Chapter 9

Appendices

9.1 Derivations and examples of chapter 5

The Jacobian as an integral

Equation (5.17) can be obtained by integrating the time derivative of the Jacobian, which can be obtained as follows

$$\begin{aligned} q(t+\delta t) &= q(t) + q\delta t \\ &= q(t) + \frac{\partial H}{\partial p}\delta t \qquad (9.1) \\ p(t+\delta t) &= p(t) + p\delta t \\ &= p(t) - \frac{\partial H}{\partial q}\delta t \qquad (9.2) \end{aligned}$$

which gives

$$\begin{aligned} \mathbf{J}(t+\delta t) &= \begin{pmatrix} \mathbf{1} + \frac{\partial^2 H}{\partial q_i \partial p_j}\delta t & \frac{\partial^2 H}{\partial p_i \partial p_j}\delta t \\ -\frac{\partial^2 H}{\partial q_i \partial q_j}\delta t & \mathbf{1} - \frac{\partial^2 H}{\partial q_i \partial p_j}\delta t \end{pmatrix} \\ &= \mathbf{1} + \mathbf{J}(t)\delta t. \qquad (9.3) \end{aligned}$$

From the last expression we can read of the time derivative of the Jacobian

$$\begin{aligned} \mathbf{J}(t) &= \begin{pmatrix} \frac{\partial^2 H}{\partial q_i \partial p_j} & \frac{\partial^2 H}{\partial p_i \partial p_j} \\ -\frac{\partial^2 H}{\partial q_i \partial q_j} & -\frac{\partial^2 H}{\partial q_i \partial p_j} \end{pmatrix} \\ &\equiv \mathbf{D}^2 H \end{aligned}$$

The differential equation for the M flow

The differential equation that drives the **M** flow can be derived in the following way. If we substitute the elements of the infinitesimal Jacobi matrix (9.3). For infinitesimal time we have

$$\begin{aligned}
\Delta\mathbf{M} &= \mathbf{M}' - \mathbf{M} \\
&= (\mathbf{J}_{pq} + \mathbf{J}_{pp}\mathbf{M})(\mathbf{J}_{qq} + \mathbf{J}_{qp}\mathbf{M})^{-1} - \mathbf{M} \\
&= ((\mathbf{J}_{pq} + \mathbf{J}_{pp}\mathbf{M}) - \mathbf{M}(\mathbf{J}_{qq} + \mathbf{J}_{qp}\mathbf{M}))(\mathbf{J}_{qq} + \mathbf{J}_{qp}\mathbf{M})^{-1} \qquad (9.4)
\end{aligned}$$

which gives

$$\begin{aligned}
\Delta\mathbf{M}(\mathbf{J}_{qq} + \mathbf{J}_{qp}\mathbf{M}) &= (\mathbf{J}_{pq} + \mathbf{J}_{pp}\mathbf{M}) - \mathbf{M}(\mathbf{J}_{qq} + \mathbf{J}_{qp}\mathbf{M}) \\
&= -(\frac{\partial^2 H}{\partial q \partial q} + \frac{\partial^2 H}{\partial q \partial p}\mathbf{M}) - \mathbf{M}(\frac{\partial^2 H}{\partial p \partial q} + \frac{\partial^2 H}{\partial p \partial p}\mathbf{M}) \qquad (9.5)
\end{aligned}$$

And since in the limit $\delta t \to 0$ we have $\mathbf{J}_{qq} \to \mathbf{1}$ and $\mathbf{J}_{qp} \to \mathbf{0}$ we get

$$\mathbf{M} = -\left(\frac{\partial^2 H}{\partial q \partial q} + \mathbf{M}\frac{\partial^2 H}{\partial p \partial q} + \frac{\partial^2 H}{\partial q \partial p}\mathbf{M} + \mathbf{M}\frac{\partial^2 H}{\partial p \partial p}\mathbf{M}\right), \qquad (9.6)$$

The volume ratio as an integral

The expression for the volume ratio (5.24) can be derived by splitting the ratio into a product over ratios of infinitesimal time evolution:

$$\begin{aligned}
\frac{V(q')}{V(q_0)} &= \prod_{t=0}^{\tau} \frac{V_{t+dt}}{V_t} \\
&= \prod_{t=0}^{\tau} \det(\mathbf{J}_{qq} + \mathbf{J}_{qp}\mathbf{M}^t) \\
&= \prod_{t=0}^{\tau} \det(\mathbf{1} + \left(\frac{\partial^2 H}{\partial q \partial p} + \frac{\partial^2 H}{\partial p \partial p}\mathbf{M}^t\right)\delta t) \\
&= \exp\{\int_0^t [\frac{\partial^2 H}{\partial p \partial q} + \frac{\partial^2 H}{\partial p \partial p}\mathbf{M}] d\tau\} \qquad (9.7)
\end{aligned}$$

Evolution of the quasi-classical wave function

If we only consider the delta functions in the kernel (9.39) and the initial wave function the calculation goes as follows. First we integrate out $\mathbf{M}$ and p in the evolved wave function:

$$\begin{aligned}
\tilde{\psi}(q,p,\mathbf{M},t) &= \int dq dp d\mathbf{M} \delta(q' - q^t(q,p))\delta(p' - p^t(q,p))\delta(\mathbf{M}' - \mathbf{M}^t(q,p,\mathbf{M})) \\
&\times \delta(p - \nabla S(q,0))\delta(\mathbf{M} - \frac{\partial^2 S(q,0)}{\partial q^2})\psi(q,0) \\
&= \int dq dp \delta(q' - q^t(q,p))\delta(p' - p^t(q,p))\delta(\mathbf{M}' - \mathbf{M}^t(q,p,\partial^2 S(q,0)/\partial q^2)) \\
&\times \delta(p - \nabla S(q,0))\psi(q,0) \\
&= \int dq \delta(q' - q^t(q,\nabla S(q,0)))\delta(p' - p^t(q,\nabla S(q,0))) \\
&\times \delta(\mathbf{M}' - \mathbf{M}^t(q,\nabla S(q,0),\partial^2 S(q,0)/\partial q^2))\psi(q,0)
\end{aligned}$$

Then when we finaly do the q integral we obtain the evolved wave function in the same form as the initial one, divided with the determinant of the Jacobian of the configuration space evolution which is just the volume ratio

$$\tilde{\psi}(q,p,\mathbf{M},t) = \frac{1}{|\det\left(\frac{\partial q^t}{\partial q}\right)|}\delta(p'-\nabla S(q',t))\delta(\mathbf{M}'-\frac{\partial^2 S(q',t)}{\partial q'\partial q'})\psi(q^{-t}(q'),0). \tag{9.8}$$

In order to get the right volume ratio in front of the old wave function we therefore have to multiply this expression with $\sqrt{|\det\left(\frac{\partial q^t}{\partial q}\right)|}$, which is just equal to the term $(V(q')/V(q))^{1/2}$ in (5.24), and hence gives the change of the sign in the trace integral in the exponent.

9.1.1 Alternative derivation of the curvature trace

In this section we shall derive the general result of the curvature integration in an arbitrary number of dimensions. The derivation of the result will follow a different approach than the on in section 5.3. Here we consider the general evolution of Lagrangian manifolds corresponding to the periodic solutions of the curvature flow.

Lagrangian manifolds

The definition of a Lagrangian manifold involves the symplectic form denoted ω, which is an antisymmetric, bilinear operator acting on vectors in phase space. If we let $\delta\mathbf{z}_1 = (\delta\mathbf{q}_1, \delta\mathbf{p}_1)$ and $\delta\mathbf{z}_2 = (\delta\mathbf{q}_2, \delta\mathbf{p}_2)$ be two small displacements in phase space, then the action of the symplectic form on them is defined by

$$\omega(\delta\mathbf{z}_1, \delta\mathbf{z}_2) = \delta\mathbf{p}_1\cdot\delta\mathbf{q}_2 - \delta\mathbf{p}_2\cdot\delta\mathbf{q}_1 \tag{9.9}$$

or, in matrix form

$$\omega(\delta\mathbf{z}_1, \delta\mathbf{z}_2) = \delta\mathbf{z}_1\cdot\omega\cdot\delta\mathbf{z}_2 \tag{9.10}$$

where ω is the unit symplectic matrix,

$$\omega = \begin{bmatrix} \mathbf{0} & \mathbf{1} \\ -\mathbf{1} & \mathbf{0} \end{bmatrix}. \tag{9.11}$$

The matrix ω is antisymmetric and orthogonal, so $\omega^t = \omega^{-1} = -\omega$. Note that only two vectors are involved in the definition of the symplectic form no matter how many dimensions in the phase space. The symplectic form is invariant under canonical transformations, in the sense that the value of the right hand side of equation (9.9) is independant of the canonical coordinates used to compute it.

We now define a Lagrangian manifold as an f-dimensional surface L in the $2f$ dimensional phasespace such that at all points $(\mathbf{x}, \mathbf{p})$ on L and for all vectors $\delta\mathbf{z}_1, \delta\mathbf{z}_2$ tangent to L at $(\mathbf{x}, \mathbf{p})$, we have

$$\omega(\delta\mathbf{z}_1, \delta\mathbf{z}_2) = 0. \tag{9.12}$$

As an example we can investigate under which conditions a surface of the form

$$L = \{(\mathbf{x}, \mathbf{p}) | \mathbf{p} = \mathbf{M}\mathbf{x}\} \tag{9.13}$$

is lagrangian. Inserting the condition (9.13) into the definition of the symplectic form yields

$$(\mathbf{M}\delta\mathbf{q}_1) \cdot \delta\mathbf{q}_2 - (\mathbf{M}\delta\mathbf{q}_2) \cdot \delta\mathbf{q}_1 = 0 \tag{9.14}$$

which implies

$$\mathbf{M}_{ij} = \mathbf{M}_{ji} \tag{9.15}$$

i.e. that that matrix $\mathbf{M}$ should be symmetric. This is going to be an important restriction in the following sections.

As we menchioned before the symplectic form is conserved by canonical transformations. As a special case this implies that the symplectic form is conserved by a Hamiltonian flow since this can be considered as a canonical transformation. This means that Lagrangian manifolds evolve into Lagrangian manifolds in Hamiltonian flows, - an important fact that we shall also use in our following considerations.

The curvature integral

We consider integrals of the type

$$\mathrm{I}(q', p', \mathbf{M}') = \int dqdpd\mathbf{M} f(q, p, \mathbf{M}; t)\delta(q' - q^t)\delta(p' - p^t)\delta(\mathbf{M}' - \mathbf{M}^t) \tag{9.16}$$

where the super script t indicates the variabel evolved in time t . Doing the q integral gives a sum over certain q values who has the possibility to end up at the correct final q' if they are provided the correct initial momentum. The p integration rules out all except one of the initial q values since the phase space flow gives unique solutions to the Hamilton equations. In this way we get a unique point (q, p) defined in phase space namely the initial condition that leads to (q', p') in time t.

Once this point is specified the flow defines a (parametrized) flow on the curvature subspace $g_{q,p(q)} : \mathbf{M} \to \mathbf{M}'$. The map works in the following way: given (q_0, p_0) and $\mathbf{M}_0$ we have defined a little fraction of a Lagrangian manifold

$$L = \{(q, p) | p(\delta q) = p_0 + \mathbf{M}_0 \delta q\}$$

where we assume that $\delta q = q - q_0$ is very small. That the manifold is really Lagrangian is ensured by the fact that $\mathbf{M}$ is symmetric since it is the second derivative of the action function $\mathbf{M} = \partial^2 S/\partial q_i \partial q_j$. This manifold can then be uniquely evolved in time t according to Hamilton Jacoby equations and from the new manifold generated in this way we can obtain $\mathbf{M}' = \mathbf{M}^t(q_0, p_0)$ as a function of the original curvature. $\mathbf{M}'$ then specifies the tangent manifold at the point (q', p') by the relation

$$\delta p' = \mathbf{M}' \delta \mathbf{q}'. \tag{9.17}$$

Example

As an example of the time evolution of the curvature matrix, we consider the free flight part of an N dimensional billard. Here we have

$$x(t) = x(0) + \mathbf{M}_0 x(0) t \tag{9.18}$$

and

$$p(t) = p(0) \tag{9.19}$$

so that we can write

$$\begin{aligned} p(t) &= \mathbf{M}_0 x_0 \\ &= \mathbf{M}_0 (\mathbf{1} + \mathbf{M}_0 t)^{-1} x(t) \end{aligned} \tag{9.20}$$

which gives

$$p(t) = \mathbf{M}_0 (\mathbf{1} + \mathbf{M}_0 t)^{-1} x(t) \tag{9.21}$$

and hence we see that the curvature matrix is a sort of generalized Sinai-Bunimowich curvature.

We are interested in the trace of the evolution operator $\mathcal{L}^t$ in (9.16)

$$\begin{aligned} \mathrm{tr} \mathcal{L}^t &= \int dq dp d\mathbf{M} \delta(q - q^t) \delta(p - p^t) \delta(\mathbf{M} - \mathbf{M}^t) \\ &\times \; e^{\frac{1}{2} \int_0^{T_p} (H_{pq} + H_{pp} \mathbf{M}) d\tau} \end{aligned} \tag{9.22}$$

Following the strategy in section 3.1 we introduce longitudinal $\mathbf{x}_\parallel$ and perpendicular $\mathbf{x}_\perp$ coordinates along the total $\mathbf{x} = (q, p, \mathbf{M})$ flow to evaluate the contribution from a prime periodic orbit to the trace. In the longitudinal direction we get

$$\int d\mathbf{x}_\parallel \delta_\parallel (\mathbf{x} - \mathbf{x}^t) = T_p \sum_{r=1}^{\infty} \delta(t - r T_p) \tag{9.23}$$

where T_p is the period of the prime periodic orbit. In the perpendicular direction we get

$$\int d\mathbf{x}_\perp \delta_\perp (\mathbf{x} - \mathbf{x}^{r T_p}) = \frac{1}{|\det(\mathbf{1} - \hat{\mathbf{J}}_p^r)|} \tag{9.24}$$

where $\hat{\mathbf{J}}_p$ is the transverse stability matrix, $\mathbf{u}(t+T_p) = \hat{\mathbf{J}}_p \mathbf{u}(t)$ of the entire flow. Since $\frac{\partial q^t}{\partial \mathbf{M}} = \frac{\partial p^t}{\partial \mathbf{M}} = 0$ it has the structure

$$\det \hat{\mathbf{J}}_p = \begin{bmatrix} \mathbf{J}_{tr} & 0 \\ * & \mathbf{J}_{\mathbf{M}} \end{bmatrix} \tag{9.25}$$

and since this is block diagonalizable the determinant splits up into a product of the usual transverse determinant and a determinant corresponding to the $\mathbf{M}$ flow

$$\det(\mathbf{1} - \hat{\mathbf{J}}_p^r) = \det(\mathbf{1} - \mathbf{J}_p^r) \cdot \det(\mathbf{1} - \mathbf{J}_{\mathbf{M}}{}_p^r). \tag{9.26}$$

We can then write the trace in a form similar to the one in [15]

$$\mathrm{Tr}\mathcal{L}^t = \sum_p T_p \sum_{r=1}^{\infty} \frac{\delta(t - rT_p)}{|\det(\mathbf{1} - \mathbf{J}_p^r)|} \Delta_{p,r}, \tag{9.27}$$

with

$$\Delta_{p,r} = \sum_{\mathbf{M}^{rT_p} = \mathbf{M}} \frac{e^{\frac{1}{2}\int_0^{rT_p}(H_{pq} + H_{pp}\mathbf{M})d\tau}}{|\det(\mathbf{1} - \mathbf{J}_{\mathbf{M}}{}_p^r)|} \tag{9.28}$$

The first point in obtaining (9.28) is then to find the periodic solutions of the $\mathbf{M}$ flow. To do this we note that the invariant manifolds W^s and W^u of the periodic orbit locally defines linear subspaces E^s and E^u which are tangent to the invariant manifolds at the periodic orbit [42] Let us say that $E^s = \mathrm{span}\{\mathbf{s}_1, \mathbf{s}_2, \ldots \mathbf{s}_N\}$ and $E^u = \mathrm{span}\{\mathbf{u}_1, \mathbf{u}_2, \ldots \mathbf{u}_N\}$. Now let us look at the mixed subspace $L = \mathrm{span}\{\mathbf{e}_1, \mathbf{e}_2, \ldots \mathbf{e}_N\}$ where the $\mathbf{e}_i$'s are taken from the union $\{\mathbf{s}_1, \mathbf{s}_2, \ldots \mathbf{s}_N, \mathbf{u}_1, \mathbf{u}_2, \ldots \mathbf{u}_N\}$. This space has the right dimension for being a Lagrangian manifold, and taking the symplectic form on two of the eigenvectors and evolving the flow for one period of the orbit we get

$$\begin{aligned} \omega(f^t(\delta\mathbf{e}_i), f^t(\delta\mathbf{e}_j)) &= \lambda_i \lambda_j \omega(\delta\mathbf{e}_i, \delta\mathbf{e}_j) \\ &= \omega(\delta\mathbf{e}_i, \delta\mathbf{e}_j) \end{aligned} \tag{9.29}$$

where the last equation follows since the symplectic form is conserved by the flow. From this it follows that the symplectic form actually vanishes on any set of eigenvectors where the product $\lambda_i \lambda_j$ is different from unity. This implies that a mixed subspace choosen in this way is actually a Lagrangian manifold. If we now let the flow evolve such a manifold for one period, we get for points on the Lagrangian manifold close to the periodic point

$$\begin{aligned} \mathcal{F}^t(x^* + \delta a_1 \mathbf{e}_1 + \ldots + \delta a_N \mathbf{e}_N) &= x^* + \mathbf{J}_p(\delta a_1 \mathbf{e}_1 + \ldots + \delta a_N \mathbf{e}_N) \\ &= x^* + \Lambda_1 \delta a_1 \mathbf{e}_1 + \ldots + \Lambda_N \delta a_N \mathbf{e}_N \\ &\in x^* + L \end{aligned} \tag{9.30}$$

so that an N-dimensional superposition of the linear subspaces is also locally invariant and hence leads to a periodic solution of the curvature flow.

If we are only looking at Hamiltonian flows where the eigenvalues of the Jacobian are non degenerate, we see that for every eigenvector of the Jacobian the symplectic form on all other eigenvectors except the one with eigenvalue Λ^{-1} must vanish due to the above argument. However, the symplectic form on an eigenvector with eigenvalue Λ and its adjoint with eigenvalue Λ^{-1} does not vanish. To see this we can consider the simple case where the phase space is 4-dimensional. Let us denote the four eigenvectors $\delta\mathbf{e}_1, \ldots, \delta\mathbf{e}_4$ with corresponding eigenvalues $\Lambda_1, \Lambda_2, \Lambda_1^{-1}, \Lambda_2^{-1}$. We can now choose a vector $\mathbf{x}$ so

$$\omega(\delta\mathbf{e}_1, \mathbf{x}) \quad \neq \quad 0 \tag{9.31}$$

Expanding $\mathbf{x}$ on the four eigenvectors and applying the linearity of ω then yields

$$\omega(\delta\mathbf{e}_1, \delta\mathbf{e}_3) \quad \neq \quad 0 \tag{9.32}$$

since all the other terms vanish due to the above argument. This means that in the non degenrate case where one can uniquely define the adjoint vector corresponding to an eigenvector by virtue of having the inverse eigenvalue, such two adjoint vectors can never lie in the same Lagrangian manifold. In this case the number of Lagrangian manifolds that are periodic solutions to our extended flow can therefore easily be counted. A given Lagrangian manifold is expanded by N eigenvectors of the Jacobian which each can be either an unstable or the adjoint stable eigenvector. This gives 2^N different possibilities and hence there are 2^N periodic solutions of the curvature flow.

By the above considerations we have now found all the periodic solutions of the curvature flow as the 2^N possible null-manifolds spanned by the eigenvectors of the Jacobian. The task is therefore now to find the eigenvalues of $\mathbf{J_M}$ as a function of the usual cycle stabilities.

2-dimensional flows

In the case of a 2-dimesional Hamiltonian flow reduced to a 2-dimensional Poincare section return map one can use the rational fraction transformation technique of the Sinai Bunimovich curvature to get the variation of $\mathbf{M}$ as illustrated in [15]. Since this procedure is restricted to two dimensions we will here show a procedure to get the $\mathbf{M}$ stabilities which can be generalised to higher dimensions. The idea is simply the following. Suppose that we have already found the periodic solutions of the $\mathbf{M}$ flow. Then we make a small variation $\delta\mathbf{M}$ of the periodic solution $\mathbf{M}_0$. This gives us a new manifold according to (9.17). Using the *usual* Jacobian we then evolve as many points on this new manifold as it takes to span it (N) for a period of time T_p and then from the evolved points we construct the linearization of the evolved manifold. From this manifold we then get the evolved curvature matrix $\mathbf{M}^t$ which is of the form

$$(\mathbf{M}_0 + \delta\mathbf{M})^t \quad = \quad \mathbf{M}_0 + \mathbf{J_M}\delta\mathbf{M} + \mathcal{O}(\delta\mathbf{M}^2) \tag{9.33}$$

where now $\mathbf{J_M}$ should be expressed in terms of the cycle stabilities.

To see how this works we first consider a simple two-dimensional example where the result is known [15]. In two dimensions we can reduce the problem to a 2-dimensional Poincaré surface of section mapping which will then have an unstable $\mathbf{u} = (u_1, u_2)$ direction of stability Λ and a stable direction $\mathbf{s} = (s_1, s_2)$ of stability Λ^{-1}. If the system is hyperbolic we have $\Lambda > 1$. In the usual Cartesian coordinates the full Jacobian can be found by the set of equations

$$\begin{aligned} \mathbf{Ju} &= \Lambda \mathbf{u} \\ \mathbf{Js} &= \Lambda^{-1} \mathbf{s} \end{aligned}$$

This gives the Jacobian

$$\mathbf{J} = (s_1 u_2 - s_2 u_1)^{-1} \begin{pmatrix} -(\Lambda u_1 s_2 - \frac{s_1 u_2}{\Lambda}) & (\Lambda - \frac{1}{\Lambda}) u_1 s_1 \\ -(\Lambda - \frac{1}{\Lambda}) u_2 s_2 & (\Lambda u_2 s_1 - \frac{s_2 u_1}{\Lambda}) \end{pmatrix}$$

The linearized stable and unstable manifolds which are both simply lines on the Poincare section see fig 9.1 are characterized by their slopes $m_s = s_2/s_1$

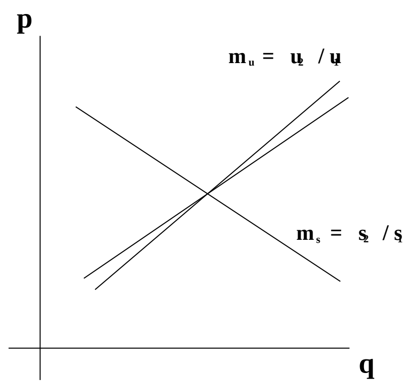

Figure 9.1: The stable and unstable manifolds in the Poincare section. A variation of the unstable manifold is shown.

and $m_u = u_2/u_1$. We now make a small variation δm_s of m_s. This corresponds to points on the line $(x, p) = (x, (m_s + \delta m_s)x)$. These points are mapped by the Jacobian into a line of slope

$$(m_s + \delta m_s)' = \frac{J_{21} + J_{22}(m_s + \delta m_s)}{J_{11} + J_{12}(m_s + \delta m_s)}. \tag{9.34}$$

According to (9.33) we expect the result to be of the form

$$(m_s + \delta m_s)' = m_s + \lambda \delta m_s + \mathcal{O}(\delta m_s^2) \tag{9.35}$$

we therefore expand the denominator to the first order in δm_s

$$\begin{aligned} (m_s + \delta m_s)' &\simeq \frac{(J_{21} + J_{22} m_s)(1 + \frac{J_{22}\delta m_s}{J_{21} + J_{22} m_s})(1 - \frac{J_{12}\delta m_s}{J_{11} + J_{12} m_s})}{J_{11} + J_{12} m_s} \\ &= \frac{J_{21} + J_{22} m_s}{J_{11} + J_{12} m_s}(1 + (\frac{J_{22}}{J_{21} + J_{22} m_s} - \frac{J_{12}}{J_{11} + J_{12} m_s})\delta m_s) \end{aligned}$$

But since m_s was an invariant of the map this simply gives

$$(m_s + \delta m_s)' = m_s + \frac{J_{22} - J_{12} m_s}{J_{11} + J_{12} m_s} \delta m_s \tag{9.36}$$

Putting in the entities from the Jacobian we get

$$(m_s + \delta m_s)' = m_s + \Lambda^2 \delta m_s \tag{9.37}$$

that is $\lambda_s = \Lambda^2$ for the stable direction. For the unstable direction everything is similar and we get $\lambda_u = \Lambda^{-2}$.

The trace integration in 2 dimensions

To get the trace of the new evolution operator $\mathcal{L}$ we have to do the integral

$$\mathrm{Tr}\mathcal{L} = \int dq dp d\mathbf{M} \mathcal{L}(q, p, \mathbf{M}, t | q, p, \mathbf{M}, 0) \tag{9.38}$$

with the kernel

$$e^{i\pi\nu + \int_0^t d\tau \frac{iL}{\hbar} + \frac{1}{2}\mathrm{Tr}\left\{\frac{\partial^2 H}{\partial p \partial q} + \frac{\partial^2 H}{\partial p \partial p}\mathbf{M}\right\}} \delta(q - q^t(q,p)) \delta(p - p^t(q,p)) \delta(\mathbf{M} - \mathbf{M}^t(q, p, \mathbf{M})), \tag{9.39}$$

where $q^t(q,p)$, $p^t(q,p)$ and $\mathbf{M}^t(q,p,\mathbf{M})$ denote the evolution of q, p and $\mathbf{M}$ from the initial coordinates q,$p = \nabla S_0(q)$ and $\mathbf{M} = \partial_i \partial_j S_0(q)$ during the time t, and ν is the Maslov index. this integral is exactly of the type (9.16.) For a 2-dimensional flow we saw that the stabilities of the $\mathbf{M}$ flow could be found in terms of the stabilities of the periodic orbits. So what is left is to evaluate the integral of the trace. Assuming the Hamiltonian to be of the form $H(q,p) = p^2/2m + V(q)$, we are left with the integral

$$\exp\{\frac{1}{2}\int_0^t d\tau \mathrm{Tr}\mathbf{M}(\tau)\} = (\frac{V(q^t)}{V(q_0)})^{1/2}. \tag{9.40}$$

Now the volume ratio in (9.40) is in configuration space but is determined by the initial $\mathbf{M}$ matrix since this gives the variations in initial momenta of the δq's spanning the initial configuration space volume element. This means that the initial $\mathbf{M}$ matrix decides if we are on the stable or the unstable manifold. Therefore the volume ratio is quite simple to determine since it is simply Λ or Λ^{-1} if we are on the unstable respectively stable manifold.

In higher dimensions we are in general on some mixed stability manifold L spanned by N phase space vectors $\mathbf{e}_i$. To determine the volume ratio we select N infinitesimal vectors $\delta\mathbf{q}_1, \delta\mathbf{q}_2 \ldots \delta\mathbf{q}_N$ spanning a small parallelepiped around the periodic point in configuration space. The volume of this is $\det(\delta\mathbf{q}_1 \delta\mathbf{q}_2 \ldots \delta\mathbf{q}_N)$. We then evolve these according to the Jacobian and evaluate the volume of the projection of the image on configuration space. By choosing $\delta\mathbf{q}_i = \pi_q(\mathbf{e}_i)$ the projection of the $\mathbf{e}_i$ vector on the q-space, we can easily calculate the image of the N $\delta\mathbf{x}_i = (\delta\mathbf{q}_i, \delta\mathbf{p}_i) = \delta\mathbf{e}_i$ vectors. The volume of the evolved parallelepiped in configuration space is then given by $\det(\Lambda_1\pi_q\mathbf{e}_1, \Lambda_2\pi_q\mathbf{e}_2 \ldots \Lambda_N\pi_q\mathbf{e}_N)$. Since multiplication of a column of a matrix by a factor Λ changes the determinant of the matrix with the same factor we get the volume ratio as the product of the stabilities of the manifold

$$\frac{V(q^t)}{V(q_0)} = \prod_{i=1}^{N} \Lambda_i \tag{9.41}$$

We now have everything we need to do the integral in 2-dimensions. The unstable and the stable manifold through the periodic orbit yields in the $\mathbf{M}$ integration respectively $|\Lambda_p|^{1/2}/(1-\Lambda_p^{-2})$ and $|\Lambda_p|^{-1/2}/(1-\Lambda_p^2)$. Putting this into (9.28) yields

$$\begin{aligned} \Delta_{p,r} &= \frac{|\Lambda_p^r|^{1/2}}{|1-\Lambda_p^{-2r}|} + \frac{|\Lambda_p^r|^{-1/2}}{|1-\Lambda_p^{2r}|} \\ &= \frac{|\Lambda_p^r|^{1/2}}{1-\Lambda_p^{-2r}} + \frac{|\Lambda_p^r|^{-5/2}}{1-\Lambda_p^{-2r}} \end{aligned} \tag{9.42}$$

which is the formula obtained in [15] for $\beta = 1/2$.

Higher dimensions

As we saw above we know how to do the trace integral in any number of dimensions, and we know what the periodic solutions of $\mathbf{M}$ are. The only thing we need now is to generalize the scheme in the above section (9.1.1). To see how it works let us consider the following simple example in 3-dimensions. Here the phase space is 6-dimensional but a Poincaré section and reduction to the energy shell reduces the problem to a 4-dimensional map. Let us further assume that the Jacobian in these coordinates is diagonal with diagonal elements: $\Lambda_1, \Lambda_2, \Lambda_1^{-1}, \Lambda_2^{-1}$. Then $\mathbf{M}_0 = \mathbf{0}$ is a periodic solution since this corresponds to a swarm of points lying in the $\mathbf{q}$-plane with zero momentum:

$$L = \{(q_1, q_2, p_1, p_2) | p_1 = p_2 = 0\}$$

We then make a small variation $\delta\mathbf{M}$ of the curvature matrix and select two vectors that spans the manifold corresponding to $\delta\mathbf{M}$. As an example we might choose:

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ \delta m_{11} \\ \delta m_{21} \end{pmatrix}, \; v_2 = \begin{pmatrix} 0 \\ 1 \\ \delta m_{12} \\ \delta m_{22} \end{pmatrix}$$

The image under the Jacobian of these two vectors is given by

$$\mathrm{span}\{\begin{pmatrix} \Lambda_1 \\ 0 \\ \delta m_{11}\Lambda_1^{-1} \\ \delta m_{21}\Lambda_2^{-1} \end{pmatrix}, \begin{pmatrix} 0 \\ \Lambda_2 \\ \delta m_{12}\Lambda_1^{-1} \\ \delta m_{22}\Lambda_2^{-1} \end{pmatrix}\}$$

from which we get the evolved curvature matrix by "division" $\mathbf{M}' = \delta\mathbf{p}'/\delta\mathbf{x}'$

$$\delta m'_{11} = \frac{\delta m_{11}}{\Lambda_1^2} \quad \delta m'_{12} = \frac{\delta m_{12}}{\Lambda_1\Lambda_2}$$

$$\delta m'_{21} = \frac{\delta m_{21}}{\Lambda_1\Lambda_2} \quad \delta m'_{22} = \frac{\delta m_{22}}{\Lambda_2^2}$$

From this we can directly get the stabilities of the periodic $\mathbf{M}$ solution corresponding to this choice of initial curvature. One should of course take care that the bonds on $\mathbf{M}$ require this to be symmetric and we should also remember to take other $\mathbf{M}$ solutions into account in the final result. The last point is not so straight forward in this example because all the other Lagrangian planes will be caustics since they have zero variation in one of the q-directions (in two dimensions this would correspond to that the invariant subspaces was the x and y axis. The x-axis is of course well described as a function of x whereas the y-axis is not given as $y = m \cdot x$!).

The general N-dimensional case

The idea of the above 2-dimensional example is good and we should try to make use of the eigenvectors of the flow. The main strategy is straight forward: we make a variation of the curvature matrix giving us a variated Lagrangian manifold. This we express in the basis of eigenvectors of the Jacobian, and then evolve it for one period by use of the cycle stabilities. Then we go back to the original curvature space and read off the eigenvalues in terms of the cycle stabilities.

This program is simple but require that we account for some details about how the δM_{ij}'s are related to the vectors spanning the variated Lagrangian manifold (these vectors are of course not eigenvectors anylonger).

To simplify the notation we start with a few definitions. Suppose we have a Lagrangian manifold: $L = \text{span}\{\mathbf{e}_1, \mathbf{e}_2, \dots \mathbf{e}_N\}$ which also can be spanned by the curvature matrix $\mathbf{M}$. This we can express by introducing the usual orthonormal basis of unit vectors in *configuration* space $\mathbf{1}_i = (0, \dots, 1, \dots, 0)$ where the '1' is on the i'th place of the only N entities. Then we also have

$$L = \text{span}\{\begin{pmatrix} \mathbf{1}_i \\ \mathbf{M}\mathbf{1}_i \end{pmatrix}\}_{i=1}^{N}$$

which also in short can be written

$$L = \text{span}\{\begin{pmatrix} \mathbf{1} \\ \mathbf{M}\mathbf{1} \end{pmatrix}\}$$

For the variated lagrangian manifold we use the notation

$$\delta L = \text{span}\{\begin{pmatrix} \mathbf{1} \\ (\mathbf{M}+\delta\mathbf{M})\mathbf{1} \end{pmatrix}\}$$

This we can also expand as before in terms of the eigenvectors but then we just have to add a small displacement vector to each of the eigenvectors to get the correct variated manifold. These displacement vectors can in turn be expanded by all the vectors we did not use to expand the original L. This reads:

$$\delta L = \text{span}\{\mathbf{e}_i + \sum_{j=N+1}^{2N} \delta\tilde{M}_{ij}\mathbf{e}_j\}_{i=1}^{N}.$$

We can also define

$$\begin{aligned} \delta\tilde{\mathbf{m}}_i &\equiv \sum_{j=N+1}^{2N} \delta\tilde{M}_{ij}\mathbf{e}_j \\ &\equiv \delta\tilde{\mathbf{M}}\mathbf{E} \end{aligned} \tag{9.43}$$

where $\mathbf{E}$ is the matrix consisting of the last N eigenvectors which does not lie in the original manifold:

$$\mathbf{E} = (\mathbf{e}_{N+1}, \mathbf{e}_{N+2}, \ldots, \mathbf{e}_{2N}). \tag{9.44}$$

Analogously we finaly define

$$\delta\mathbf{m}_i = \begin{pmatrix} \delta m_{1i} \\ \delta m_{2i} \\ \vdots \\ \delta m_{Ni} \end{pmatrix}$$

whereas

$$\delta\mathbf{M} = \begin{pmatrix} \delta m_{11} & \ldots & \delta m_{1N} \\ \vdots & & \vdots \\ \delta m_{N1} & \ldots & \delta m_{NN} \end{pmatrix}.$$

which ends our initial definitions. The geometrical interpretation of the defined vectors is indicated on figure 9.2.

Since we can interpret $\delta\mathbf{M}$ and $\delta\tilde{\mathbf{M}}$ as vectors themselves in $\mathbf{R}^{N^2}$, there exist an invertible matrix $\mathbf{P}$ such that

$$\vec{\delta}\tilde{\mathbf{M}} = \mathbf{P}\vec{\delta}\mathbf{M} \tag{9.45}$$

and

$$\vec{\delta}\mathbf{M} = \mathbf{P}^{-1}\vec{\delta}\tilde{\mathbf{M}}. \tag{9.46}$$

where the entities of the matrices are now written as single column of length N^2. This simply corresponds to a rescaling and rotation of the matrices and

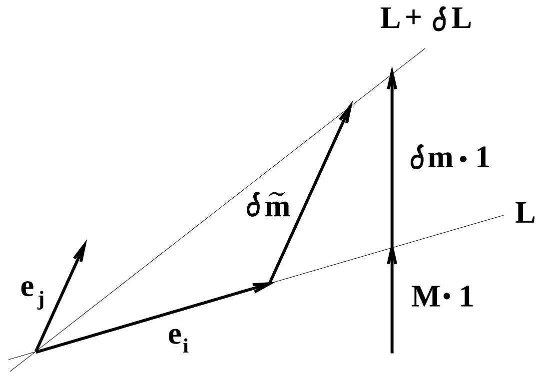

Figure 9.2: The geometrical significance of the introduced vectors. The $\delta\tilde{\mathbf{m}}_i$ vectors connects the original Lagrangian manifold L to the variated manifold along the directions not used to span L. The $\delta\mathbf{m}_i$ vectors on the contrary connects L to $L+\delta L$ in vertical direction.

is therefore obviously a welldefined transformation. We note that the map $\mathbf{P}$ only gives a valid relation between $\delta\mathbf{M}$ and $\delta\tilde{\mathbf{M}}$ around vectors of the form

$$\delta\mathbf{e}_i + \sum_{j=N+1}^{2N} \delta\tilde{M}_{ij}\delta\mathbf{e}_j \tag{9.47}$$

Now we evolve the variated Lagrangian manifold by applying the Jacobian on the $\mathbf{e}_i + \delta\tilde{\mathbf{m}}_i$ vectors

$$\mathbf{J}(\mathbf{e}_i + \sum_{j=N+1}^{2N} \delta\tilde{M}_{ij}\mathbf{e}_j) \;=\; \Lambda_i\mathbf{e}_i + \sum_{j=N+1}^{2N} \delta\tilde{M}_{ij}\Lambda_j\mathbf{e}_j \tag{9.48}$$

This image we can not immideately map back to the δM_{ij}'s because this map was only defined in the neighbourhood of of the $\mathbf{e}_i$ vectors. The evolved manifold can however also be described by

$$\begin{aligned} L+\delta L' &= \text{span}\{\Lambda_i\mathbf{e}_i + \sum_{j=N+1}^{2N} \delta\tilde{M}_{ij}\Lambda_j\mathbf{e}_j\}_{i=1}^{N} \\ &= \text{span}\{\mathbf{e}_i + \sum_{j=N+1}^{2N} \delta\tilde{M}_{ij}\frac{\Lambda_j}{\Lambda_i}\mathbf{e}_j\}_{i=1}^{N} \end{aligned} \tag{9.49}$$

where the latter is a description of the same manifold in a form that can be mapped directly back to the δM_{ij} description by $\mathbf{P}$. We then have that the evolved $\delta\tilde{M}_{ij}$ is given in diagonal form by

$$\delta\tilde{M}'_{ij} \;=\; \frac{\Lambda_j}{\Lambda_i}\delta\tilde{M}_{ij} \tag{9.50}$$

or

$$\vec{\delta}\tilde{\mathbf{M}}' = \left[\frac{\Lambda_j}{\Lambda_i}\right]\vec{\delta}\tilde{\mathbf{M}} \tag{9.51}$$

where we have introduced the vector form of the matrices $\vec{\delta}\tilde{\mathbf{M}} = (\delta M_{11}, \ldots, \delta M_{1N}, \ldots, \delta M_{NN})$, and where $[\Lambda_j/\Lambda_i]$ is a diagonal matrix, having the eigenvalues of the original manifold in the denominator. This implies

$$\begin{aligned} \vec{\delta}\mathbf{M}' &= \mathbf{P}^{-1}\vec{\delta}\tilde{\mathbf{M}}' \\ &= \mathbf{P}^{-1}\left[\frac{\Lambda_j}{\Lambda_i}\right]\mathbf{P}\vec{\delta}\mathbf{M} \end{aligned} \tag{9.52}$$

where we now got rid of the tilded coordinates by mapping back with $\mathbf{P}$. The eigenvalues of the $\vec{\delta}\mathbf{M}$ map is given by the zeros of the determinant

$$\begin{aligned} \det(\mathbf{P}^{-1}\left[\frac{\Lambda_j}{\Lambda_i}\right]\mathbf{P} - \lambda\mathbf{1}) &= \det(\left[\frac{\Lambda_j}{\Lambda_i}\right] - \lambda\mathbf{1}) \\ &= \prod_{i=1}^{N}\prod_{j=N+1}^{2N}(\frac{\Lambda_j}{\Lambda_i} - \lambda) \\ &= 0. \end{aligned} \tag{9.53}$$

This gives the N^2 eigenvalues of the $\vec{\delta}\mathbf{M}$ map

$$\lambda_{ij} = \frac{\Lambda_j}{\Lambda_i} \tag{9.54}$$

where the indices run like $i = 1, \ldots, N$ representing the first N eigenvectors used to span the original manifold, and $j = N+1, \ldots, 2N$ denotes the remaining N eigenvectors. We now must recall that the $\mathbf{M}$ integration takes place over the space of *symmetric* matrices, since the curvature matrix $\mathbf{M}$ is the second derivative of the action and therefore is symmetric. We therefore also only can allow for symmetric variations of the solution. That $\delta\mathbf{M}$ is symmetric implies that we only have the $N(N+1)/2$ symmetric eigenvectors $\delta\mathbf{M} = \delta_{i,j} + \delta_{j,i}$, where $i = 1, \ldots, N$ and $j = i, \ldots N$ with the stabilities

$$\lambda_i = \frac{\Lambda_{i+N}}{\Lambda_j}, \quad i = 1, \ldots, N \quad j = j, \ldots, N \tag{9.55}$$

We can now write down the general result of the $\mathbf{M}$ integration

$$\begin{aligned} \Delta_{p,r} &= \int d\mathbf{M} e^{\int_0^t d\tau \frac{1}{2}\mathrm{Tr}(H_{pq} + H_{pp}\mathbf{M})}\delta(\mathbf{M} - \mathbf{M}^t(\mathbf{M})) \\ &= \sum_{l=1}^{2^N}\prod_{i_l=1}^{N}|\Lambda_{i_l}|^{r/2}\prod_{j=i_l}^{N}|1 - \Lambda^r_{i_l+N}\Lambda^r_{i_l}|^{-1}, \end{aligned} \tag{9.56}$$

where l labels the periodic $\mathbf{M}$ solutions.

In the simple 2-dimensional case the above formula would reduce to

$$\begin{aligned}\Delta_{p,r} &= \int d\mathbf{M} e^{\int_0^t d\tau \frac{1}{2}\mathrm{Tr}\mathbf{M}} \delta(\mathbf{M} - \mathbf{M}^t(\mathbf{M})) \\ &= \sum_{l=1}^{2} \prod_{i_l=1}^{1} |\Lambda_{i_l}^{r/2}| \prod_{j=1}^{1} |1 - \Lambda_{i_l+N}^r / \Lambda_{i_l}^r|^{-1} \\ &= \frac{|\Lambda_p^r|^{1/2}}{|1 - \Lambda_p^{-2r}|} + \frac{|\Lambda_p^r|^{-1/2}}{|1 - \Lambda_p^{2r}|} \\ &= \frac{|\Lambda_p^r|^{1/2}}{1 - \Lambda_p^{-2r}} + \frac{|\Lambda_p^r|^{-5/2}}{1 - \Lambda_p^{-2r}} \end{aligned} \tag{9.57}$$

which is the previously obtained result.

Example: solution of the usual second order equation

To see how the formula (5.62) works let us try first to solve a usual second order equation. To be specific we can try

$$-x^2 + x + 2 = 0$$

that has the roots $x_1 = 2$ and $x_2 = -1$. First we should get the equation on the form (5.44) which implies $B = -1, C = -2$ and for instance $A = 1$ and $D = 0$ since we are more free to choose the two latter constants. $\mathbf{J}$ then looks like

$$\mathbf{J} = \begin{bmatrix} 1 & -1 \\ -2 & 0 \end{bmatrix}$$

To diagonalize $\mathbf{J}$ we have to solve the characteristic second order equation

$$y^2 - y - 2 = 0$$

which yields $y = 2, y = -1$. It is obvious that we have now solved the original equation already, but in the N-dimensional case we would at this point have solved an equation

$$a_0 + a_1 y + \ldots + a_{2N} y^{2N} = 0$$

in stead of dealing with $N \times N$ matrices! The eigenvectors are easily found to be

$$v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \text{ and } v_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

so that the matrix that diagonalizes **J** is given by the inverse

$$\mathbf{T}_1^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \text{ , or } \mathbf{T}_2^{-1} = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}$$

since we are allowed to permute the columns. The linear fixpoint equation (5.58) now yields

$$\tilde{m} = 1 \cdot \tilde{m} \cdot 2^{-1} \text{ , or } \tilde{m} = 2 \cdot \tilde{m} \cdot 1^{-1}$$

which both has the solution $\tilde{m} = 0$. This finaly gives us the solutions

$$m = -T_{pp}^{-1} T_{pq}$$

which yields respectively $m = -1$ and $m = 2$.

Example: solution of a second order matrix equation

Here we try with a bit more complicated 2-dimensional example. Let the matrices be given by

$$\mathbf{A} = \begin{bmatrix} 5 & 0 \\ 0 & 2 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 0 & 2 \\ 0 & -2 \end{bmatrix}$$

and

$$\mathbf{C} = \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix}, \mathbf{D} = \begin{bmatrix} 1 & -4 \\ 1 & 6 \end{bmatrix}$$

This gives

$$\mathbf{J} = \begin{bmatrix} 5 & 0 & 0 & 2 \\ 0 & 2 & 0 & -2 \\ 0 & -1 & 1 & -4 \\ 0 & 1 & 1 & 6 \end{bmatrix}$$

which turns out to be diagonalized by

$$\mathbf{T}_1 = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & 3 \\ 1 & 1 & 1 & 4 \end{bmatrix}$$

where the rows can be permuted arbitrarily. Since the two lower rows of $\mathbf{T}$ determines the solution there will be $K_{4,2} = 6$ different solutions of the form

$$\tilde{\mathbf{M}} = -\mathbf{T}_{pp}^{-1}\mathbf{T}_{pq}$$

We note here that permutations within the rows does not change the $\mathbf{M}$ solution since

$$\begin{aligned} -(\mathbf{P}\mathbf{T}_{pp})^{-1}\mathbf{P}\mathbf{T}_{pq} &= -\mathbf{T}_{pp}^{-1}\mathbf{P}^{-1}\mathbf{P}\mathbf{T}_{pq} \\ &= \tilde{\mathbf{M}} \end{aligned} \tag{9.58}$$

For $\mathbf{T}_1$ the solution is

$$\mathbf{M}_1 = -\begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ -1 & 0 \end{bmatrix}$$

The remaining solutions are

$$\mathbf{M}_2 = \begin{bmatrix} 1 & -1 \\ -1/2 & 0 \end{bmatrix}, \mathbf{M}_3 = \begin{bmatrix} 1/3 & 1/3 \\ -1/3 & -1/3 \end{bmatrix}$$

$$\mathbf{M}_4 = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}, \mathbf{M}_5 = \begin{bmatrix} 0 & 1/2 \\ 0 & -1/2 \end{bmatrix}$$

and finaly

$$\mathbf{M}_6 = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$$

That all these are in fact solutions can easily be verified by direct substitution into (5.44).

Example: The periodic M solutions generated by a symplectic Jacobian

As a demonstration of the considerations about symplectic matrices we shall in this example diagonalize a symplectic 4 by 4 Jacobian by a symplectic rotation. This leads us to $2^2 = 4$ symmetric $\mathbf{M}$ solutions using the results obtained in section (5.3.1).

Let $\mathbf{J}$ be the Jacobian

$$\mathbf{J} = \begin{bmatrix} -13 & -11 & -8 & -2 \\ -49 & -263/6 & -76/3 & -41/3 \\ 60 & 52 & 33 & 14 \\ 87 & 155/2 & 46 & 23 \end{bmatrix}$$

The symplectic matrix $\mathbf{T}$ given by

$$\mathbf{T} = \begin{bmatrix} -8 & -6 & -4 & -2 \\ 6 & 6 & 3 & 2 \\ 1 & 3/2 & 0 & 1 \\ 3 & 4 & 1 & 2 \end{bmatrix}$$

diagonalizes $\mathbf{J}$ and we have

$$\mathbf{TJT}^{-1} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & -1/3 \end{bmatrix}$$

where we note that the eigenvalues enters in the right sequence according to (5.70) in order for the diagonalized Jacobian to be symplectic. Using eq. (5.62) we get the solutions

$$\mathbf{M}_1 = -\begin{bmatrix} 1 & 1 \\ 1 & 3/2 \end{bmatrix}$$

for the identity permutation $\tau(1234) = 1234$,

$$\mathbf{M}_2 = -\frac{1}{3}\begin{bmatrix} 5 & 2 \\ 2 & 5 \end{bmatrix}$$

with the permutation $\tau(1234) = 3214$,

$$\mathbf{M}_3 = -\begin{bmatrix} 4/3 & 1 \\ 1 & 3/2 \end{bmatrix}$$

with $\tau(1234) = 1432$, and finaly

$$\mathbf{M}_4 = -\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

with $\tau(1234) = 3412$.

So this is an example where $\mathbf{J}$ can be diagonalized by symplectic rotations in four different ways each leading to different $\mathbf{M}$ solutions of the fixpoint equations.

Example: The Hamilton Jacobi equation near a periodic orbit

As another application of our solution to the fixpoint equation of the rational fraction transformation of the curvature matrix, we can create the solution to the Hamilton Jacobi equation in the neighbourghood of a periodic orbit going through (x_0, p_0). In that case we simply have

$$S(\mathbf{x}) \quad = \quad S(\mathbf{x}_0) + \mathbf{p}_0(\mathbf{x}-\mathbf{x}_0) + (\mathbf{x}-\mathbf{x}_0)^{tr}\mathbf{M}(\mathbf{x}-\mathbf{x}_0) + \mathcal{O}(||\mathbf{x}-\mathbf{x}_0||^3) \tag{9.59}$$

As we shall see in section 7.1 this can be used to speed up the calculation of $\hbar$ corrections to the Gutzwiller-Voros zeta function.

9.2 Derivations and examples of chapter 6

The geometrical contribution to the semiclassical propagator

The expression for the geometrical part reads

$$G_{\text{geo}}(k\vec{r}, k\vec{r'}) \quad = \quad -\frac{i}{8}\int_{-\infty+i\epsilon}^{+\infty+i\epsilon} d\nu\, e^{i\nu\theta}\left(H_\nu^{(1)}(kr')H_\nu^{(2)}(kr) - H_\nu^{(1)}(kr')\frac{H_\nu^{(2)}(ka)}{H_\nu^{(1)}(ka)}H_\nu^{(1)}(kr)\right)$$

We first discuss the saddle point approximation of the first term. The saddle-point condition here is

$$\theta + \arccos\frac{\nu_{S1}}{kr} - \arccos\frac{\nu_{S1}}{kr'} = 0 \tag{9.60}$$

The geometrical interpretation of this is that the two points $\vec{r}$ and $\vec{r'}$ should be on the same side on the line joining $\vec{r'}$ with $\vec{r}$ as measured from the point of closest approach to the center of the disk (see fig. 9.3). If $\vec{r'}$ and $\vec{r}$ are on

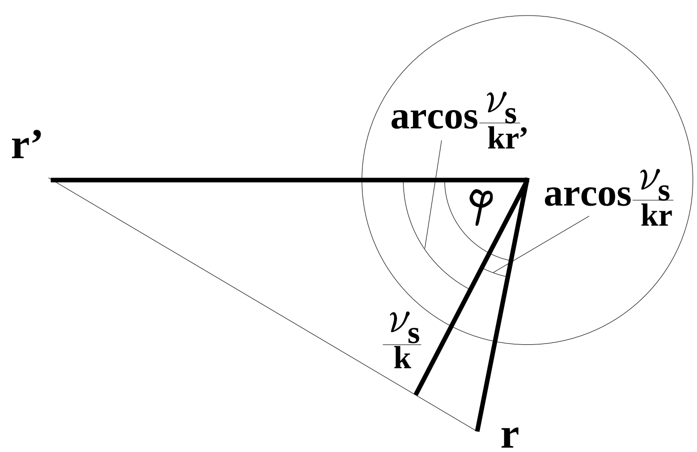

Figure 9.3: the line joining $\vec{r'}$ with $\vec{r}$. The point of closest approach to the center of the disk is indicated.

opposite sides with respect to this point the first summand in (9.60) wil have a vanishing contribution in the semiclassical limit. So in case there is a real saddle the contribution of the first summand in (6.33) reads

$$-\frac{i}{8}\int_{-\infty+i\epsilon}^{+\infty+i\epsilon} d\nu\, e^{i\nu\theta} H_\nu^{(1)}(kr')H_\nu^{(2)}(kr) \sim -\frac{i}{8}\frac{2}{\pi}\frac{e^{i\sqrt{k^2r'^2-\nu_{S1}^2}-i\sqrt{k^2r^2-\nu_{S1}^2}}}{(k^2r'^2-\nu_{S1}^2)^{\frac{1}{4}}(k^2r^2-\nu_{S1}^2)^{\frac{1}{4}}}\times$$
$$\times\int d\tilde{\nu}\, e^{-i\frac{1}{2}(\tilde{\nu}-\nu_{S1})^2\left(\frac{1}{\sqrt{k^2r^2-\nu_{S1}^2}}-\frac{1}{\sqrt{k^2r'^2-\nu_{S1}^2}}\right)}$$
$$\sim -\frac{i}{4}\sqrt{\frac{2}{\pi}}\frac{e^{iL_{\text{direct1}}-i\pi/4}}{\sqrt{kL_{\text{direct1}}}} \tag{9.61}$$

where $L_{\text{direct1}} = \sqrt{(kr')^2-\nu_{S1}^2} - \sqrt{(kr)^2-\nu_{S1}^2} = |\vec{r'} - \vec{r}|$ is the geometrical distance between $\vec{r'}$ and $\vec{r}$. Note that the result (9.61) is exactly what we get if we insert the semiclassical Debye approximation is the expression (6.21) for the free propagator, as it also should in the semiclassical limit $kr \gg 1$.

The evaluation of the second summand in (9.60) is slightly more tedious because it involves the ratio of the two Hankel functions $H_\nu^{(2)}(ka)/H_\nu^{(1)}(ka)$. For the first summand we deformed the integration path in order to pick up a saddlepoint at the real ν axis if it existed, and otherwise we received a vansihing contribution which we then neglected. In the evaluation of the second summand we shall now see that we are guarantied the existence of a real saddle $\nu_R < ka$ and in addition a saddle $\nu_{S2} > ka$ if the saddle in the first summand was nonexisting. In order to use this the path is deformed so that starting from the left-upper asymptotic region in the ν plane it goes via the saddle $\nu_R < ka$ on the real ν axis to the first zero of the Hankel function $H_\nu^{(2)}(ka)$ in the lower complex ν plane. Secondly, from there the path passes to the right asymptotic region which lies (depending on the existence of the other saddle ν_{S2} on the real axis) in the upper or lower part of the ν plane. We first discuss the contribution from the first saddle $\nu_R < ka$ after the Debye asymptotic condition has been inserted for *all* Hankel functions appearing in the second geometrical summand in (9.60). The saddle is determined from the condition

$$\theta + 2\arccos\frac{\nu_R}{ka} - \arccos\frac{\nu_R}{kr'} - \arccos\frac{\nu_R}{kr} = 0 \tag{9.62}$$

which is guaranteed to exist and to be real-valued when $\vec{r}$ lies in the light region of $\vec{r'}$ and when $r' > r > a$. The existence of the saddle becomes obvious when we note the geometrical significance of the saddle point condition. We see that this corresponds to the geometrical reflection off the disk of the ray from $\vec{r}$ to $\vec{r'}$ see fig(9.4). The semiclassical result for the second geometrical summand in (9.60) therefore reads

$$+\frac{i}{4}\sqrt{\frac{2}{\pi}}\frac{e^{ikL_{\text{refl}}-i\pi/4}}{\sqrt{kR_{\text{eff}}}} \tag{9.63}$$

where where

$$\begin{aligned} L_{\text{refl}} &= d' + d \\ R_{\text{eff}} &= d' + d + \frac{2dd'}{\sqrt{a^2-b^2}} \end{aligned} \tag{9.64}$$

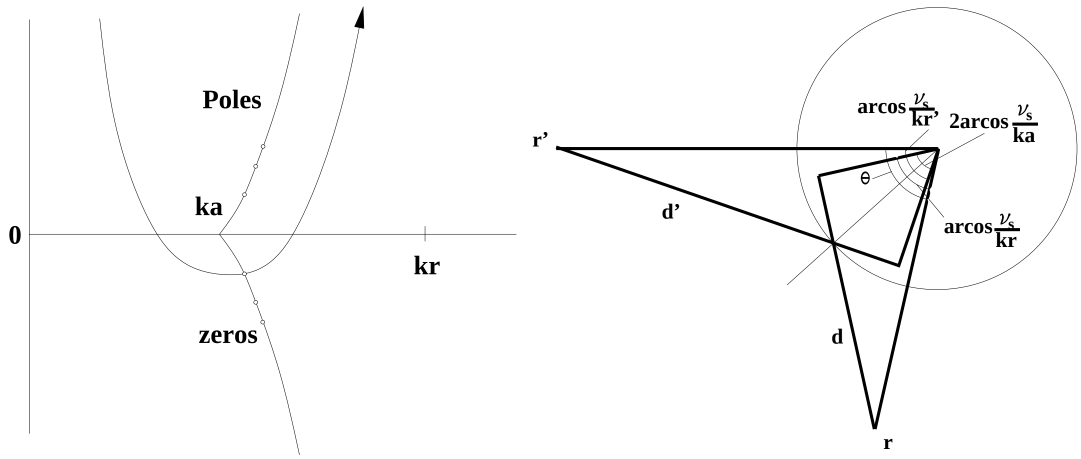

Figure 9.4: The reflection condition leads to the saddle point $\nu_R < ka$.

$$= \quad d' + d + \frac{2dd'}{a\cos\varphi} \tag{9.65}$$

with

$$\begin{aligned} d' &\equiv \sqrt{r'^2 - b^2} - \sqrt{a^2 - b^2} \\ d &\equiv \sqrt{r^2 - b^2} - \sqrt{a^2 - b^2} \\ b &\equiv a|\sin\varphi| \ , \end{aligned}$$

where φ is the angle of incidence measured with repect to the normal at the point of reflection.

For $\nu > ka$ there might exist yet another saddle if the condition

$$\theta - \arccos\frac{\nu_{S2}}{kr'} - \arccos\frac{\nu_{S2}}{kr} = 0 \tag{9.66}$$

can be met for a real ν_{S2}. In the limit $\nu \gg ka$ the ratio of the two a-dependent Hankel function becomes $H_\nu^{(2)}(ka)/H_\nu^{(1)}(ka) \approx -1$. and the remaining evaluation of the integral with respect to the saddle ν_{S2} follows completely the one presented for the geometrical saddle ν_{S1} with the result

$$-\frac{i}{4}\sqrt{\frac{2}{\pi}}\frac{e^{iL_{\mathrm{direct2}}-i\pi/4}}{\sqrt{kL_{\mathrm{direct2}}}} \tag{9.67}$$

where $L_{\mathrm{direct2}} = \sqrt{(kr')^2 - \nu_{S2}^2} + \sqrt{(kr)^2 - \nu_{S2}^2}$ is the distance $|\vec{r}' - \vec{r}|$ of the direct geometrical path between $\vec{r}'$ and $\vec{r}$ in this situation. The only difference to the above discussed case of the first geometrical summand in (6.33) is that here $\vec{r}$ and $\vec{r}'$ lie on opposite sides on the line connecting these points with respect to the line's point of closest approach to the origin. This case and the case of the first summand therefore exclude each other and we get no double counting: either the direct geometrical semiclassical propagator comes from the first summand in (6.33) or from the second saddle of the second summand, but not from both. Finally as we assumed $r' > r$ there is only the special situation left where $\vec{r}$ coincides with the point of closest approach to the origin of the line joining $\vec{r}'$ are and $\vec{r}$. Since this is just a single point which depends on the choice of the origin of the coordinate system and which can be approached from both sides ($\vec{r}$ on the same or opposite side as $\vec{r}'$) with the same result, we do not have to deal further with this problem.

The stability-cumulant expansion relation for the 2-disk system

The stability reads

$$\Lambda = \frac{R - a + \sqrt{R^2 - 2Ra}}{a} \tag{9.68}$$

The identity that should be prooven is

$$\begin{aligned} \frac{1}{2}\sqrt{\frac{a}{2R}}\left(1 + \frac{1}{\sqrt{1 - 2a/R}}\right) &= \frac{1}{\sqrt{\Lambda}(1 - \Lambda^{-2})} \\ &\equiv S \end{aligned} \tag{9.69}$$

We first note that

$$S = \frac{1}{2}\left(\frac{1}{\sqrt{\Lambda} + \frac{1}{\Lambda}} + \frac{1}{\sqrt{\Lambda} - \frac{1}{\Lambda}}\right) \tag{9.70}$$

and therefore calculate

$$\frac{1}{\sqrt{\Lambda} + \frac{1}{\Lambda}} = \frac{R + \sqrt{R^2 - 2aR}}{\sqrt{a(R - a + \sqrt{R^2 - 2aR})}} \tag{9.71}$$

and

$$\frac{1}{\sqrt{\Lambda} - \frac{1}{\Lambda}} = \frac{R - 2a + \sqrt{R^2 - 2aR}}{\sqrt{a(R - a + \sqrt{R^2 - 2aR})}} \tag{9.72}$$

and the sum S is therefore reduced to

$$S = \frac{\sqrt{a(R - a + \sqrt{R^2 - 2aR})}(R - a + \sqrt{R^2 - 2aR}}{2(R^2 - 2aR + (R - a)\sqrt{R^2 - 2aR}} \tag{9.73}$$

pulling out hte square root we get

$$\begin{aligned} S &= \frac{1}{2}\frac{\sqrt{a(R - a + \sqrt{R^2 - 2aR}}(R - a + \sqrt{R^2 - 2aR})}{\sqrt{R^2 - 2aR}(\sqrt{R^2 - 2aR} + R - a)} \\ &= \frac{1}{2}\sqrt{\frac{a(R - a + \sqrt{R^2 - 2aR})}{R^2 - 2aR}} \end{aligned} \tag{9.74}$$

Now we are in a position where we can easily check if the two expressions coincides. This would imply

$$\sqrt{\frac{R - a + \sqrt{R^2 - 2aR}}{R^2 - 2aR}} = \frac{1}{\sqrt{2R}}\left(1 + \frac{1}{\sqrt{1 - 2a/R}}\right) \tag{9.75}$$

By squaring both sides we obtain

$$\frac{2R(R-a+\sqrt{R^2-2aR})}{R^2-2aR} = 1+\frac{1}{1-2a/R}+\frac{2}{\sqrt{1-2a/R}} \tag{9.76}$$

or

$$\frac{R^2-Ra+R\sqrt{R^2-2aR}}{R^2-2aR} = \frac{R^2-Ra+R\sqrt{R^2-2aR}}{R^2-2aR} \tag{9.77}$$

which proves the identity since both sides were positive from the beginning.

9.3 A program that calculates $C_l^{p(1)}$

In this section we present the FORTRAN code that calculaties the first order $\hbar$ correction in the case of two-dimensional billard systems. In this version the program finds the correction contribution associated with a single periodic orbit, in the full 3-disk domain. The program is easily generalized to arbitrary 2-dimensinal billard systems, by specifying the length segments (Li) and the bouncing angles (angles) of the periodic orbit, together with the expansion coefficients of the billard wall (C2,C3 and C4) at the bouncing points. The program first solves locally the Hamilton Jacobi equation in terms of the expansion coefficients $S_y(t), S_{y^2}(t), S_{y^3}(t), S_{y^4}(t)$. These solutions are then used to drive the amplitude equation (7.31) which is then finaly used to calcuate the correction term by means of the integral (7.128),(7.135).

```
program hbarcorrection
implicit none
integer Npmax,Norbit       !Max number of flight segments,
                                   !number of orbits.
parameter(Npmax=40,Norbit=1)
* Note all the arrays can only handle Npmax orbit segments!!

real*8 Li(Norbit,Npmax)    ! length(orbit,segment)
real*8 angles(Norbit,Npmax)! bounceangle(orbit,teta_i)

integer i,j,iteration,Np   !counting dummies, iteration =
                                   !numbers of iterations of the P.O.
                                   !N_p =
                                   !topological length of orbit.

real*8 teta,A,B,C,D,E,l    !Current bouncing angle, integration
                                   !constants and index of
                                   !the wave funciton.
```

```
real*8 C2,C3,C4            !Expansion coefficients of the
                                   !circle.

real*8 Syy,Syyy,Syyyy      !Discontinuous evolution variables
real*8 A0,Ay,Ayy,Ayplus    !of phase and amplitude.
real*8 Sy2minus(Npmax),Sy3minus(Npmax),Sy4minus(Npmax)
real*8 Sy2plus(Npmax),Sy3plus(Npmax),Sy4plus(Npmax)

real*8 A0array(Npmax)
real*8 Ayarray(Npmax)      !Arrays for the amplitude coefficients.

real*8 Aarray(Npmax)       !Arrays for the integration
real*8 Barray(Npmax)       !constants.
real*8 Carray(Npmax),Darray(Npmax),Earray(Npmax),Int

* Arrays for the value of the phase coefficients just before and
* after a bounce:
real*8 Sy2minus(Npmax),Sy3minus(Npmax),Sy4minus(Npmax)
real*8 Sy2plus(Npmax),Sy3plus(Npmax),Sy4plus(Npmax)

real*8 t,t0,t0array(Npmax) !Time variables, array for t0 constants.
  real*8 lambda,pi           !Stability of the orbit and pi.
parameter(pi = 3.14159265358979323846264338327950d0 )

* Internal functions:
real*8 Sy2,Sy3,Sy4         !Continuous time evolution functions
        real*8 Afct,Ayfct,Ayyfct   !of phase and amplitude
real*8 Ayplusfct           !Function that gives Ay right after bounce.
real*8 IntegralDt,Int      !The integral and its accumulated
                                   !value.

* External functions:
real*8 SmxyyypSpxyyy,SmxxyymSpxxyy,SmyyymSpyyy
real*8 SmxyymSpxyy,SmxyypSpxyy,SmyymSpyy,SmyypSpyy
*Above functions gives: Sm(inus)... (p(lus)/m(inus)) Sp(lus)....

real*8 segn                !gives the sign: segn(l) = (-1)^l.



* The evolution functions:

Sy2(t,t0) = 1.0d0/(t + t0)
  Sy3(A,t,t0) = A/((t + t0)**3.0d0)
Sy4(A,B,t,t0) = B/(t + t0)**4.0d0 - 3.0d0/(t + t0)**3.0d0
     &                   + 3.0d0*A*A/(t + t0)**5.0d0
```

```
Afct(E,t0,t,l) = E*dexp((l+0.5d0)*dlog( t0/(t + t0)))
Ayfct(A,C,E,t0,t,l) = (E/dexp((l+1.5d0)*dlog(t + t0)))
     &   *(C + (l+1.d0)*(l+1.d0)*(A/2.0d0)*(t0**(l+0.5d0))/(t+t0))
Ayyfct(A,B,C,D,E,t0,t,l) = (E/dexp((l+2.5d0)*dlog(t + t0)))
     &   *(D
     &   + ( ((l+2.0d0)**2.0d0)*A*C/2.0d0
     & +(l*l+3.d0*l+2.0d0)*(1.5d0+l)*(B/6.0d0)*(t0**(l+0.5d0)))
     &   /(t+t0)
     &   + ((l+2.0d0)*(l+1.0d0)*(0.5d0*(l+2.0d0)
     &   *(l+1.0d0)+l+1.5d0)
     &   *(A*A/4.0d0)*(t0**(l+0.5d0)))/((t + t0)**2.0d0) )

        Ayplusfct(A0,Ay,C2,teta,l) =  -segn(l)*(Ay -
     &                      C2*(dsin(teta)/dcos(teta)**2.0d0)*A0
     &    *(l+1.0d0)*(l+1.0d0) )

* The integral:
IntegralDt(A,B,C,D,t0,t,l) =
     &  ( (l+0.5d0)*(l+1.5d0)
     &    + D/(t0**(l+0.5d0)))*t/(t0*(t + t0))
     & +( ((l+2.d0)**2.d0)*(A*C/4.d0)/(t0**(l+0.5d0))+
     &    + (l+2.d0)*(l+1.d0)*(1.5d0+l)*B/12.0d0 )
     &   *(t*(t + 2.0d0*t0)/(t0*(t + t0))**2.0d0)
     &   +(l+2.d0)*(l+1.0d0)
     &   *((l*l+3.d0*l+2.0d0)*0.5d0+l+1.5d0)*(A*A/12.d0)
     &   *(1.0d0/t0**3.0d0 - 1.0d0/(t + t0)**3.0d0)

**********************************************************************
* Main program:
**********************************************************************
* initialisations :

5       continue
iteration = 0
write(*,*) 'l='
read(*,*) l

C2 = 1.0d0  !
C3 = 0.0d0  ! Constants for the circle
C4 = 3.0d0  !

t=0

enddo


* Orbit input: Np = number of flight segments,Li = length of flight
```

```
* segments, angles = bouncing angles.
******************************************************************
  open(unit = 13, file = 'orbit.segm', status = 'old')

  read(13,*) Np

  do i = 1, Np
  read(13,*) Li(1,i)
  read(13,*) angles(1,i)
  enddo
  Li(1,Np+1) = Li(1,1)         ! To ensure periodic
  angles(1,Np+1) = angles(1,1) ! boundary conditions
******************************************************************

Syy  =  5.0d0     ! First the phase
Syyy  = 1.0d0     ! coefficients are initialized
        Syyyy = -1.0d0    ! arbitrarily.

10      continue
*
*  Here is the orbit loop for solving the HJ equation around the P.O.
*
*First calculating the constants t0,A and B:

do j=1,Np

t=0.0d0

t0 = 1.0d0/Syy
A = Syyy*(t0**3.0d0)
B = (Syyyy + 3.0d0/t0**3.0d0
     &                 - 3.0d0*A*A/t0**5.0d0)*(t0**4.0d0)


        Aarray(j) = A
        Barray(j) = B
        t0array(j) = t0


* Then evolving the phase coefficients in continuous time:

t = Li(1,j)   !flight time = length of j'th orbit segment.

Syy = Sy2(t,t0)
Syyy = Sy3(A,t,t0)
Syyyy = Sy4(A,B,t,t0)
```

```
Sy2minus(j+1) = Syy    !
Sy3minus(j+1) = Syyy   !minus indicates the value
Sy4minus(j+1) = Syyyy  !just before next bounce.

* Then the discontinuous jumps at the bouncing points:

teta = angles(1,j+1)    !teta = the current bouncing angle.

Syyyy = Syyyy
     &      - 4.0d0*dsin(teta)/dcos(teta)
     &                     *SmxyyypSpxyyy(Syyy,Syy,C2,C3,teta)
     &      + 6.0d0*(dsin(teta)/dcos(teta))**2.0d0
     &                     *SmxxyymSpxxyy(Syy,C2,teta)
     &      + 6.0d0*dsin(teta)/dcos(teta)**2.0d0*C2
     &                     *SmyyymSpyyy(Syyy,Syy,C2,C3,teta)
     &     + 12.0d0*(1.0d0/(2.0d0*dcos(teta)**1.0d0)
     &       - dsin(teta)**2.0d0/dcos(teta)**3.0d0)*C2
     &                     *SmxyypSpxyy(Syy,C2,teta)
     &      +3.0d0*dsin(teta)**2.0d0/dcos(teta)**4.0d0*C2**2.0d0
     &                     *SmyymSpyy(Syy,C2,teta)
     &      -4.0d0*dsin(teta)/dcos(teta)**3.0d0*C3
     &                     *SmyypSpyy(Syy,C2,teta)
     &      + 2.0d0*C4/dcos(teta)**3.0d0

        Syyy = -Syyy + 2.0d0*C3/dcos(teta)**2.0d0
     &        - 3.0d0*dsin(teta)/dcos(teta)**2.0d0*C2
     &                                  *SmyypSpyy(Syy,C2,teta)
     &        + 3.0d0*dsin(teta)/dcos(teta)
     &                                *SmxyymSpxyy(Syy,C2,teta)

Syy = Syy + 2.0d0*C2/dcos(teta)

Sy2plus(j+1) = Syy
Sy3plus(j+1) = Syyy
Sy4plus(j+1) = Syyyy

enddo
* End of orbit loop for the HJ equation. Then to ensure periodicity:

        Sy2plus(1) = Sy2plus(Np+1)
        Sy3plus(1) = Sy3plus(Np+1)
        Sy4plus(1) = Sy4plus(Np+1)
!to be able to read Sy(1) = Sy(Np+1)
Sy2minus(1) = Sy2minus(Np+1)
Sy3minus(1) = Sy3minus(Np+1)
Sy4minus(1) = Sy4minus(Np+1)
```

```
iteration = iteration + 1
if(iteration.ge. 25) then
goto 100     ! At least in the 3-disk case it turns
else                 ! out that 25 iterations is even more than
goto 10      ! sufficient.
endif

*This ends the solution of the HJ equation.

100     continue

* Now for the evolution of the amplitudes:


        iteration = 0
        A0 = 1.0d0            ! A0 is set to 1, by normalization conditions.
        Ay = 3.0d0            ! Ay and Ayy are initialized arbitrarily.
        Ayy = 5.0d0


110     continue

* start of orbit loop:
do j = 1,Np

A0array(j) = A0

t0 = t0array(j)
        A = Aarray(j)
        B = Barray(j)

        teta = angles(1,j+1)    !teta = the current bouncing angle.
t = Li(1,j)             !t = current flight time = length of curent
!orbit segment.

* First the calculation of the integration constants.
        E = A0
        C =(t0**(l+1.5d0))*Ay/E-(A/2.0d0)
     &    *(t0**(l-0.5d0))*((l+1.0d0)**2.0d0)
        D = dexp((l+2.5d0)*dlog(t0))*Ayy/E
     &     -((l+2.0d0)**2.d0)*A*C/(2.d0*t0)
     &    -B*(l+2.d0)*(l+1.d0)*(l/3.d0+0.5d0)*(t0**(l-0.5d0))/2.0d0
     &  - ((((l+2.d0)*(l+1.d0))**2.d0)/8.d0
     &      +(l+2.d0)*(l+1.d0)*(l+1.5d0)/4.d0 )
```

```
     &    *A*A*dexp((l-1.5d0)*dlog(t0) )

Earray(j) = E
Carray(j) = C
Darray(j) = D


* then the continous evoln. of the amplitudes:

        A0 = Afct(E,t0,t,l)
        Ay = Ayfct(A,C,E,t0,t,l)
Ayarray(j+1) = Ay
        Ayy = Ayyfct(A,B,C,D,E,t0,t,l)

* then finaly the jumps for the amplitudes:


Ayplus = Ayplusfct(A0,Ay,C2,teta,l)

Ayy = segn(l)*Ayy + dtan(teta)*
     &              ((segn(l)*Sy2minus(j+1)*Ay + Sy2plus(j+1)*Ayplus)
     &            *(l+2.d0)*(l+1.5d0)
     &                    + segn(l)*(Sy3minus(j+1) + Sy3plus(j+1))*A0
     &         *(l+2.0d0)*(l+1.0d0)*(l+1.0d0)/2.0d0)
     &            + dtan(teta)**2.0d0*A0
     &     *(l+1.0d0)*(l+2.0d0)*(l+0.5d0)*(l+1.5d0)/2.0d0
     &     *segn(l)*(Sy2minus(j+1)**2.0d0 - Sy2plus(j+1)**2.0d0 )
     &      + C2*dsin(teta)/dcos(teta)**2.0d0*(segn(l)*Ay - Ayplus)
     &     *(l+2.0d0)/2.0d0
     &            - C2/dcos(teta)*0.5d0*A0
     &            *segn(l)*(Sy2minus(j+1) + Sy2plus(j+1) )
     &            *(l+2.0d0)*(l+1.0d0)*(l+0.5d0)
     &      +segn(l)*2.0d0*l*(l+1.0d0)*(l+2.0d0)*A0
     &       *(C2*C2*l*dtan(teta)**2.d0/4.0d0 -
     &       C3*dtan(teta)/6.0d0)/dcos(teta)**2.0d0
     &   +l*(l+2.d0)*(l+1.0d0)*(l+0.5d0)*A0*C2*segn(l)*Sy2minus(j+1)
     &       *dtan(teta)**2.d0/dcos(teta)
     &      + segn(l)*l*(l+2.d0)*C2*Ay*dtan(teta)/dcos(teta)


Ay = Ayplus

        A0 = segn(l)*A0          ! just for completenes

enddo
*End of the individual orbit loop
*For periodicity we redefine:
```

```
        A0array(Np+1) = A0
Ayarray(1) = Ayarray(Np+1)

*And after running through the orbit we renormalize the A coeffs.:

  Ayy = Ayy/A0
  Ay = Ay/A0
  A0 = A0/A0

*(Lambda is just the stability of the orbit included to check the
*validity of final result)
  lambda = 1.0d0/(dabs(A0)**(1.d0/(0.5d0+l)))

  write(*,*) A0,Ay,Ayy !we check the convergence visualy.

        iteration = iteration + 1
        if(iteration.ge. 25) then
                goto 200           !Also for the amplitude
        else    !25 iterations seems to be
                goto 110    !ok in the 3-disk system.
        endif

200     continue

* Now we should have everything to do the integration to find a(1)

Int = 0.0d0
do j = 1, Np
t=Li(1,j)
teta = angles(1,j)
A0 = A0array(j+1)
         t0 = t0array(j)
         A = Aarray(j)
               B = Barray(j)
C = Carray(j)
D = Darray(j)
E = Earray(j)
write(*,*) A,B,C,D,E,t0
Int = Int + IntegralDt(A,B,C,D,t0,t,l)
enddo

write(*,*) 'C(1) = ',Int/2.0d0
        write(*,*) 'lambda ',lambda      !To be checked with correct stability
close(13)  !to ensure convergence has set in.
goto 5
end
```

```
*******************************************************
* The bouncing functions:
*******************************************************
real*8 function SmyypSpyy(Syy,C2,teta)
implicit none
real*8 Syy,C2,teta

SmyypSpyy = 2.0d0*(Syy + C2/dcos(teta))

return
end
*******************************************************
real*8 function SmyymSpyy(Syy,C2,teta)
        implicit none
        real*8 Syy,C2,teta

        Syy=Syy
SmyymSpyy = -2.0d0*C2/dcos(teta)

        return
        end
*******************************************************
real*8 function SmxyymSpxyy(Syy,C2,teta)
        implicit none
        real*8 Syy,C2,teta

SmxyymSpxyy = -Syy**2.0d0
     &            + ( Syy + 2.0d0*C2/dcos(teta))**2.0d0

        return
        end
*******************************************************
real*8 function SmxyypSpxyy(Syy,C2,teta)
implicit none
        real*8 Syy,C2,teta

 SmxyypSpxyy = -Syy**2.0d0
     &            -  ( Syy + 2.0d0*C2/dcos(teta))**2.0d0

        return
        end
*******************************************************
        real*8 function SmyyymSpyyy(Syyy,Syy,C2,C3,teta)
        implicit none
        real*8 Syyy,Syy,C2,C3,teta,Spyyy
```

```
SmyyymSpyyy = Syyy - Spyyy(Syyy,Syy,C2,C3,teta)

return
end
**********************************************************
        real*8 function SmxxyymSpxxyy(Syy,C2,teta)
        implicit none
        real*8 Syy,C2,teta


SmxxyymSpxxyy = 2.0d0*(Syy**3.0d0
     &                   - ( Syy + 2.0d0*C2/dcos(teta))**3.0d0)

return
end
**********************************************************
real*8 function SmxyyypSpxyyy(Syyy,Syy,C2,C3,teta)
        implicit none
        real*8 Syyy,Syy,C2,C3,teta,Spyyy

SmxyyypSpxyyy = -3.0d0*(Syy*Syyy + (Syy + 2.0d0*C2/dcos(teta))*
     &                           Spyyy(Syyy,Syy,C2,C3,teta))

return
end
**********************************************************
        real*8 function Spyyy(Syyy,Syy,C2,C3,teta)
implicit none
real*8 Syyy,Syy,C2,C3,teta,SmyypSpyy,SmxyymSpxyy

        Spyyy = -Syyy + 2.0d0*C3/dcos(teta)**2.0d0
     &       - 3.0d0*dsin(teta)/dcos(teta)**2.0d0
     &                          *C2*SmyypSpyy(Syy,C2,teta)
     &       + 3.0d0*dsin(teta)/dcos(teta)*SmxyymSpxyy(Syy,C2,teta)

        return
end
**********************************************************
real*8 function segn(l)
implicit none

real*8 l,pi
parameter(pi = 3.14159265358979323846264338327950d0 )

        segn = dsin(pi/2.0d0 + l*pi)
return
```

```
end
************************************************************
```