georgia tech PHYS 4267/7224

introduction to nonlinear dynamics and chaos

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take-home final exam

due no later than 10:50am, Thursday, May 3

[delivered either to Jonathan or Predrag, 5th floor Howey]

1 Fluttering flame front

The only road to intuition about chaotic dynamics is by experimentation. Try to work through the essential steps in this take-home exam, applying the techniques learned in the course to a real-life research problem. In what follows, I have indicated which steps are exam questions, and which are optional. I do not expect you to get through the whole length of the exam in the time allotted; do as much as feels right. You might start, for example, with exercise 6 which is easier than the main, turbulent theme.

Star student Henriette Roux would like to understand turbulence. How does she get started? The steps are:

2 Thinking

1. **Have a dream**: In case at hand, Hopf[1] and Spiegel[2, 3, 4] vision of turbulence: the dynamics drives the fluid through a repertoire of unstable patterns. As we watch a turbulent system evolve, every so often we catch a glimpse of a familiar pattern. For any finite spatial resolution, for a finite time the system follows approximately a pattern belonging to a finite alphabet of admissible patterns, and the long term dynamics can be thought of as a walk through the space of such patterns, just as chaotic dynamics with a low dimensional attractor can be thought of as a succession of nearly periodic (but unstable) motions.

   In this exam we apply this vision to the flame flutter of gas burning on your kitchen stove. We are happy if in a few days of analysis we succeed in simulating the system numerically, and develop intuition about turbulence deploying basic nonlinear dynamics notions: the equilibria, stabilities, stability eigenvectors, bifurcations, periodic orbits, onset of chaos.

2. **Pose the problem**: A flame front is described by the Kuramoto-Sivashinsky [KS] equation

$$u_t = -\frac{1}{2}(u^2)_x - u_{xx} - u_{xxxx} \,. \tag{1}$$

Here $t \geq 0$ is the time and $x \in [0, L]$ is the periodic space coordinate. In what follows we use interchangeably the "dimensionless system size" $\tilde{L}$, or the periodic domain size $L = 2\pi\tilde{L}$, as the system parameter. The subscripts x and t denote the partial derivatives with respect to x and t; $u_t = du/dt$, u_{xxxx} stands for 4th spatial derivative of the "velocity of the flame front" $u = u(x, t)$ at position x and time t. The term $(u^2)_x$ makes this a *nonlinear system*.

Please read the chapter "Turbulence?" [5]

3. **Rethink**: As the "flame front velocity" $u(x, t) = u(x + 2\pi, t)$ is periodic on the $x \in [0, 2\pi]$ interval, expand it in a spatial Fourier basis:

$$u(x, t) = \sum_{k=-\infty}^{+\infty} a_k(t) e^{ikx/\tilde{L}} \,. \tag{2}$$

Since $u(x, t)$ is real,

$$a_k = a^*_{-k} \,. \tag{3}$$

Show that substituting (2) into (1) yields the infinite ladder of evolution equations for the complex Fourier coefficients $a_k(t)$: Exam question

$$\dot{a}_k = v_k(a) = ((k/\tilde{L})^2 - (k/\tilde{L})^4)\, a_k - i\frac{k}{2\tilde{L}} \sum_{m=-\infty}^{+\infty} a_m a_{k-m} \,. \tag{4}$$

As $\dot{a}_0 = 0$, the solution integrated over space is constant in time. We set this average velocity to zero, $a_0 = \int dx\, u(x, t) = 0$. The coefficients a_k are in general complex functions of time. Use (3) to further simplify the tower of evolution equations.

4. The constant solution $u(x, t) = 0$ is an equilibrium point of (1). For this "laminar" equilibrium the stability matrix is diagonal,

$$A_{kj}(a) = \left(k^2/\tilde{L}^2 - k^4/\tilde{L}^4\right) \delta_{kj} \,, \tag{5}$$

and so is the Jacobian matrix $\mathbf{J}^t_{kj} = \delta_{kj} e^{(k/\tilde{L})^2(1-(k/\tilde{L})^2)t}$.

Show that from (5) it follows that the $|k| < \tilde{L}$ long wavelength modes of this equilibrium are linearly unstable, and the $|k| > \tilde{L}$ short wavelength modes are stable. For $\tilde{L} < 1$, $u(x, t) = 0$ is the globally attractive stable equilibrium, i.e., the dissipation is so strong that any flame front burns out. Exam question

5. Determine the most unstable mode a_k. It sets the mean wavelength in the plots to follow. Exam question

6. Starting with $\tilde{L} = 1$ the solutions go through a rich sequence of bifurcations. **What** kind of bifurcation takes place as $\tilde{L} < 1 \to \tilde{L} > 1$? As $\tilde{L}$ increases, are there any further bifurcations from the $u(x,t) = 0$ equilibrium, and if so, of what type? Exam question

3 Why are you doing this?

A theory of turbulence should predict measurable properties of turbulent flows, such as their mean energies and their energy dissipation rates.

Please read the section “Energy budget" of chapter “Turbulence?" [5].

1. The time-dependent average velocity-squared

$$E = \frac{1}{L}\int_0^L dx\, \frac{u^2}{2} \tag{6}$$

has a physical interpretation as the average “kinetic energy” density of the flame front. Derive the power / dissipation energy rate equation Exam question

$$\dot{E} = P - D\,, \qquad P = \left\langle (u_x)^2 \right\rangle\,, \quad D = \left\langle (u_{xx})^2 \right\rangle. \tag{7}$$

KS is a far-from equilibrium system: the power P pumped in by the anti-diffusion u_{xx} is balanced by the hypervicosity u_{xxxx} dissipation rate D. In principle, these are experimentally observable quantities, used in what follows as flow diagnostics.

This is all H. Roux can extract from the problem by thinking, unassisted by experimentation. Next,

4 Computing

1. Implement a numerical simulator for your problem. Some options:

   (a) Divide the x interval into a grid of N points, replace space derivatives (1) by approximate discrete derivatives, and integrate a finite set of first order differential equations for the discretized spatial components $u_j(t) = u(jL/N, t)$.

   (b) Integrate numerically the Fourier modes (4), truncating the ladder of equations to N modes, set $a_k = 0$ for $k > N$. N has to be sufficiently large that no harmonics a_k important for the dynamics with $k > N$ are truncated. On the other hand, computation time increases with the increase of N. Empirically, for this exploration $N = 16, 32, 64, 128$ truncations were sufficiently accurate.

(c) Use Davidchack implementation of Kassam and Trefethen code[6], `ChaosBook.org/extras/#PDEs`.

5 Fishing

From here on we turn to numerical experimentation. Take L sufficiently large so that the dynamics can be spatiotemporally chaotic, but not so large that we would be overwhelmed by many short wavelength modes needed in order to accurately represent the dynamics.

My advice: start on *terra firma*, small system size $\tilde{L} = 1$, low truncation N, and increase $\tilde{L}$ a little bit, integrate until the trajectory has settled down; then increase $\tilde{L}$ a little bit again, restart from the trajectory just computed, integrate until has settled down. Repeat. Sometimes stop incrementing the trajectory, increment N instead and check how sensitive is your attractor to truncation number N. I have not test-run this calculation, but I believe you will sail through a sequence of bifurcations and enter chaos, perhaps via period-doubling route. This "adiabatic" approach has advantage of (almost) always starting you close to the attractor, thus avoiding long transients typical of random starting conditions.

The problem with high-dimensional truncations of (4) is that the dynamics is difficult to visualize. The simplest (but not the smartest) visualization is to examine trajectory's projections onto any three Fourier coefficient axes a_i, a_j, a_k (real or imaginary parts). Better are projections onto 2 or 3 basis vectors - see Davidchack code for examples.

1. **Plot** a long trajectory for $L = 22$, using the same vector basis as Davidchack. Is your dynamics qualitatively the same as in his plots? Exam question

2. Plot several interesting long orbits to get some sense for the attractors for different values of system size L Exam question

3. If you are integrating in the Fourier space, track also the evolution of $u(x, t)$, by inverse Fourier transform of (2).

4. **Plot** a spatiotemporal solution $u(x, t)$ for the chaotic, $\tilde{L} = 22$ attractor. Hopf wanted us to see *recurrent* patterns, that is to say, the unstable spatiotemporally periodic solutions of our equations. This can be done, but is hard work. Other solutions exhibit the same overall gross structure - a few wiggles here and there, continuously in flux and yet so alike. Exam question

5. Explore bifurcation sequences by plotting the time-averaged values $(\overline{E}, \overline{P}, \overline{D})$ of (E, P, D) defined in (7), for small increments of $1 < L < 25$ (I have not test-run this, but am hopeful). Exam question

6. find some numerically stable equilibria and/or periodic orbits (if any) within $1 < L < 25$ window of system sizes Optional

7. Drastic truncations tend to mangle the dynamics. **Do you get** any chaos for small N, something like $N < 9$ nd $L < 22$? Optional

8. **Do you get** any stable periodic orbits for $\tilde{L} = 22$? (if you do, we would love to see them) Optional

9. determine values of system sizes L for stable cycles $\rightarrow$ unstable cycles bifurcations (estimate by trial and error) Optional

10. diagnose L values at which chaos sets in, or vanishes again Optional

11. Try to find numerically some equilibria for $\tilde{L} > 1$ (hard). As explained in section "Equilibria of equilibria" of chapter "Turbulence?" [5], these are periodic orbits of a 3*d* dynamical system, and they might be to difficult to find within a week take-home exam. We provide a pre-computed set of equilibria within the matlab code, `ChaosBook.org/extras/#PDEs` for you to use. Optional

12. **Try** to estimate stability eigenvalues of these eigenvalues by numerical experimentation (how fast do they spiral out, *etc.*). Optional

13. Estimate the leading Lyapunov exponent for the turbulent flow at $\tilde{L} = 22$ Optional

14. **Poincaré sections**: One of the first recommended steps in analysis of chaotic flows is to view the dynamics to a Poincaré section. Fix (arbitrarily) the Poincaré section to be the hyperplane $\mathrm{Re}\, a_1 = 0$, and integrate (4) with the initial conditions $a_1 = 0$, and arbitrary values of the coordinates $a_2, \ldots, a_N$, where N is the truncation order. When a_1 becomes 0 the next time, the coordinates $a_2, \ldots, a_N$ are mapped into $(a'_2, \ldots a'_N) = P(a_2, \ldots, a_N)$. Does the long-time attractor look confined to a smaller subspace, foliated as a nice fractal? Optional

6 How strange is the Hénon attractor?

Let us switch gears for a moment, and perform a numerical experiment that will enable you to do a part of this exam even if all your integration programs are in shambles.

You might have wondered why I often state values of system parameters 5 significant figures, if all we want is to get a qualitative feeling for the flame front flutter?

The problem is that it is extremely hard to prove that an attractor is chaotic. Adding an extra dimension to a truncation of the system (4) introduces a small perturbation, and this can (and often will) throw the system into a totally different asymptotic state. A chaotic attractor for $N = 15$ can become a period three window for $N = 16$, and so on.

Numerical studies indicate that for $a = 1.4$, $b = 0.3$ the attractor of the Hénon map (see figure 5.32 in the Tél and Gruiz book[7])

$$\begin{aligned} x_{n+1} &= 1 - a x_n^2 + b y_n \\ y_{n+1} &= x_n \,. \end{aligned}$$

is "strange". Reproduce the Hénon picture of his "strange attractor" by numerical iteration of the map. Next, repeat the numerical experiment for the map with parameter $a = 1.39945219$ (right: all digits are significant - it is a craftily designed perturbation precise to 10^{-8}!). If you wait long enough (100,000's of iterations), the attractor should undergo a dramatic change. What do you get? Exam question

Exam question

The moral of this numerical experiment is that "strange attractors" are not structurally stable. If we compute, for example, the Lyapunov exponent $\lambda(\tilde{L}, N)$ for the strange attractor of the N-modes truncation of the system (4), there is no reason to expect $\lambda(\tilde{L}, N)$ to smoothly converge to the limit value $\lambda(\tilde{L}, \infty)$ as $N \to \infty$.

Now, have a Carlsberg, perhaps the best beer in some parts of Copenhagen, and a good summer.

References

[1] E. Hopf. A mathematical example displaying features of turbulence. *Comm. Appl. Math.*, 1:303–322, 1948.

[2] D. W. Moore and E. A. Spiegel. A thermally excited nonlinear oscillator. *Astrophys. J.*, 143:871, 1966.

[3] N. H. Baker, D. W. Moore, and E. A. Spiegel. Aperiodic behavior of a nonlinear oscillator. *Quatr. J. Mech. and Appl. Math.*, 24:391, 1971.

[4] E. A. Spiegel. Chaos: a mixed metaphor for turbulence. *Proc. Roy. Soc.*, A413:87, 1987.

[5] P. Cvitanović, R. Artuso, R. Mainieri, G. Tanner, and G. Vattay. *Chaos: Classical and Quantum*. Niels Bohr Institute, Copenhagen, 2005. ChaosBook.org.

[6] A. K. Kassam and L. N. Trefethen. Fourth-order time stepping for stiff PDEs. *SIAM J. Sci. Comp.*, 2004.

[7] T. Tél and M. Gruiz. *Chaotic Dynamics: An Introduction Based on Classical Mechanics*. Cambridge University Press, Cambridge, 2006.