Comparison between cycle expansion and adjoint equations

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Abstract

In this article ideas of cycle expansion and of adjoint equations are compared with each other. Both methods give information about the spectrum of evolution operators $\mathcal{L}$. We show how cycle expansion can benefit from exact mathematical results for numerical checks. On the other hand we derive with the help of complex cycle expansion certain conjectures for *exact* forms of nontrivial evolution operators. We perform detailed numerical studies.

1 Introduction

In this article ideas of cycle expansion and of adjoint equations are compared with each other. Both methods give information about the spectrum of evolution operators $\mathcal{L}$, which have been introduced to the dynamical systems literature by Ruelle [10].

In section 2 we give a short exposure of the cycle expansion method to introduce notations, more material can be found in [2]; a nice application to the theory of random matrices can be found in [9]. We propose the study of complex cycle expansions as a tool as well as a new point of view in the theory of complex dynamical systems.

In section 3 we derive an adjoint equation for the eigenfunctions of the adjoint $\mathcal{L}^\star$ of $\mathcal{L}$ [4]. To this end we have to go from real to complex dynamical systems. In section 4 the example of a quadratic map evolution is considered. For this map one has complete control over the eigenvalues of $\mathcal{L}$, as far as only polynomial weights are considered, as well numerically via cycle expansion as theoretically via the adjoint equation. In addition, these results follow neatly from sum rules for the stabilities along complex cycles [1]. In section 5 we deal with a claim of [4] that in their setup under special conditions transcendental evolution operators can be systematically approached by polynomial evolution operators. Here we compare with numerical results from cycle expansion. In particular we analyze whether it is possible to get the physical escape rates for the system with these approximations. In section 6 we study the complex cycle expansion and use mathematical results for numerical checks. With the help of cycle expansion we derive conjectures for the exact form of spectral determinants for certain (real) chaotic quadratic maps which can't been done in the mathematical frame. We end with a short conclusion.

2 Cycle expansion

Deterministic low dimensional dynamical systems and their strange sets can be analyzed in terms of unstable periodic orbits – called cycles – which build up a skeleton for the phase space at asymptotic times. While this analysis can be performed for flows as well as for maps, we want to restrict ourselves to maps. Let $R : \mathbb{R} \to \mathbb{R}$ be a map. Iterated application of R to a point $x_0 \in \mathbb{R}$ produces a sequence $x_1, x_2, \dots$ which is called orbit p or trajectory of x_0. If this orbit closes it is called a periodic orbit p, with period n_p defined as the smallest $n \geq 1$ such that $R^n x_0 = x_0$.

2.1 Real cycle expansion

One can look at the orbits of points x_0 in a finite enclosure V. In general, some trajectories will escape from V after some iterations of R. To make this quantitative one defines an escape rate γ_n by

$$e^{-n\gamma_n} = \frac{\int_V dx dy \delta(y - R^n x)}{\int_V dx} = \frac{\int_V dx \chi(R^n x \in V)}{\int_V dx}\,, \tag{1}$$

where

$$\chi(R^n x \in V) = \begin{cases} 1 & \text{if } R^n x \in V, \\ 0 & \text{else.} \end{cases} \tag{2}$$

That means that γ_n measures the relative volume of the points which stay in V after n iterations.

Definition (1) leads to another more general and more important notion, the evolution operator $\mathcal{L}$, which is defined by

$$(\mathcal{L}f)(y) = \int_{\mathbb{R}} dx \delta(R(x) - y) \left|R'(x)\right| Q(x) f(x) = \sum_{x:R(x)=y} Q(x) f(x)\,. \tag{3}$$

Here f is a function $\mathcal{L}$ acts on and the function Q is called the weight of the evolution operator. A natural and physically interesting weight is the choice $Q(x) = 1/\left|R'(x)\right|$. With this weight the largest eigenvalue σ of the evolution operator gives the asymptotic escape rate γ:

$$\gamma = \log \frac{1}{\sigma}\,. \tag{4}$$

Heuristically this can be seen by expanding f in a basis of eigenfunctions ϕ_k to the eigenvalues σ_k. Then we get:

$$e^{-n\gamma_n} = \frac{\int_V dy (\mathcal{L}^n f)(y)}{\int_V dy f(y)} = \sum_k f_k \sigma_k^n \frac{\int_V dy \phi_k(y)}{\int_V dy f(y)}\,. \tag{5}$$

In this sum clearly the largest eigenvalue dominates the behaviour.

The characteristic values of $\mathcal{L}$ (that means the inverse of the eigenvalues) are determined by the zeros of the Fredholm determinant

$$\begin{aligned} \det(I - z\mathcal{L}) &= \exp\left(\operatorname{tr}\log(I - z\mathcal{L})\right) \\ &= \exp\left(-\sum_{n=1}^{\infty} \frac{z^n}{n} \operatorname{tr}\mathcal{L}^n\right)\,. \end{aligned} \tag{6}$$

The traces of $\mathcal{L}^n$ can be expressed in terms of fixed points of R^n:

$$\mathrm{tr}\mathcal{L}^n = \int dx \delta(R^n x - x) \left| R^{n\prime}(x) \right| Q(n,x) = \sum_{x: x=R^n x} \frac{Q(n,x) \left| R^{n\prime}(x) \right|}{\left| 1 - R^{n\prime}(x) \right|} . \tag{7}$$

Here we define for the whole article for any function $Q(x)$ the iterated product of Q by

$$Q(n,x) = \prod_{i=0}^{n-1} Q(R^i x) . \tag{8}$$

Rather than thinking in terms of fixed points of R^n we want to think in terms of the periodic orbits which are defined by the fixed points. First we notice that for a fixed point x of R^n the stability $R^{n\prime}(x)$ and the product $Q(n,x)$ are only properties of this periodic orbit q. Therefore we give a special symbol to these things:

$$\Lambda_q = R^{n\prime}(x) , \quad Q_q = Q(n,x) . \tag{9}$$

Furthermore if q is a r-times repeat of a shorter orbit p of length n_p, with $n = rn_p$, which we call reducible orbit, contrary to the other "prime" orbits, we get

$$R^{n\prime}(x) = \Lambda_p^r , \quad \text{and} \quad Q(n,x) = Q_p^r . \tag{10}$$

Therefore we can express the sum over fixed points as a sum over periodic orbits, and if we are cautious and prevent overcounting we get

$$\mathrm{tr}\mathcal{L}^n = \sum_{p: p \, \text{prime orbit}} n_p \sum_{r \geq 1} \delta_{n, rn_p} \frac{\left| \Lambda_p \right|^r Q_p^r}{\left| 1 - \Lambda_p^r \right|} . \tag{11}$$

Now we can insert this into (6) to get the cycle expansion for the Fredholm determinant:

$$\det \left(I - z\mathcal{L} \right) = \exp \left(- \sum_{p: p \, \text{prime orbit}} \sum_{r \geq 1} \frac{1}{r} z^{rn_p} \frac{\left| \Lambda_p \right|^r Q_p^r}{\left| 1 - \Lambda_p^r \right|} \right) . \tag{12}$$

If we approximate $\frac{1}{\left| 1 - \Lambda_p^r \right|}$ by $\left| \Lambda_p \right|^{-r}$, which is well justified for expanding hyperbolic maps, we get

$$\det \left(I - z\mathcal{L} \right) = \exp \left(- \sum_p \sum_{r \geq 1} \frac{1}{r} (z^{n_p} Q_p)^r \right) = \prod_p (1 - z^{n_p} Q_p) =: \zeta^{-1}(z) . \tag{13}$$

This product is called ζ-function because of direct formal analogy with the Riemann ζ-function.

Rather than computing prime orbits $x_0, \ldots, x_{n_p-1}$ from explicitly given numbers one adopts an abstract point of view and introduces the notion of symbolic dynamics. This is often given by a labelling of the different branches of R^{-1} with symbols of an (finite or infinite) alphabet. An orbit then corresponds to a sequence of letters, for example 0011. It is reducible if the symbolic sequence is reducible, for example in the case 010101. The prime cycles corresponding to a symbolic sequence can in the easiest cases be obtained by backward iteration of the map R, using the branches of R^{-1} which correspond to the respective symbol. In the simplest case the alphabet consists of only two letters 0 and 1. This is for example the alphabet for simple quadratic maps. In table 1 we display all prime cycles with two letters up to length eight.

0	11010	1100000	1111100	11110000	11110100
1	11110	1010000	1101010	11001000	10101100
10	100000	1110000	1111010	10101000	11101100
100	110000	1001000	1110110	11101000	11011100
110	101000	1101000	1111110	10011000	10111100
1000	111000	1011000	10000000	11011000	11111100
1100	110100	1111000	11000000	10111000	11101010
1110	101100	1100100	10100000	11111000	11011010
10000	111100	1010100	11100000	10100100	11111010
11000	111010	1110100	10010000	11100100	11110110
10100	111110	1101100	11010000	11010100	11111110
11100	1000000	1011100	10110000	10110100	

Table 1: Prime cycles up to length 8, 2 symbols

2.2 Complex cycle expansion

Cycle expansion is an extremely good, highly converging and widely established numerical tool in the theory of real low dimensional dynamical systems. However, there are certainly dynamical systems in which not all solutions to the equation $R^n x = x$ are real. To get the real escape rates and spectrum of the evolution operators as defined above one has to sum only over a subset of the prime orbits. This so called pruning can be very complicated first of all because the pruning rules can be very complicated; furthermore the convergence of the cycle expansion might be bad even if one knows the correct rules for generating the orbits.

Intuitively the complex orbits which are ”near” the real line might be responsible for this behaviour, but so far nothing has been found out about this subject.

We take this problem as a motivation for the analysis of complex dynamical systems not only with abstract mathematical methods but also with the cycle expansion. In the numerical sections we show that complex cycle expansion works as well in the complex regime as it does in the real one. Furthermore there is no pruning involved, which follows in the algebraic case from the fundamental theorem of algebra. It turns out that the only problem is the actual determination of the prime orbits.

The formalism is almost the same as in the real case, because the Fredholm determinant can again be expanded as a function of the traces

$$\det(I - z\mathcal{L}) = \exp\left(-\sum_{n\geq 1} \frac{1}{n} z^n \mathrm{tr}\mathcal{L}^n\right) , \tag{14}$$

but for the definition of the traces one has to use complex integration. The evolution operator is defined on $\mathbb{C}$ as

$$(\mathcal{L}f)(z) := \sum_{\omega : R\omega = z} Q(\omega) f(\omega) , \tag{15}$$

where the sum includes real and complex pre-images ω of z. Therefore:

$$(\mathcal{L}^n f)(z) = \sum_{\omega : R^n\omega = z} Q(n,\omega) f(\omega) = \oint_\gamma \frac{d\omega}{2\pi i} \frac{Q(n,\omega)(R^n)'(\omega)}{R^n\omega - z} f(\omega) . \tag{16}$$

The trace follows from this:

$$\begin{aligned} \mathrm{tr}\mathcal{L}^n &= \oint_\gamma \frac{d\omega}{2\pi i} \frac{Q(n,\omega)(R^n)'(\omega)}{R^n\omega - \omega} = \sum_{\omega : R^n\omega = \omega} \frac{Q(n,\omega)(R^n)'(\omega)}{(R^n)'(\omega) - 1} \\ &= \sum_p n_p \sum_{r\geq 1} \delta_{n, r n_p} \frac{(Q_p\Lambda_p)^r}{\Lambda_p^r - 1} . \end{aligned} \tag{17}$$

The stability and the weight function now take in general complex values. The combination of (14) and (17) is the complex cycle expansion for the Fredholm determinant.

2.3 Sum rules

Sum rules [1] are simple relations between the stabilities of periodic orbits in the complex plane. These rules, for example

$$\sum_{\omega : R^n \omega = \omega} \frac{1}{R^{n\prime}(\omega) - 1} = 0 \tag{18}$$

for any polynomial R, show a high degree of correlation between the various orbits which may lie at completely different points of the phase space. These rules can be derived easily by writing the trace $\mathrm{tr}\mathcal{L}^n$ as the contour integral (17) and expanding in powers of ω. If R is a quadratic map one can – with a bit more labour – establish a recursion relation between various traces and compute from this information the complete spectral determinant [1]. However, in sections 3 and 4 we show how this can be done more easily.

If the Fredholm determinant is known one can always recursively compute the traces $\mathrm{tr}\mathcal{L}^n$ similar to a calculation we perform in section 5. However, we can also compute sum rules directly from the Fredholm determinant. Because sum rules might be useful for numerical checks for the stabilities and cycles let us demonstrate this briefly.

Let us for example assume we had a quadratic spectral determinant

$$\det\left(I - z\mathcal{L}\right) = 1 + az + bz^2 = \exp\left(-\sum_{n\geq 1} \frac{z^n}{n} \mathrm{tr}\mathcal{L}^n\right). \tag{19}$$

Then we can take the logarithm and expand:

$$\begin{aligned} \sum_{n\geq 1} \frac{z^n}{n}\mathrm{tr}\mathcal{L}^n &= \sum_{n\geq 1} \frac{(-1)^n}{n}(az+bz^2)^n \\ &= \sum_{n=1}^{\infty}\sum_{m=0}^{n} \binom{n}{m} \frac{(-1)^n}{n} a^{n-m} b^m z^{m+n} \\ &= \sum_{N=1}^{\infty} z^N \sum_{n=1}^{\infty}\sum_{m=0}^{n} \delta_{N,m+n} \binom{n}{m} \frac{(-1)^n}{n} a^{n-m} b^m \\ &= \sum_{N=1}^{\infty} z^N \sum_{k=0}^{[N/2]} \binom{N-k}{k} \frac{(-1)^{N-k}}{N-k} a^{N-2k} b^k , \end{aligned} \tag{20}$$

from which we deduce that

$$\mathrm{tr}\mathcal{L}^n = n \sum_{k=0}^{[n/2]} \binom{n-k}{k} \frac{(-1)^{n-k}}{n-k} a^{n-2k} b^k . \tag{21}$$

For a linear spectral determinant ($b = 0$) this sum rule reduces to

$$\mathrm{tr}\mathcal{L}^n = (-a)^n \,. \qquad (22)$$

Sum rules for maps that are related to number theory and plenty ideas for applications and further steps to go can be found in [1].

3 The adjoint equation

3.1 General theory

A different, mathematically more sophisticated way to information about the spectrum of evolution operators is given by the solution of an equation for the eigenfunctions of the adjoint $\mathcal{L}^\star$ of $\mathcal{L}$ [4]. However, for this scheme to work, the iterating map R and the weight Q are restricted to be rational functions of the complex variable z on the whole complex sphere. The evolution operator is defined as the sum over all complex pre-images of z:

$$(\mathcal{L}f)(z) = \sum_{\omega:R\omega=z} Q(\omega)f(\omega)\,. \qquad (23)$$

The main idea of [4] is the following: Define the evolution operator $\mathcal{L}$ on such a space that the adjoint operator $\mathcal{L}^\star$ has the same, all in all well behaved spectrum. It turns out that by demanding this, $\mathcal{L}^\star$ acts on measures $f^\star$ which can be identified with analytic functions f which vanish at infinity, and which are analytic everywhere except around the closure of all repelling periodic orbits of R. This closure plays a special role in the theory of complex dynamical systems and is called Julia set J. The duality between f and $f^\star$ is expressed by

$$f^\star(g) = \oint_\gamma \frac{dz}{2\pi i} f(z)g(z)\,, \qquad (24)$$

for functions g which are holomorphic around J. Because the spectrum of $\mathcal{L}$ and $\mathcal{L}^\star$ is the same, it is equivalent to analyze $\mathcal{L}^\star$ instead of $\mathcal{L}$.

Now the equation

$$f^\star = \lambda\mathcal{L}^\star f^\star \qquad (25)$$

for an eigenmeasure with characteristic value λ can be written as an adjoint equation. Let us supply the derivation of this equation here. We have

$$\begin{aligned}(\mathcal{L}^\star f^\star)(g) &= f^\star(\mathcal{L}g) = \oint_\gamma \frac{dz}{2\pi i} f(z) \sum_{\omega:R\omega=z} Q(\omega)g(\omega) \\ &= \oint_\gamma \frac{dz}{2\pi i} f(R(z))R'(z)Q(z)g(z)\,. \qquad (26)\end{aligned}$$

For the last equality we actually need two steps: a substitution $y = R(z), dy = R'(z)dz$; to do this properly one has to go on the Riemann surface of R^{-1}, which gives the sum. Furthermore a contour deformation has to be used. We express the left hand side of (25) by (24) and get

$$\oint_\gamma \frac{dz}{2\pi i} g(z)\big(f(z) - \lambda R'(z)Q(z)f(R(z))\big) = 0\,. \tag{27}$$

That means there exists a functions ϕ which is holomorphic around J and which satisfies the adjoint equation

$$f(z) - \lambda R'(z)Q(z)f(R(z)) = \phi(z)\,. \tag{28}$$

From a more detailed analysis of the function spaces one can conclude that ϕ has to be rational, with no poles on J, of course. Because f is holomorphic outside J one can conclude that ϕ can have poles only when $R'Q$ has poles and the poles of ϕ have at most the order of the poles of $R'Q$. If we find a function ϕ, f and a $\lambda \in \mathbb{C}$ such that (28) is fulfilled and all functions live in the right spaces we have a solution of the eigenmeasure problem (25) and therefore determined a characteristic value of $\mathcal{L}$.

The main technical advance in [4] is a general prescription for the solution of (28). Before we turn to a short description of this algorithm let us define the operator K of weighted composition by

$$(Kf)(z) = R'(z)Q(z)f(R(z)) \tag{29}$$

for brevity. The algorithm is divided into three steps.

(i) First one solves equation (28) for f in a function space which is bigger than the function space in which f has to be (holomorphic outside J); in this first step we allow in addition for poles at poles of $R'Q$ and their pre-images. In this step we treat ϕ as a known function. The (finitely many) coefficients of ϕ are determined later. For the cycle expansion educated physicist it is important to note, that one solves (28) for f in the attracting basin of an attracting (not repelling) periodic orbit $a_0, \ldots, a_{p-1}$ with period p. This might also be an attracting fixed point at infinity. The solution in such a basin is given as a power series

$$\begin{aligned} f(\lambda, z) &= \sum_{n=0}^{\infty} \lambda^{pn} \phi_p(\lambda, R^{pn}z)(R'Q)(pn, z) \\ \phi_p(\lambda, z) &= \sum_{l=0}^{p-1} \lambda^{l} \phi(R^{l}z)(R'Q)(l, z)\,. \end{aligned} \tag{30}$$

(ii) Even from the few things we have actually mentioned about function spaces one can deduce that a eigenfunction f and the function ϕ cannot have poles both at on point $z \in \mathbb{C}$. In a second step one determines the coefficients of ϕ so that f is holomorphic at the poles of ϕ. These conditions can be translated to a linear equation system. From a pole $z = c$ of order L of $R'Q$ we get the equations

$$0 = \Delta_l(\lambda) = \mathrm{Res}_{z=c}[z^l f(\lambda, z)],\ l = 0, 1, \dots, L-1\,. \tag{31}$$

The nontrivial solvability condition for this system for all poles of $R'Q$ gives an equation for the characteristic value λ.

(iii) The case $\phi = 0$, that means λ is a characteristic value of K itself, is special. One has to check whether K has eigenfunctions in the correct spaces. To do this one solves the equation

$$f(z) = \lambda (Kf)(z) \tag{32}$$

in the attractive basins of attracting cycles $a_0, \dots, a_{p-1}$ of period p by iteration. Local linearization theorems ("Koenigs function" [6]) give the characteristic value of K. If Λ_p is the stability of the cycle, the characteristic values are all p roots

$$\lambda = \left((R'Q)(p, a_0)\Lambda_p^m\right)^{-\frac{1}{p}}\,, \quad m = 0, 1, 2, \dots\,. \tag{33}$$

3.2 Application to polynomial evolution

In the case that both R and Q are polynomials we can get remarkably explicit expressions for the characteristic values of $\mathcal{L}$. In this case, the problem is an algebraic one, that means the Fredholm determinant is a polynomial. Let q and r be the degrees of Q and R respectively. Then the degree of $R'Q$ is $r - 1 + q$. In the adjoint equation

$$f(z) - \lambda f(R(z))R'(z)Q(z) = \phi(z) \tag{34}$$

ϕ is then a polynomial. ϕ can't have poles because $R'Q$ doesn't have poles. At infinity f goes to zero at least as z^{-1}. Therefore the degree of ϕ is not greater than $q - 1$. If we focus our attention to the attracting fixed point infinity, we can expand f as

$$f(z) = \sum_{k=1}^{\infty} \frac{f_k}{z^k}\,. \tag{35}$$

Inserted into the adjoint equation (34) we get

$$\sum_{k\geq 1} \frac{f_k}{z^k} - \lambda R'(z)Q(z) \sum_{k\geq 1} \frac{f_k}{R(z)^k} = \phi(z)\,. \tag{36}$$

To get f_l we multiply with z^{l-1} and use Cauchy's formula (let ρ be a large number):

$$f_l = \lambda \sum_{k\geq 1} f_k \oint_{|z|=\rho} \frac{dz}{2\pi i} \frac{z^{l-1}R'(z)Q(z)}{R(z)^k}\,. \tag{37}$$

If k is too big, the integral is zero because the Laurent series contains only powers smaller than minus one. One can then find a value $l^\star$ for l such that the corresponding system closes. This value equals the smallest integer larger than $\frac{q}{r-1}$. That means that the evolution problem is equivalent to the eigenvalue problem of the matrix

$$B_{lk} = \oint_{|z|=\rho} \frac{dz}{2\pi i} \frac{z^{l-1}R'(z)Q(z)}{R(z)^k} \quad l,k = 1,\ldots,l^\star\,. \tag{38}$$

If there are attracting periodic orbits $a_0, \ldots a_{p-1}$ of period p we have the additional characteristic values [4]

$$\lambda = (R'Q)(p, a_0)^{-\frac{1}{p}}\,. \tag{39}$$

If we look hard at equation (36) for $\phi = 0$ we see that all necessary cancellations can only occur if $\frac{q}{r-1}$ is an integer. Then we know from step (iii) of the solution algorithm that $\lambda = \frac{1}{c}$ is one of the characteristic values of $\mathcal{L}$, where c is the leading coefficient of $R'Q$.

4 Fredholm determinants for quadratic maps

As a warm up and a first demonstration of the power of the methods developed above we calculate the Fredholm determinant for a quadratic map with a general polynomial weight.

Let $R(z) = z^2 + c$, with $c \in \mathbb{C}$ a complex number, be the iterating map. (Note that the set of all c for which the corresponding Julia set is connected forms the famous Mandelbrot set.) Furthermore we define the weight Q as a polynomial

$$Q(z) = \sum_{n=0}^{N} a_n z^n\,. \tag{40}$$

The matrix B_{lk} in (38) then reads

$$B_{lk} = \sum_{n=0}^{N} a_n \oint_\gamma \frac{dz}{2\pi i} \frac{2z^{l+n}}{(z^2+c)^k} \quad l,k = 1,\dots,N+1\,. \tag{41}$$

If $l+n$ is even, the coefficient of a_n is zero. Therefore we can solve the problem by substituting $u = z^2$ in (41).

$$\begin{aligned} B_{lk} &= \sum_{n=0:\,n+l\,\text{odd}}^{N} a_n \oint_\gamma \frac{dz}{2\pi i} \frac{2zz^{l+n-1}}{(z^2+c)^k} \\ &= \sum_{n=0:\,n+l\,\text{odd}}^{N} 2a_n \frac{1}{(k-1)!} \left(\frac{d}{du}\right)^{k-1} [u^{\frac{l+n-1}{2}}]_{u=-c} \\ &= \sum_{n=0:\,n+l\,\text{odd},\,n\geq 2k-1-l}^{N} 2a_n \binom{\frac{l+n-1}{2}}{k-1} (-c)^{\frac{l+n-1}{2}-k+1}\,. \end{aligned} \tag{42}$$

We observe that the last column is zero except for the element $B_{N+1,N+1}$ which equals $2a_N$. This corresponds to the unique solution of (36) for $\phi = 0$, as noted at the end of the last section. Therefore a factor $1 - 2a_N z$ can be factorized from the Fredholm determinant. In table 2 we display the first few characteristic polynomials of the matrix B for growing degree N without this factor. With a little computer program we could obtain the Fredholm

N	Fredholm determinants
2	$1-2\left(a_0+a_1-ca_2\right)z+4\,a_0a_1z^2$
3	$1+2\left(-a_0-a_1+(c-1)a_2+2\,ca_3\right)z$
	$+4\left(a_0a_1+a_0a_2+(-2\,c-1)a_0a_3+a_1a_2-ca_2{}^2+c^2a_2a_3\right)z^2$
	$+8\left(a_0{}^2a_3-a_0a_1a_2\right)z^3$
4	$1+2\left(-a_0-a_1+(-1+c)a_2+(-1+2\,c)a_3+\left(-c^2+3\,c\right)a_4\right)z$
	$+4\left(a_0a_1+a_0a_2-2\,ca_0a_3-2\,ca_0a_4+a_1a_2+a_1a_3+\left(-c^2-1-3\,c\right)a_1a_4\right.$
	$\left.-ca_2{}^2+\left(1+c^2-c\right)a_2a_3+3\,c^2a_2a_4-2\,ca_3{}^2+4\,c^2a_3a_4-2\,c^3a_4{}^2\right)z^2$
	$+8\left(a_0{}^2a_3-a_0a_1a_2-a_0a_1a_3+(3\,c+1)a_0a_1a_4-a_0a_2a_3+(1+2\,c)a_0a_3{}^2\right.$
	$+\left(-c-3\,c^2\right)a_0a_3a_4+a_1{}^2a_4-a_1a_2a_3-ca_1a_2a_4+c^2a_1a_3a_4-c^3a_1a_4{}^2$
	$\left.+ca_2{}^2a_3-c^2a_2a_3{}^2+c^3a_2a_3a_4\right)z^3$
	$-16\left(a_0{}^2a_3{}^2+a_0a_1{}^2a_4-a_0a_1a_2a_3\right)z^4$

Table 2: Fredholm determinants for the quadratic map evolution operators

determinant for an evolution operator with general polynomial weight of degree up to 10 in reasonable time as a function of the coefficients of the weight and of c. If we specialize to weights of the form $Q(z) = R'(z)^N = 2^N z^N$, the Fredholm determinants can be computed more efficiently because the size of the system reduces by a factor of two. This is due to the constraint in the sum in (42) that $n + l$ has to be odd. A factor $(1 - 2^{N+1} z)$ factorizes in this case from the determinant, and the results of our computation are displayed in table 3. These results with the specific weight chosen above

N	Fredholm determinants
1	1
2	$1 + cz$
3	$1 + 2\,cz$
4	$1 - c\,(-3 + c)\,z - 2\,c^3 z^2$
5	$1 - c\,(-4 + 3\,c)\,z - 8\,c^3 z^2$
6	$1 + c\,(c-1)\,(c-5)\,z - c^3\,(20 - 5\,c + 3\,c^2)\,z^2 - 8\,c^6 z^3$

Table 3: Fredholm determinants for quadratic maps with general weight of the form $R'(z)^N$.

have now been obtained by many methods including sum rules [1]. The finiteness of this type of spectral determinants has been conjectured in [3] before it could be proved.

As a conclusion let us state that we obtained in this section general expressions for the spectral determinant for quadratic maps with polynomial weights. This is a slight generalization of the results in [1]. Furthermore we know from the theoretical arguments above that for some values of the parameter c, namely exactly for those for which the quadratic map has an attracting periodic orbit, there are additional characteristic values which are related to the stabilities of the attracting periodic orbits as in formula (39). One theorem in the mathematical theory for iteration of polynomial maps states that a quadratic map has at most one attracting periodic orbit (in $\mathbb{C}$) [7]. An example is the map $z^2 - \frac{7}{4}$ which has an attracting periodic orbit of period 3.

5 Approximation of transcendental weights

In the last paragraph we demonstrated that the mathematical tools coming from the adjoint equation are very powerful in the case of polynomial maps with polynomial weights. The direct application of this philosophy to physical problems (that means real periodic orbits plus the unsigned version of the traces) would be desirable but is expected to be very complicated. Here we show that however we can get some information about the spectrum of the evolution operator for weights $|R'(z)|^{-s}$ if the Julia set is part of the real axis. This means we want to treat problems in which all periodic orbits are real (no pruning of orbits). This family of weights includes the physically most interesting weight with $s = 1$ for which the escape rate is included in the spectrum. As an example we choose iteration with quadratic maps. These problems can also be attacked for any weight function with the cycle expansion in a systematic way and to high accuracy. Therefore we compare numerical results between the two approaches.

5.1 Expansion of the weight

The idea in the analytic approach is to expand the weight function

$$Q(z) = |R'(z)|^{-s} \tag{43}$$

with $s \in [0, 2]$, for example into a power series which yields systematic approximations of it. To do this, we have to know, where we expect the Julia set to be. Only with this knowledge can we say something about convergence of the series.

Let $R(z) = z^2 + c$. We search for an interval $[-x_0, x_0]$ such that

$$R([-x_0, x_0]) \supseteq [-x_0, x_0] \text{and} R(x_0) = x_0 \,. \tag{44}$$

That means that there is no pruning and x_0 is the largest fixed point of R, see figure 1. This fixed point

$$x_0 = \frac{1}{2} + \sqrt{\frac{1}{4} - c} \tag{45}$$

has to be real, which means that c is also real and $c < \frac{1}{4}$. From (44) it follows that we have to demand that $R(0) = c < -x_0$. This is fulfilled if $c < -2$ which we now take as the range for the parameter c. In this range the Julia set is real and Cantorian because c is outside the Mandelbrot set. We know that $J \subseteq [-x_0, x_0]$. After one iteration we get

$$J \subseteq M_J := [-x_0, -\sqrt{-x_0 - c}] \cup [\sqrt{-x_0 - c}, x_0] \,. \tag{46}$$

(For this we just see that the equation $R(z) = -x_0$ is solved by $z = x_1 = \pm\sqrt{-x_0 - c}$). This means that we look for an approximation to the weight $|R'(z)|^{-s}$ which is good on M_J. Let z be in M_J. Then z^2 is in $[x_1^2, x_0^2] = [\,|c| - x_0,\, |c| + x_0]$. We know that $|c|$ is bigger than x_0. Therefore:

$$\frac{z^2}{|c|} \in [1 - \frac{x_0}{|c|}, 1 + \frac{x_0}{|c|}] \subsetneq (0, 2) \,. \tag{47}$$

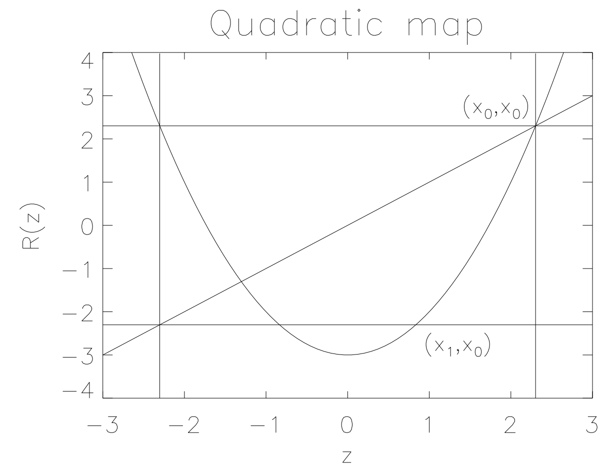

Figure 1: Schematic plot of the map R

If we define x as

$$x = \frac{z^2}{|c|} - 1 \in [-\frac{x_0}{|c|}, \frac{x_0}{|c|}], \tag{48}$$

then x is a good (small) expansion parameter. For smaller values of $\frac{x_0}{|c|}$ the expansion will converge faster. This is the case for $c \to -\infty$.

Now let us write the weight for real values of z as

$$\begin{aligned} |R'(z)|^{-s} &= 2^{-s}\,|z|^{-s} = 2^{-s}\,|c|^{-s/2}\big(1+x\big)^{-s/2} \\ &= 2^{-s}\,|c|^{-s/2}\big(1 + \sum_{m\geq 1}(-1)^m \tfrac{\frac{s}{2}(\frac{s}{2}+1)\cdots(\frac{s}{2}+m-1)}{m!} x^m\,, \end{aligned} \tag{49}$$

by binomial expansion. We approximate this power series by finite polynomials

$$q_{s,N}(z^2) = 2^{-s}\,|c|^{-s/2}\left(1 + \sum_{m=1}^{N}(-1)^m \tfrac{\frac{s}{2}(\frac{s}{2}+1)\cdots(\frac{s}{2}+m-1)}{m!}\big(\tfrac{z^2}{|c|} - 1\big)\right)\,. \tag{50}$$

The approach of these polynomials to the correct weight for $s = 1$ is shown in figure 2 and 3. In both cases the polynomials $q_{1,N}$ are plotted for N between 4 and 8. They are displayed in the relevant and, to make it comparable, rescaled range in the variable x, see equation (48). Furthermore the correct weight $|R'(z)|^{-1}$ is depicted. This function is approximated from below. One

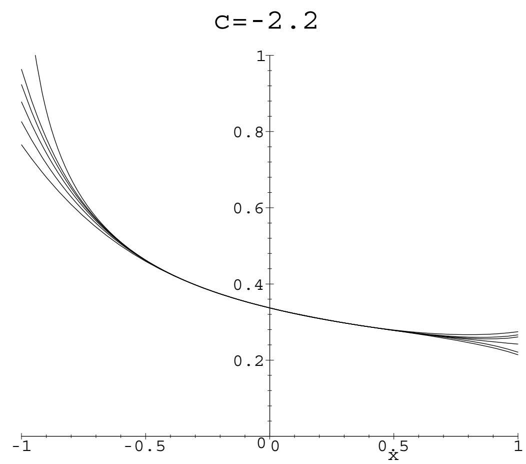

Figure 2: Approximation of the weight $|R'(z)|^{-1}$ by the polynomials $q_{1,N}$ for $N = 4 \dots 8$

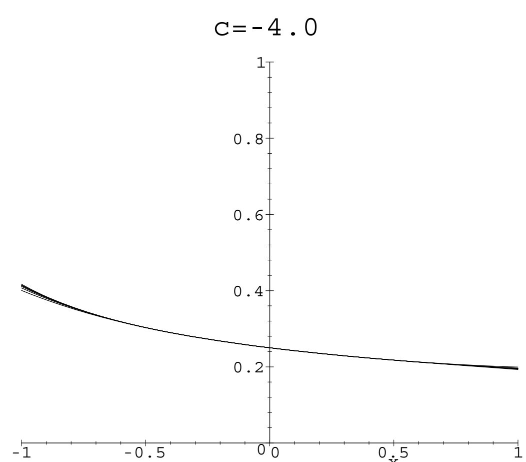

Figure 3: Approximation of the weight $|R'(z)|^{-1}$ by the polynomials $q_{1,N}$ for $N = 4 \dots 8$

can see clearly that the quality of the approximation is poorer if c approaches -2. If the cycle expansion with the weight $|R'(z)|^{-s}$ deviates from the systematic approximation, periodic orbits at the border of the interval in (48) must be held responsible.

Let us write down the matrix B_{lk} for this problem:

$$B_{lk} = \oint_{|z|=\rho} \frac{dz}{2\pi i} \frac{2z^l q_{s,N}(z^2)}{(z^2+c)^k} \quad l,k = 1,\dots,2N\,. \tag{51}$$

This is zero if l is even. Therefore the problem reduces to the odd rows and columns of B. As in section 4 we can now perform a substitution $u = z^2$ to get

$$\begin{aligned} B_{2i+1,2j+1} &= \oint_{|u|=\rho^2} \frac{du}{2\pi i} \frac{u^i q_{s,N}(u)}{(u+c)^{2j+1}} \\ &= \frac{1}{(2j)!} \left(\frac{d}{du}\right)^{2j} [u^i q_{s,N}(u)]_{u=-c}\,. \end{aligned} \tag{52}$$

These derivatives can now be computed along the lines of section 4.

5.2 Numerical study

First we turn to the computation of the spectrum of the physical evolution operator with weight $|R'(z)|^{-s}$ with the cycle expansion. We show that the cycle expansion for the Fredholm determinant converges super-exponentially. Therefore the limits we get for the escape rate are treated as the correct values for this system. Then we compute for certain orders of the approximation to this weight the "correct" spectrum of the transfer operators with the cycle expansion. Here we can compare with the exact data from our mathematical theory. Finally we compute the spectrum of the transfer operator with increasing order of the approximation. We conclude with a discussion.

The application of cycle expansion is straightforward. First we compute the prime cycles with symbols 0 and 1 for the two branches of the inverse map by backward iteration. The convention we adopted was

$$\begin{aligned} \text{symbol} &= 0 \Rightarrow \text{iteration with } -\sqrt{z-c} \\ \text{symbol} &= 1 \Rightarrow \text{iteration with } \sqrt{z-c}\,. \end{aligned}$$

This converges rapidly to the unstable periodic orbits. As an example we display in table 4 the prime orbits for $c = -2.1$ which correspond to the symbolic sequences up to length 6 in the order shown in table 1. The accuracy of these orbits is machine precision. The prime orbits are the data necessary

Symbols	Periodic orbits for $c = -2.1$					
0	-1.032971					
1	2.032971					
10	0.661895	-1.661895				
100	1.258096	-1.832511	-0.517193			
110	0.423594	1.588582	-1.920568			
1000	0.935156	-1.742170	-0.598189	-1.225484		
1100	1.328589	1.851645	-1.987874	-0.334851		
1110	0.311742	1.552978	1.911277	-2.002817		
10000	1.082917	-1.784073	-0.562074	-1.240132	-0.927291	
11000	0.887684	1.728492	-1.956653	-0.378612	-1.312017	
10100	1.187301	-1.813092	0.535638	-1.623465	-0.690315	
11100	1.349471	1.857275	1.989290	-2.022199	-0.278928	
11010	0.721466	1.679722	-1.944151	0.394777	-1.579486	
11110	0.273239	1.540532	1.908018	2.002003	-2.025340	
100000	1.009409	-1.763352	-0.580214	-1.232796	-0.931238	-1.081093
110000	1.103113	1.789724	-1.972238	-0.357438	-1.320062	-0.883141
101000	0.952473	-1.747133	0.594026	-1.641349	-0.677238	-1.192796
111000	0.869476	1.723217	1.955305	-2.013779	-0.293635	-1.344011
110100	1.170002	1.808315	-1.976946	0.350791	-1.565500	-0.731095
101100	1.297858	-1.843328	0.506628	1.614506	-1.927305	-0.415566
111100	1.355018	1.858768	1.989665	2.022292	-2.030343	-0.263926
111010	0.742676	1.686024	1.945771	-2.011410	0.297640	-1.548432
111110	0.262527	1.537051	1.907106	2.001776	2.025284	-2.031080

Table 4: Periodic prime orbits up to length 6 for the quadratic map $R(z) = z^2 + c$ for $c = -2.1$. Each line builds an orbit, the lines have to be read from right to left for the map R.

n	$\mathrm{tr}\mathcal{L}^n$	a_n
1	0.652328	6.52328e-01
2	0.740741	1.57604e-01
3	0.732861	4.89483e-02
4	0.658829	8.02233e-03
5	0.588177	2.82232e-04
6	0.529521	5.09918e-07

Table 5: Traces of powers of the evolution operator up to power six, computed with the cycle expansion (11) from the data in table 4 and coefficients of the corresponding spectral determinant.

for the computation of the traces of the evolution operator along the lines of equation (11). In table 5 we show the traces for $c = -2.1$ and maximal cycle length equal to 6. From the traces the corresponding coefficients of the spectral determinant can be computed recursively. If we make the ansatz

$$\det(I - z\mathcal{L}) = 1 - \sum_{n\geq 1} a_n z^n \,, \tag{53}$$

we have from equation (6) that

$$z\frac{d}{dz}\log\det\left(I - z\mathcal{L}\right) = \frac{-\sum_{n\geq 1} n a_n z^n}{1 - \sum_{n\geq 1} a_n z^n} = -\sum_{n\geq 1} z^n \mathrm{tr}\mathcal{L}^n \,. \tag{54}$$

Therefore:

$$\begin{aligned} \sum_{n\geq 1} n a_n z^n &= \Big(1 - \sum_{n\geq 1} a_n z^n\Big) \sum_{n\geq 1} z^n \mathrm{tr}\mathcal{L}^n \\ &= \sum_{n\geq 1} z^n \Big(\mathrm{tr}\mathcal{L}^n - \sum_{m=1}^{n-1} a_m \mathrm{tr}\mathcal{L}^{n-m}\Big) \end{aligned} \tag{55}$$

from which by comparison of coefficients follows:

$$a_n = \frac{1}{n}\Big(\mathrm{tr}\mathcal{L}^n - \sum_{m=1}^{n-1} a_m \mathrm{tr}\mathcal{L}^{n-m}\Big) \,. \tag{56}$$

The first six coefficients of the spectral determinant for our example $c = -2.1$ are displayed in table 5. One can see that the coefficients are decreasing rapidly. That means that we can expect that the zeros of the polynomial build from the first p coefficients of the spectral determinant give a good approximation to the characteristic values of the Fredholm determinant. The smallest characteristic value is connected to the escape rate via equation (4). For $c = -2.1$ we observed that at a maximal cycle length of 10 we can get the escape rate to machine precision. For smaller values of c the convergence is of course faster, because the escape rate will be higher. The results for various values of c are shown in table 6. In figure 4 we demonstrate the fast convergence to the "true" values for $c = -2.1$ and $c = -4.0$. The regime $c \to -\infty$ can be explored analytically by approximating the map R by tent maps, which can be exactly solved. Then the largest eigenvalue grows as $\sqrt{|c|}$.

Now we have computed the spectrum of the evolution operator $\mathcal{L}$ with weights $q_{1,N}(z^2)$ for N between 4 and 8 with the cycle expansion and from

c	Escape rate γ
-2.1	0.10592146763303689
-2.5	0.27270219256420158
-3.0	0.40682253879594737
-3.5	0.50902785837629816
-4.0	0.59282440725868102
-4.5	0.66418559147385337
-5.0	0.72645657248190221
-10.0	1.1133488223181551

Table 6: Escape rate for the quadratic map $R(z) = z^2 + c$ for different values of c computed with cycle expansion.

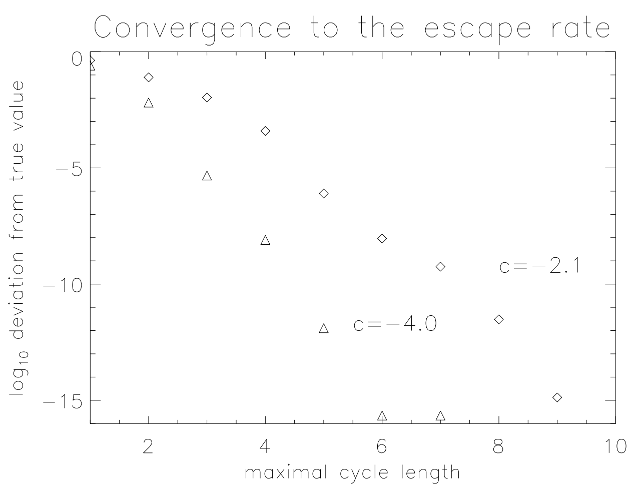

Figure 4: Convergence to the escape rate as computed in various orders of cycle expansion.

$N=4$	$N=6$	$N=8$
cycle expansion		
1.818787	1.812102	1.810089
-17.259650	-19.871638	-23.654760
(-27.245194,-136.719434)	-75.792816	-40.671498
(-27.245194,136.719434)	-406.381235	591.620215
1872.461572	(883.268581,0.000000)	-655.970106
756939002.269680	-4188.408456	-18478.169706
	46830.484571	
exact characteristic values		
1.81878664	1.81210219	1.81008864
-17.2596502	-19.8716375	-23.6547355
(-27.245194,-136.719433)	-75.7927967	-40.6715136
(-27.245194,136.719433)	-406.429527	591.384681
	883.856043	-655.520084
	-4090.40103	(-3817.86,-21283.1)
		(-3817.86,21283.1)
		-25697.5741

Table 7: Characteristic values of the transfer operator for $c = -4.0$ with polynomial weight $q_{1,N}(z^2)$ for N between 4 and 8 as computed with cycle expansion of order 7 and with the exact mathematical method.

the matrix B_{lk} in (52) for an exemplary value $c = -4.0$. The results are shown in table 7. We found that we got the bet values for the higher eigenvalues for an order of 7 of the cycle expansion. In table 7 we display the best approximation to the spectrum. We see from the table that we get the first four eigenvalues with a reasonable accuracy. The conclusion is that it is much harder to compute the whole spectrum with cycle expansion than only the first few eigenvalues. At this point a systematic approximation of transcendental weights by polynomials combined with an exact evaluation seems to be reasonable if the numerical work in doing the approximation is not too big. In other words, the numerical applicability of the approximation depends on the convergence of the spectrum to the "true" spectrum. Therefore let us turn to this question and find out how precise we can determine the spectrum as function of the order N of the polynomial $q_{s,N}$. As a (crude) measure for the convergence to the "true" spectrum we measure the approach to the "true" escape rate which we can get from the cycle expansion. This

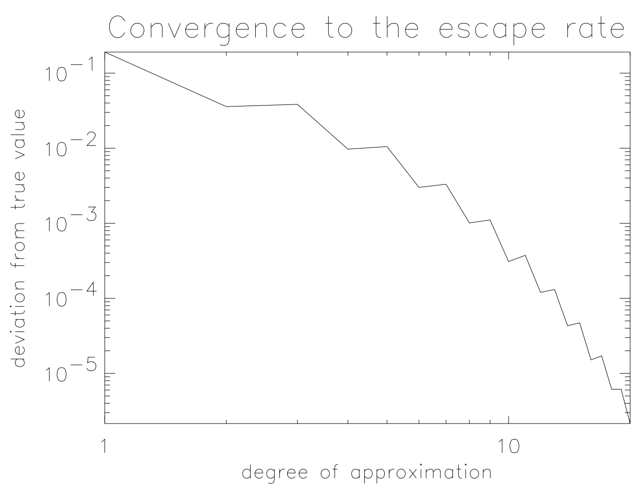

Figure 5: Convergence of the escape rate as computed by approximating the weight $|R'(z)|^{-1}$ by the polynomials $q_{1,N}$ to its "true" value as computed by cycle expansion.

convergence for $c = -4.0$ up to $N = 20$ is displayed in figure 5 in a log-log plot. One can see that the convergence is stronger than exponentially. If one goes from an even to an odd degree N one loses accuracy. This is quite surprising. One has to compute high order approximations to the weight even at $c = -4.0$ where we expect that the approximation should work fine. That means that in principle the approximation of transcendental weights works fine, for numerical purposes however one has to go to high orders. We can affirm our statement by the observation that at $N = 20$ only the first three characteristic values have assumed their asymptotic values.

Let us end this somewhat lengthy discussion of the approximation of transcendental weights by polynomials by indicating some open questions. Is it possible to solve any real problem in this way? Can the convergence be improved if one chooses different approximations, for example in a basis of other orthogonal polynomials? How can this be generalized to complex problems, that means problems in which the Julia set is not a subset of $\mathbb{R}$?

6 Application of complex cycle expansion to quadratic maps

In this section we want to demonstrate the applicability of the complex cycle expansion to the theory of complex dynamical systems. To this end we show first that complex cycle expansion works at all by computing the finite Fredholm determinants of section 4 for the quadratic map . Then we compute the spectrum for evolution operators with weight $R'(z)^{-2}$ where R is again the quadratic map. This has the advantage that we can develop the mathematical theory of section 3 further and compare with the exact formulas obtained with these methods. With the help of complex cycle expansion we find some conjectures for the *exact* form of non-finite and nontrivial spectral determinants for parameters c of the quadratic map at the border of the Mandelbrot set where the mathematical methods fail.

6.1 Complex cycle expansion

As we have already noted in section 2 the complex cycle expansion has almost no formal differences to the real one. However there is a big practical difference. In the complex case it is much more complicated to find the periodic prime orbits. This is because there is no apparent reason why the abstract symbolic sequences should should be related to the explicitly found periodic orbits in the same way as for real problems where we have good reasons for believing in this relation. Directly stated this means that we iterate backwards and relate the two branches of the quadratic map to the symbols 0 and 1 respectively the cycle we determine might come up as reducible although the symbolic sequence is prime. Furthermore it might happen that we iterate with three symbols and determine a cycle of period six with where we cycle through three numbers first and then through the three complex conjugates afterwards. Shortly speaking we could observe all sorts of strange behaviour.

If we search for the periodic orbits with the Newton–Raphson method we have even less control over their positions. One idea to solve this problem in general is to continue the labelling of orbits one gets for the not-pruned, completely real case to the case of complex periodic orbits.

For the backward iteration of the quadratic map we need to take square roots of complex numbers. The square root is defined on a complex plane which is cut along a line from zero to infinity. In our case we always choose a straight line cut which forms an angle α with the real positive axes. If we tune the angle α we can find a value for which we find all periodic orbits up to a certain period.

symbols	Periodic orbits (real part, imaginary part)			
0	(1.300243,-0.624811)			
1	(-0.300243,0.624811)			
10	(-1.000000,1.000000)	(-0.000000,-1.000000)		
100	(-1.290491,0.779282)	(0.096797,-1.140114)	(1.058088,-1.011312)	
110	(-0.852419,0.791527)	(-0.112077,0.930043)	(0.100103,-0.349425)	
1000	(-1.321031,0.671800)	(0.141702,-1.158063)	(1.073409,-1.005238)	(1.293807,-0.774937)
1100	(-1.182279,0.786928)	(-0.097587,1.091697)	(-0.134762,-0.340218)	(0.778529,-0.860738)
1110	(-1.057248,0.729149)	(-0.130658,1.036494)	(0.050004,0.364908)	(0.586115,-0.541781)

Table 8: Periodic prime orbits for the quadratic map $R(z) = z^2 + i$ in the complex plane up to length 4.

As an example table 8 shows all periodic prime orbits up to length 4 for the quadratic map $R(z) = z^2 + i$. Note that i lies inside the Mandelbrot set, therefore the corresponding Julia set is connected. For values of c outside the Julia set the search for prime orbits was in general easier. With the prime orbits we can compute the Fredholm determinant up to order 4. If we choose the weight 1 or in general a polynomial, we know the exact result from the theory developed in section 3. For weight 1 the spectral determinant is simply equal to $1 - 2z$. If we compute the spectral determinant with the complex cycle expansion we get for $c = i$ the contents of table 9. We take the equality as an indicator for the fact that we determined the periodic prime orbits up to order 8 correctly.

Therefore we suggest to do the comparison with the finite and known spectral determinants to find out whether all periodic orbits could be determined or not. It might be interesting to note that in the case $c = i$ we had to tune α to $\alpha = 2.24$.

6.2 Nontrivial example

Once we know that we determined all periodic orbits up to a certain order p correctly we can study evolution operators with any weight function we

n	Coefficient a_n
0	(1.000000,0.000000)
1	(-2.000000,-0.000000)
2	(-0.000000,-0.000000)
3	(-0.000000,0.000000)
4	(0.000000,0.000000)
5	(-0.000000,0.000000)
6	(-0.000000,-0.000000)
7	(0.000000,-0.000000)
8	(-0.000000,0.000000)

Table 9: Coefficients of the spectral determinant in complex cycle expansion up to order 8 for the map $R(z) = z^2 + i$.

choose. From that point on we can use the full power of cycle expansion. Here we want to demonstrate this and choose the weight $Q(z) = R'(z)^{-2}$. First we solve the corresponding adjoint equation.

In the case $R(z) = z^2 + c$ and $Q(z) = R'(z)^{-2}$ the adjoint equation (34) is

$$f(z) - \lambda \frac{f(R(z))}{R'(z)} = \phi(z)\,. \tag{57}$$

ϕ can only have poles when $R'Q(z) = \frac{1}{R'(z)} = \frac{1}{2z}$ has poles, that means ϕ is holomorphic except possibly at zero. Furthermore we see from (57) and the behaviour of f at infinity that ϕ tends to zero as z goes to infinity. Because ϕ is only determined up to a constant we know that either $\phi(z) = \frac{1}{z}$ or $\phi(z) \equiv 0$. Bearing this in our mind we can easily solve step (i) of the solution algorithm for equation (57) in the attractive basin of the fixed point infinity if $\phi(z) = \frac{1}{z}$ by writing equation (30) as

$$f(\lambda, z) = \sum_{n=0}^{\infty} \lambda^n \phi_1(\lambda, R^n z) \frac{1}{R'}(n, z) = \sum_{n=0}^{\infty} \frac{\lambda^n}{R^n(z) R^{n\prime}(z)}\,. \tag{58}$$

From equation (31) we now get the condition for λ (step (ii)):

$$\begin{aligned} 0 &= \mathrm{Res}_{z=0}[f(\lambda, z)] \\ &= \sum_{n=0}^{\infty} \lambda^n \mathrm{Res}_{z=0}[\frac{1}{R^n(z) R'(z) R'(R(z)) \ldots R'(R^{n-1}(z))}] \\ &= \sum_{n=0}^{\infty} \frac{(\lambda/2)^n}{R(0) R^2(0) \ldots R^n(0)} =: \Delta(\lambda)\,. \end{aligned} \tag{59}$$

If the parameter c lies outside the Mandelbrot set M then we know that 0 iterates to infinity. Upon iteration of equation (57) we deduce from $\frac{f(R^n z)}{R^{n\prime}(z)} \to 0$ as $n \to \infty$ for large absolutes of z that ϕ cannot be identical to zero. Otherwise f would also be identical to zero and could not be an eigenfunction. Then we know already that $\phi(z) = \frac{1}{z}$ and the characteristic values λ are determined by (59). Their computation thus only involves forward iterates of zero. The same holds true after meromorphic continuation of $\Delta(\lambda)$ if c lies in the interior of M and has an attracting periodic orbit of period p; the points where $\Delta(\lambda)$ has poles can be given explicitly [4]. If 0 lies on the attracting orbit $\Delta(\lambda)$ reduces to a polynomial.

As an example for the fast convergence of the power series (59) for $\Delta(\lambda)$ for values of c outside the Mandelbrot set we show in table 10 the first

n	$(\Re(a_n), \Im(a_n))$
0	(1.0000e+00,0.0000e+00)
1	(0.0000e+00,-2.5000e-01)
2	(-1.2500e-02,2.5000e-02)
3	(-7.3529e-04,1.8382e-04)
4	(-1.0136e-07,-1.1165e-06)
5	(-1.0629e-12,4.7894e-12)
6	(-1.3943e-22,1.2594e-22)
7	(-4.8341e-43,-2.6476e-43)
8	(9.3505e-84,-1.5974e-84)

Table 10: The first eight coefficients a_n of the spectral determinant of R for $c = 2i$.

eight coefficients of the spectral determinant $\Delta(\lambda)$ at $c = 2i$. They decrease rapidly with the order, therefore we can compute the spectrum from the first few coefficients to a high precision. The leading characteristic values for our example are displayed in table 11.

With the complex cycle expansion the spectrum can also be determined. We got the best approximation at the sixth order of the expansion. To higher order roundoff errors deteriorate the result. Our best values are displayed in the third column of table 11.

Now we want to test the complex cycle expansion for a chaotic map which has a pruned symbolic dynamic in the real case. If we choose the parameter c such that $z = 0$ iterates after N steps to the less repelling fixed point of $R(z) = z^2 + c$ then the map R is chaotic [5, 8]. The solution to this for $N = 3$ is $c = -1.5436890126920763616$. Around this point the Mandelbrot set M is homeomorphic to a line, that means this value for c lies at the boundary of

n	λ_n by (59)	λ by cycle expansion
1	(-1.422853,-3.360150)	(-1.422853,-3.360150)
2	(8.071310,-0.934992)	(8.071310,-0.934992)
3	(-33.25845,31.81435)	(-33.25845,31.81435)
4	(128.6820,640.5525)	(128.6820,640.5525)
5	(217599.9,-70144.07)	(217434.1,-70679.45)

Table 11: Characteristic values corresponding to the Fredholm determinant in table 10 compared with the values from complex cycle expansion to order 6.

M. Therefore the spectrum cannot be determined directly from (59). The complex cycle expansion shows, that this is a hard problem to solve. The

n	a_n	n	a_n
0	1.000000	9	0.005139
1	-0.323899	10	-0.003062
2	-0.192961	11	0.001824
3	0.114955	12	-0.001087
4	-0.068484	13	0.000647
5	0.040799	14	-0.000386
6	-0.024306	15	0.000230
7	0.014480	16	-0.000137
8	-0.008626		

Table 12: Spectral determinant for a chaotic quadratic map up to order 16 in complex cycle expansion.

coefficients for the spectral determinant are depicted in table 12. One can see that they don't decrease rapidly enough to yield a good estimate for the spectrum. We observe, that all coefficients are real.

Another way to get information about the Fredholm determinant for this value of c might be to approximate c from c values outside the Mandelbrot set M and use formula (59) to compute the spectrum. It is not obvious that this converges to the correct Fredholm determinant at $c = -1.5436890126920763616$. In figure 6 we show that it indeed does. Here we add small imaginary parts to c and study the behaviour of the coefficients of the Fredholm determinant. In the figure the absolute values of the coefficients are depicted on a logarithmic scale. With every reduction of the imaginary part by a factor of ten we get three more coefficients of the actual

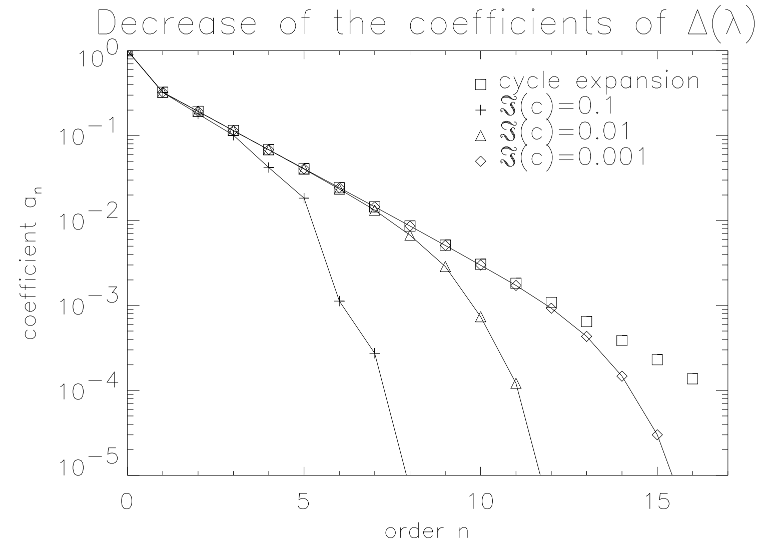

Figure 6: Decrease of the coefficients of the spectral determinant as computed with equation (59) for small imaginary parts added to $c = -1.543689\ldots$ compared to the cycle expansion of order 16 at real c.

chaotic spectral determinant. However we didn't find it particularly practical to compute approximately 70 orders of (59) and to compute the zeros of the resulting polynomial.

However, figure 6 shows that the coefficients a_n of the spectral determinant are decreasing approximately as βq^n for some β and q. Furthermore we see from table 12 that the coefficients alternate from order two on. Therefore we propose to study this system by measuring q and β and by calculating the resulting geometrical series:

$$\begin{aligned} \Delta(z/q) &= 1 - \beta z - \beta \sum_{n\geq 2} (-z)^n = 1 - \beta z - \beta \frac{z^2}{1+z} \\ &= \frac{1 + (1-\beta)z - 2\beta z^2}{1+z}, \end{aligned} \tag{60}$$

if we assume that the scaling form holds from element a_1 on. In table 13 we show that the scaling assumption holds true to an unexpected accuracy by computing the ratio of respectively two coefficients which should be equal to $1/q$. Because in higher order of cycle expansion there are big influences from roundoff errors we did this computation to eighth order of cycle expansion. The cycles are computed with an absolute precision of 10^{-14}. With the same

n	$\frac{a_n}{a_{n+1}} = \frac{1}{q}$
1	1.6785735104283492
2	-1.6785735104283219
3	-1.6785735104283410
4	-1.6785735104282917
5	-1.6785735104282846
6	-1.6785735104283179
7	-1.6785735104283059

Table 13: Ratios of respectively two neighbouring coefficients of the Fredholm determinant for the chaotic square map as computed by the cycle expansion to order eight.

accuracy the scaling assumption is valid. I conclude that this cannot be good luck and conjecture therefore that (60) is the *exact* form of the spectral determinant in this case.

From the data in table 12 and 13 we can compute the parameter β and get

$$\beta = 0.543689012692088 = -1 - c\,. \tag{61}$$

The spectral determinant computed with these parameters is depicted in

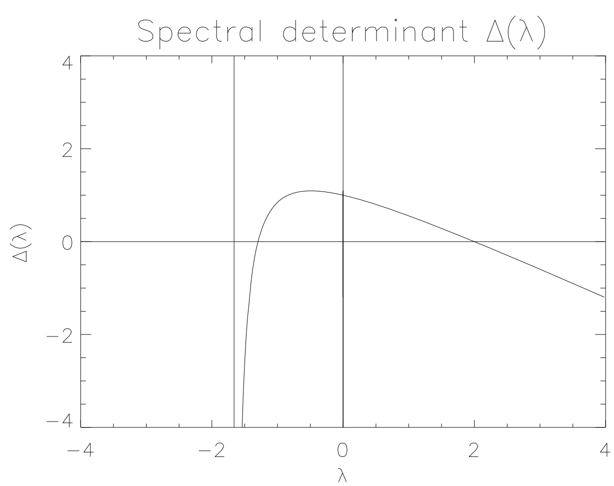

Figure 7: Spectral determinant for the chaotic quadratic map.

figure 7. The zeros are at $z = 2$ and at $z = -1.2955977$, the pole lies at $z = -\frac{1}{q} = -1.6785735$.

Once the scaling $a_n = \beta q^n$ for the coefficients of the spectral determinant is known it is possible to compute exact relations for the parameters β, q and c from the traces of $\mathcal{L}$ and $\mathcal{L}^2$. To order z we have

$$\beta q = \mathrm{tr}\mathcal{L}\,. \tag{62}$$

The trace can be computed analytically:

$$\begin{aligned} \mathrm{tr}\mathcal{L} &= \sum_{\omega:R\omega=\omega} \frac{1}{R'(\omega)(R'(\omega)-1)} \\ &= \frac{1}{(1+\sqrt{1-4c})\sqrt{1-4c}} - \frac{1}{(1-\sqrt{1-4c})\sqrt{1-4c}} \\ &= -\frac{1}{2c}\,. \end{aligned} \tag{63}$$

Therefore the first relation is

$$\beta q = -\frac{1}{2c}\,. \tag{64}$$

The analytical computation of $\mathrm{tr}\mathcal{L}^2$ involves a bit more effort but is straightforward. The result is

$$\mathrm{tr}\mathcal{L}^2 = \frac{c-1}{4c^2(c+1)}\,. \tag{65}$$

That means that we get the second condition from the second order of cycle expansion:

$$-a_2 = \beta q^2 = -\frac{1}{8c^2}\left(1 - \frac{c-1}{c+1}\right) = -\frac{1}{4c^2(c+1)}\,. \tag{66}$$

The combination of (64) and (66) yields

$$\begin{aligned} q &= \frac{1}{2c(c+1)}\,, \\ \beta &= -(c+1)\,. \end{aligned} \tag{67}$$

This confirms the numerical found (61).

The same analysis holds true with slight modifications if we choose c such that the critical point $z = 0$ maps to the less repelling fixed point after $N > 3$ iterations. From the examples below we deduce an infinite family of

N	c
4	-1.8929109879078194211
5	-1.9739320445949196125
6	-1.6975553932375475482
	-1.8186201342243301121
	-1.9935450866059057468

Table 14: Values for c for which the critical point $z = 0$ iterates after N steps to the less repelling fixed point of R for $N = 4, 5, 6$.

conjectures for the form of spectral determinants for (real) chaotic quadratic maps with values of c at the border of the Mandelbrot set.

Some values for c for which the critical point iterates after N steps to the fixed point as discussed above are depicted in table 14. To be concrete these values of c are the solutions of the equation

$$R^N(0) = \frac{1}{2}(1 - \sqrt{1-4c}), \tag{68}$$

which is purely algebraic. Of course solutions for $N = 3$ are also solutions for higher values of N etc, and in the table only the new solutions are depicted. For higher values of N the numerical determination of the solutions of (68) is naturally more difficult since more and more solutions are generated which cumulate at $c = -2$.

Now we do the complex cycle expansion for these values of c. The resulting coefficients of the spectral determinant and their successive quotients can be found in table 15. For brevity we show only six digits for the results which were obtained to 14 digits. One can clearly see that the coefficients a_n of the spectral determinant assume also a scaling form for the values of c depicted in table 14 which iterate after $N > 3$ steps to the less repelling fixed point. However, for $N > 3$ the scaling form of the coefficients a_n is only assumed for $n \geq N - 2$:

$$a_n = \beta q^{N-3+n}, \quad \text{for}\, n \geq N-2\,. \tag{69}$$

I conjecture that the spectral determinant has in general the following form:

$$\begin{aligned}\det(1 - z\mathcal{L}) &= 1 + \sum_{n=1}^{N-3} a_n z^n - \beta q z^{N-2} - \beta q^2 z^{N-1} + \beta q^3 z^N - \ldots \\ &= 1 + \sum_{n=1}^{N-3} a_n z^n - \beta q z^{N-2} - \frac{\beta q^2 z^{N-1}}{1+qz}\,, \end{aligned} \tag{70}$$

Coefficients a_n of Fredholm determinant					
n	$c = -1.892911$	-1.973932	-1.697555	-1.818620	-1.993545
0	1.000000	1.000000	1.000000	1.000000	1.000000
1	-0.264143	-0.253302	-0.294541	-0.274934	-0.250809
2	-0.078140	-0.065879	-0.124369	-0.092337	-0.063314
3	-0.040534	-0.019129	0.210531	-0.116064	-0.016407
4	0.021027	-0.009649	-0.065370	0.03495	-0.004743
5	-0.010908	0.004867	-0.036497	0.018625	-0.002377
6	0.005658	-0.002455	0.020377	-0.009925	0.001191
7	-0.002935	0.001238	-0.01137	0.005289	-0.000597
8	0.001523	-0.000625	0.006352	-0.002819	0.000299
Quotients $\frac{a_n}{a_{n+1}}$ of coefficients					
n	$c = -1.892911$	-1.973932	-1.697555	-1.818620	-1.993545
1	3.380402	3.844951	2.368278	2.977518	3.961354
2	1.927737	3.443961	-0.590741	0.795567	3.859072
3	-1.927737	1.982571	-3.220623	-3.320777	3.459128
4	-1.927737	-1.982571	1.791097	1.876540	1.995694
5	-1.927737	-1.982571	-1.791097	-1.876540	-1.995694
6	-1.927737	-1.982571	-1.791097	-1.876540	-1.995694
7	-1.927737	-1.982571	-1.791097	-1.876540	-1.995694

Table 15: Results for the spectral determinant from complex cycle expansion to order eight for the values of c in table 14 and their successive coefficients.

with a simple pole at $z = -\frac{1}{q}$. The information which is necessary to compute the complete spectral determinant are the traces $\mathrm{tr}\mathcal{L}^n$ for n from 1 to $N-1$. The infinite problem of computing all coefficients of the Fredholm determinant is therefore reduced to a finite problem, similar to the polynomial evolution with a polynomial weight, where this can be proved mathematically.

Let us illustrate this way for determining the spectral determinant for some other examples. We know that $c = i$ is also a point at the border of the Mandelbrot set. As above we can't apply the mathematical tools of [4] to get the Fredholm determinant. Therefore we use the complex cycle expansion to determine the spectral determinant $\Delta(z) = 1 + \sum_{n\geq 1} a_n z^n$ up to order eight. The result can be found in table 16. In the third column we have listed the

n	a_n	$\frac{a_n}{a_{n+1}}$
0	(1.000000,0.000000)	
1	(-0.000000,-0.500000)	(-2.000000,2.000000)
2	(-0.125000,0.125000)	(0.000000,-2.000000)
3	(-0.062500,-0.062500)	(-2.000000,2.000000)
4	(-0.000000,0.031250)	(0.000000,-2.000000)
5	(-0.015625,-0.000000)	(-2.000000,2.000000)
6	(0.003906,0.003906)	(0.000000,-2.000000)
7	(-0.001953,0.001953)	(-2.000000,2.000000)
8	(0.000977,0.000000)	(0.000000,-2.000000)

Table 16: Spectral determinant for the map $R(z) = z^2 + i$ as computed with complex cycle expansion up to order 8; quotients of successive coefficients.

ratios $\frac{a_n}{a_{n+1}}$. Again we observe a structure in these numbers. If we multiply two successive ratios with each other we see that

$$\frac{a_{2n-1}}{a_{2n+1}} = 4 + 4i \quad \text{and} \quad \frac{a_{2n}}{a_{2n+2}} = 4 + 4i \,. \tag{71}$$

Again we conjecture that this holds true to any order of cycle expansion. Then we can write the spectral determinant as

$$\begin{aligned} \Delta(z) &= 1 + \sum_{n\geq 1} a_n z^n = 1 + \sum_{n\geq 1} \left(a_{2n-1} z^{2n-1} + a_{2n} z^{2n}\right) \\ &= \sum_{n\geq 1} \Big(\frac{(4+4i)a_1}{z} \frac{z^{2n}}{(4+4i)^n} + (4+4i)a_2 \frac{z^{2n}}{(4+4i)^n} \Big) \\ &= \frac{a_1 z + a_2 z^2}{1 - \frac{z^2}{4+4i}} = \frac{-iz/2 - \frac{z^2}{4+4i}}{1 - \frac{z^2}{4+4i}} \,. \end{aligned} \tag{72}$$

The zeros of this function lie at 0 and at $2 - 2i$.

One can perform the same analysis for $c = -i$. Of course one gets exactly the complex conjugate result:

$$\Delta(z) = \frac{iz/2 - \frac{z^2}{4-4i}}{1 - \frac{z^2}{4-4i}} \,. \tag{73}$$

We note that the conjectures (70) and (72) are compatible with the (mathematically unjustified) application of (59) to these problems. That means we can solve for all coefficients in the conjectures by just comparing them to this formula. That means that the value of q in the example (70) is given by $1/q = 1 - \sqrt{1-4c}$, and the parameter β is 2^{-N+2} times the inverse of the product of all images of 0, $R(0) \ldots R^{N-2}(0)$. Despite the simplicity of this result, we feel that it is not unnecessary labour to find these conjectures by using the complex cycle expansion. This is completely independent from formulas like (59). Therefore it is adapted to use cycle expansion if we don't know, whether we can apply the mathematical theory or not. However, as a final result let us state that we conjecture from the results in this section that formula (59) holds true also for all values of c on the boundary of the Julia set after analytic continuation.

7 Conclusion

In this report we have analyzed the relation between cycle expansion and adjoint equations in great detail for many different examples. We established the complex cycle expansion as a tool in the theory of complex dynamical systems and vice versa. Exact mathematical results can be very helpful for numerical checks. Typical problems which can be attacked in an efficient way in this spirit are checks whether all complex orbits have been determined with the correct stabilities. On the other hand we could demonstrate that complex cycle expansion can lead one to new and exciting conjectures about the general form for nontrivial spectral determinants. These conjectures are the main results of our computations.

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