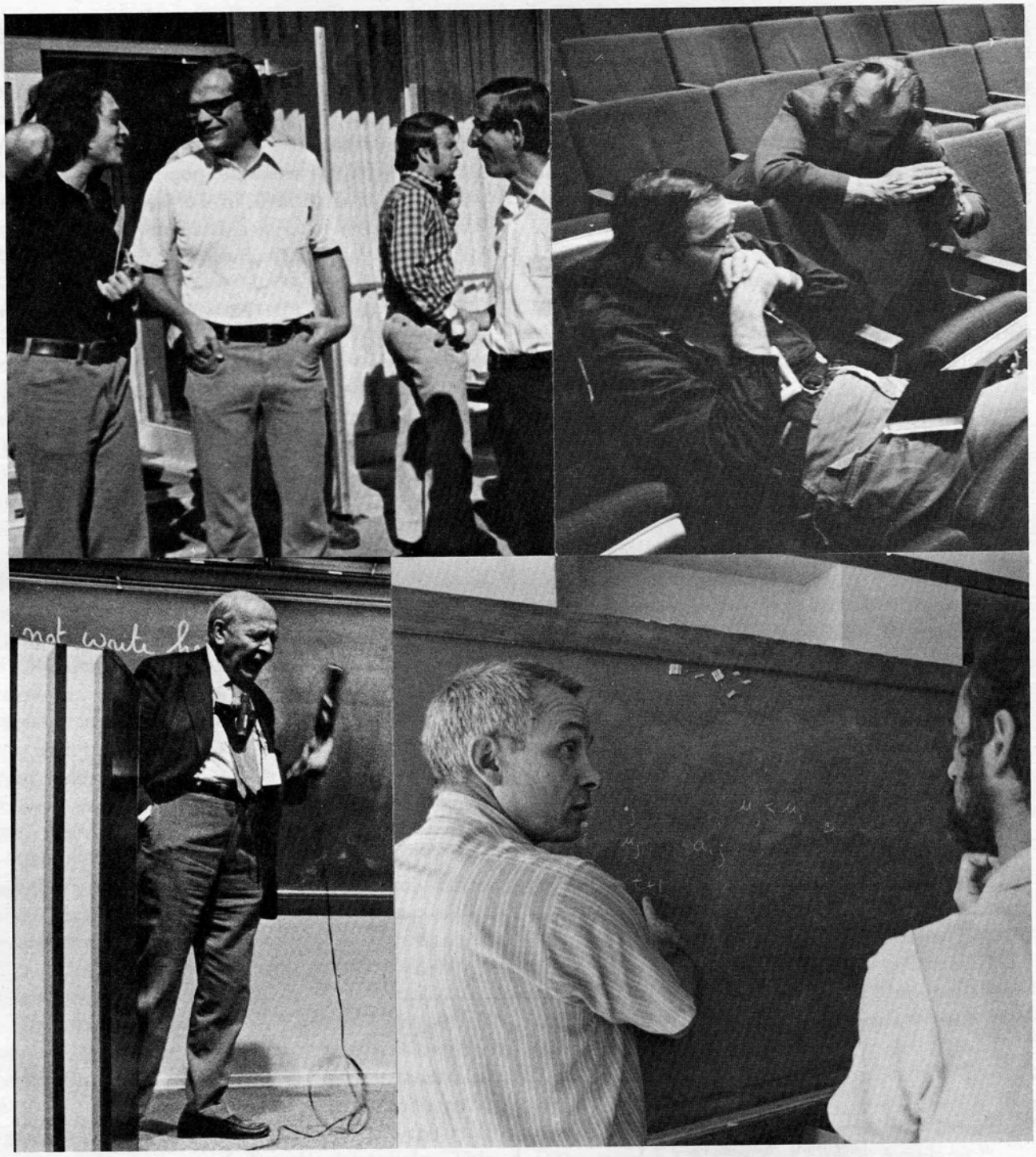

Fig. I.2.

Some candid shots of the participants at the Second Los Alamos Workshop on Mathematics in the Natural Sciences. Upper right: G.-C. Rota (ℓ) and N. Metropolis of T-7, conference organizers. Lower right: B. Wood of T-11 (ℓ) discusses his talk on numerical statistical mechanics with LASL consultant E. Lieb of Princeton. Lower left: M. Kac from the Rockefeller Institute at the podium. Upper left: Speaker M. Feigenbaum of T-DOT (ℓ) with R. Dashen and speaker F. Dyson from the Institute for Advanced Study.

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UNIVERSALITY IN COMPLEX DISCRETE DYNAMICS

Mitchell J. Feigenbaum, T-DOT

During the past year, a framework for the analysis of high iterates of functions of a general class has been fully cast. The theory exhibits a variety of novel features suggestive of a quite unorthodox

viewpoint toward complex dynamics such as one encounters in the evolution of a discrete population. As the problem analyzed is of a generally abstract nature, this example is useful to set forth our ideas.

Consider a population (say of fish in a controlled pond) that exists for a certain duration, spawns, and becomes extinct prior to the first signs of a new, mature population. Since there is a finite interval of time during which the mature population is nonexistent, it is impossible to describe the population through any analytic differential equation. Thus, one is led to consider a discrete equation

$$p_{n+1} = F(p_n, \alpha_i) ,$$

which relates the population of the n + 1st generation solely to that of the nth generation, but dependent upon certain parameters α_i (for example, the usable nutrient level in the pond, intrinsic reproductive capacity of the species, etc.) We simplify the situation by assuming that all factors that affect the population can be reflected in a single parameter ω. That is,

$$p_{n+1} = F(p_n, \omega) . \qquad \text{(III.3.1)}$$

It remains to specify the form of F. We shall specify only the two most salient features.

(1) When the population is very dilute, each individual is virtually independent from another: there is no competition for nutrient, no mutual predation, no mutual toxicity, etc. In this regime, each has ω progency, so that

$$p_{n+1} = \omega p_n + 0(p_n^2) , \qquad \text{(III.3.2)}$$

with deviations of order p_n^2, and negligible for sufficiently small p_n. Equation (III.3.2) has the elementary solution of geometric growth

$$p_n = \omega^n p_0$$

which grows arbitrarily large for $\omega > 1$, or systematically approaches extinction for $\omega < 1$. Only $\omega > 1$ is interesting, in which case, even if Eq. (III.3.2) is initially valid, it must ultimately fail as p_n grows.

(2) Evidently, if either through untampered evolution or external imposition, p_n is made so large that no individual can receive sufficient nutrient to survive, p_{n+1} will be very small.

We include a third, technical specification:

$$F(p,\omega) = \omega f(p) . \qquad \text{(III.3.3)}$$

Certainly Eq. (III.3.3) is correct for very small p. We really mean that

$$F(p,\omega) = \omega f(p,\omega) ,$$

but that the internal ω-dependence is negligible in some sense. This technical assumption is actually a minor restriction.

Beyond 1 and 2 (and technical 3) we assume little else: rather, we consider all F's that reflect the fundamental mechanisms of independence in the dilute regime, strong competition in the dense regime, and the least specific variation (i.e. no other mechanisms) in the intermediate regime. Thus, f of Eq. (III.3.3) looks like Fig. III.3.1 and is smooth (although break-points reflect specific mechanisms, we allow them also). Equations (III.3.1) and (III.3.3) combined

$$p_{n+1} = \omega f(p_n) \qquad \text{(III.3.4)}$$

together with Fig. III.3.1 to specify f (no equation for f is declared) completely model all mechanisms determinative of the population's growth. Yet, it would seem that in order to actually calculate the p_n's a quite exact form of f should be entered into Eq. (III.3.4), with different f's, each resembling Fig. III.3.1, leading to different p_n's. *In fact, such a precise specification is superfluous*! Equation (III.3.4) together with Fig. III.3.1 determines the ex-

Fig. III.3.1.
General form of the function f(p) of Eq. (III.3.3).

act sequence of p_n's (when, as is usual, the p_n's vary in a complex, almost chaotic fashion) apart from the "unit" of population; this universal information (universal since specific details of f are irrelevant) is highly detailed including absolute, universal numbers which must leave their imprint on the dynamics of the population. What is so impressive and unexpected, though, is that situations modeled in a certain fashion [say recursively, as Eq. (III.3.4)] that incorporate all salient qualitative features can receive a perfectly precise quantitative description independent of the exact, analytic specification. That is, we have a formula: qualitative similarity implies identical quantitative description, this quite opposite to the accustomed scientific process. While Eq. (III.3.4) is rather too simple, the nature of these results leads to speculation as to just how specific an equation (e.g., the Navier-Stokes equation) is necessary to determine hydrodynamic turbulence; just how much detailed neural information is required to comprehend the workings of a vast neural network, etc. These considerations are hardly idle since too many mechanisms of too detailed a nature obfuscate the workings of a system insensitive to detail, and present equations at best arduous, if not humanly impossible, to penetrate.

With these illustrative considerations and vaguely philosophical outlook presented, let us now describe with some precision the general problem solved. The context is Eq. (III.3.4) together with Fig. III.3.1 with a few technical details fixed. In this setting, Metropolis et al.[1] proved several years ago that a very rich qualitative behavior is universal over f. To describe this behavior, a few terms must be introduced. The simplest behavior of Eq. (III.3.4) is the approach to a value p* that satisfies

$$p^* = \omega f(p^*) \ .$$

Upon "attaining" the value p*, the system forever remains at that value (zero population growth). p* is the "fixed point" of ωf (at a fixed value of ω) or a "one-cycle." Another possibility is that Eq. (III.3.4) reproduces some value every k generations. The causal relation of p_{n+1} to p_n then implies that there are k distinct values each of this nature: the system forever runs through these k values in unchanging order. Such a set of points is a "k-cycle" of Eq. (III.3.4). Finally, there can be infinite "cycles": this is a sequence p_i of never repeating values that the system can move through, and that can be very sparsely distributed in the interval of possible values of p.

For any k-cycle, if placed exactly at a point of the set, the system will periodically return. However, should it be slightly perturbed away from the cycle, then either the system will reapproach the cycle pattern, or diverge away from it; the cycle is called stable or attractive in the first case, and unstable or repellent in the second. Should the system be started at some value p_0, it, in general, will not be in any k-cycle. Should some particular k-cycle be attractive for virtually all p_0's (such an attractor is called "global") then after some transient period, the system's behavior will be arbitrarily close to the periodic motion through that cycle. This is the fundamental simplification in solving Eq. (III.3.4): while an exact solution issuing out of an arbitrary p_0 might be impossible to calculate, providing there exists a global attractor, the calculation of the elements of that stable cycle suffices for the long-term behavior of Eq. (III.3.4). Thus, the problem of the dynamics of Eq. (III.3.4) reduces to the computation of the elements of each global attractor for each value of ω.

The results of Metropolis et al. can now be stated. For each ω there exists a unique stable attractor which is, moreover, global. As ω is increased, the ordering of the sizes of these attractors is universal. More specifically, if a cycle of order k is stable at ω_0, then it remains stable for some interval of ω above ω_0, becoming unstable at a value Ω_0 after which a 2k cycle is stable. This change of stability size is termed a "bifurcation." As ω is further increased, this phenomenon recurs with successive bifurcations doubling the size of the stability set, until at a value ω_∞ an infinite attractor is stable. Now, the range of ω in which all we discuss occurs is strictly bounded above (with appropriate normalization, bounded by 1).

We are ready to discuss our results.[2,3] We term a cycle many times bifurcated from a fundamental one "highly bifurcated," and our results apply to *all* highly bifurcated cycles (e.g., 2^n-cycles, $3x2^n$-cycles, etc.) First, the bifurcation values Ω_n increase with n and are bounded above, meaning that they converge. If we ask how $\Omega_n \to \omega_\infty$, a wonderful result obtains. Each Ω_n depends upon the specific choice of f,

as does an ω_∞. However, convergence is geometric, and the rate δ is *independent* of f:

$$\omega_\infty(f) - \Omega_n(f) \propto \delta^{-n} \quad n \gg 1 .$$

[There is a sole proviso: δ depends only on the nature of the maximum. Unless special constraints exist—which a priori should *not* in the population context—this maximum is a "normal" quadratic (or parabolic) maximum:

$$f(\bar{x}) - f(x) \propto (x - \bar{x})^2 \quad \text{for } x \text{ near } \bar{x} .$$

We assume normality in the rest of the discussion.] Its value is

$$\delta = 4.66920160910299097 \ldots$$

It is to be emphasized that δ depends on no analytic structure of f: rather, it is an artifact of the structural form of Eq. (III.3.4). Since Ω_n is a measure of the nutrient level of our illustrative population for a certain dynamical behavior, δ is in principle measurable.

The next universal feature is a rescaling ratio. If ω is set at Ω_n, $n >> 1$, the attractor includes many points scattered about $p = \bar{p}$ (the abscissa of the maximum of f). At $\omega = \Omega_{n+1}$ there are twice as many points. Looking again near $\bar{p}$, one observes that the points are identically distributed except that all mutual distances are reduced by the factor α (also, points that had been located to the left of $\bar{p}$ are now found to the right) where α is again universal over f:

$$\alpha = 2.5029078750958928485 \ldots$$

Moreover, whatever f, the points for corresponding $\Omega_n(f)$'s are identically distributed in local clusters: apart from a scale length for a cluster determined by a specific f, the mutual distances between points in a cluster are universal over f. These universal locations of elements of any highly bifurcated cycle for any f are determined by fixed points of a universal function g_0. Indeed, there is an infinite hierarchy of such functions embodying the same information, but organized in varying cluster sizes:

$$g_r(p) = \lim_{n\to\infty} (-\alpha)^n \, (\omega_{r+n} f)^{(2^n)} \, (p/(-\alpha)^n) ,$$

where

$$u^{(2^n)}(x) = u^{(2^{n-1})}\,(u^{(2^{n-1})}(x)) .$$

(For a given f, the limit of high iterates on the right converges not to g_r, but rather to a magnification of one such g_r absolutely normalized. The magnification is determined by f and is the sole fashion by which a given f modifies this limit.) (ω_{r+n} is that value of ω for which $\bar{p}$ belongs to a 2^{r+n}-cycle, and the above g_r's determine elements of this attractor: there is a trivial modification to obtain the distribution at bifurcation values.) These functions can be obtained without any reference to f through the fundamental equation of the theory of highly bifurcated attractors:

$$g_{r-1}(p) = -\alpha g_r(g_r(p/\alpha)) . \qquad \text{(III.3.5)}$$

As r approaches infinity, the g_r approach a limiting function g which describes the attractor elements at a level of infinite size clusters (the same as determined by a g_r but with "statistical" scatter). By Eq. (III.3.5), g obeys

$$g(p) = -\alpha g\,(g\,(p/\alpha))$$

which can be solved for g and α. The number δ is determined through the solution of the functional eigenvalue problem

$$L(u) = -\alpha(u(g(p/\alpha)) + g'(g(p/\alpha))u(-p/\alpha)) = \nu u(p) .$$

We conjecture (with computational and self-consistency evidence) that the spectrum of L contains, inter alia, a unique eigenvalue ν greater than 1. This eigenvalue is exactly δ. Calling its eigenfunction h, it follows that

$$g_r(p) \sim g(p) - \delta^{-r} h(p) \quad r \gg 1 .$$

Asymptopia occurs to 10 figures for r = 6. Setting g_6 to its asymptotic value then determines g_0 through successive application of Eq. (III.3.5). Thus the theory of highly bifurcated attractors of Eq. (III.3.4) (chaotic behavior) is totally determined in clusters by the functional equations (III.3.5) free of any reference to the specific f of Eq. (III.3.4). Conversely, the approach to the asymptotic regime for any

specific f has been worked out, and the various scales set by it as well as ω values for bifurcations, etc., can be trivially estimated to good precision.

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SIGNIFICANCE ARITHMETIC: ALGORITHMS AND AXIOMATICS

N. Metropolis (T-7) with G.-C. Rota

The ambition of significance arithmetic is to be able to specify the standard deviations of computational results in the general case where input quantities have quite disparate magnitudes and accuracies. Many problems arising in the natural sciences are of this character, although too often their initial data are idealized by artificially extending their precision to that of a standard computer word.

Our approach is twofold. In the first, definitive algorithms are sought for problems of modest complexity. A new approach to arithmetic processing is implied. Such studies provide a basis for structuring an axiomatic approach to arithmetic; the progress in this second approach has been encouraging.

THE ROLE OF VARIANCE AND ITS RULES

The basic idea is a simple one. Every arithmetic operation performed in a computer is accompanied by a calculation of the resultant variance as a function of the variances of the inputs. If the inputs are not statistically independent, covariances must be examined; we postpone at first this consideration.

Define the number of significant digits of a computer quantity x as

$$s_x = \log_2 \frac{|x|}{\sigma_x} \quad . \tag{III.3.6}$$

where σ_x is the associated standard deviation, i.e., the square root of the variance, written $\langle x \rangle$. If $E(\cdot)$ represents expected value, then

$$\langle x \rangle = E(x^2) - E^2(x) \quad . \tag{III.3.7}$$

It is well known that, if x, y are statistically independent:

$$\begin{aligned}
&\text{addition:} && \langle x + y \rangle = \langle x \rangle + \langle y \rangle \\
&\text{multiplication:} && \langle xy \rangle = x^2 \langle y \rangle + y^2 \langle x \rangle + \langle x \rangle \langle y \rangle \\
&\text{reciprocal:} && \left\langle \frac{1}{x} \right\rangle \cong \frac{1}{x^4} \langle x \rangle, \text{ for } \langle x \rangle < x^2 \ ,
\end{aligned} \tag{III.3.8}$$

where quantities outside the brackets are expected values of the variables. For the division process, the last two rules are combined in an obvious manner.

An important observation is that the above rules do not depend on the detailed structure of the distribution function associated with each operand, apart from the natural assumption that the first and second moments exist.

ERROR CORRELATION

If there is no error correlation, the above rules for computing variance would be reliable. Unfortunately, as a calculation proceeds, the operands