Towards reducing continuous symmetry of baroclinic flows

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Baroclinic flows

1 Introduction

The concept of baroclinic instability is perhaps one of the harder ones to grasp in geophysical fluid mechanics. However, it is also one of the most fundamental concepts on this field, as it is the main driver for the large scale circulations in the atmosphere and important circulations in the ocean. It is by this mechanism that the atmosphere redistributes heat from low latitudes to high latitudes, that it sustains synoptic weather system and by which some oceanic eddies develop. Its importance can not be overstated.

On the other hand, one could argue that the study of baroclinic flows remains in its early stages. Although there exist a very well developed linear theory for the onset of this type of instability, there still much room to explore the nonlinear regimes. A way to approach this studies is given by recent advances in the dynamical approaches to study of turbulence [1, 2, 3]; making use of the symmetries of the system, and finding periodic orbits and fixed points, as a way to understand the manifold of this type of setups.

In this study we introduce the physics and present some of the nonlinear theory methods that might be used to analyze baroclinic flows. We will briefly mention insights from stability theory, but the emphasis would not be on them. Great papers and books have already been written about it.

The work is divided as follows. In sect. 2 we introduce the problem in a qualitative matter, hoping that this simple approach illuminates the underlying physical principles. In sect. 3 we use Navier-Stokes equation to derive the vorticity equation and explicitly expose the the baroclinic term which is the cause of this instability. In sect. 4 we introduce the QG-Equations, which are a simplified set of equations suitable to study of geophysical flows. Sect. 5 introduces some basic techniques for showing how the stability of such flows might be addressed. Finally, the remainder of this study is devoted to show how nonlinear techniques can be used to identify important properties of this types of flows.

2 Qualitative Examples

The simple example which we develop here illustrates the mechanism initiating baroclinic flows. To start with, let us ignore the effects due to earths rotation and concentrate only in inertial frames. That is, let's start not by treating baroclinic instability perse, but the process by which a fluid adjust to equilibrium given a uneven distribution in density. A great example of this is Marsigli's experiment to explain undercurrent flows in the Bosphorus river from the Mediterranean to the Black Sea (see ref. [4]), and we will begin with a gedanken experiment based on this.

Consider the situation shown in figure 1.a, where two fluids of different densities, initially separated at x_0, are suddenly allowed to interact. The situation is clearly unstable; a pressure gradient would exist at all levels, except for the surface, going from the heavier fluid to thee lighter one. This would create both subsurface and surface currents, one due to the pressure gradient in the bottom, and the other due to mass conservation in the surface. Intuitively we can imagine that the system would finally settle to a configuration where the heavier fluid would lay on the bottom and the lighter one on top. That is, a configuration where the potential energy is minimized.

Thinking about this problem in terms of surface of constant pressure and density we can understand the instability that causes this type of behavior. In the initial configuration, the isopycnals are orthogonal to the isobars (see figure 1.b), so that the mentioned pressure gradient is generated at x_0. Later, as the denser fluid starts to settle in the lower layer, this pressure gradient starts to spread out; but it will always exist as long as there is a inclination is the isopycnals. At the end, when all the transient motions are settled and equilibrium is reached, both the isobars and the isopycnals are parallel; leaving the system in a lower potential energy state. A fundamental concept can be extracted from this:

> *In the absence of rotation, and an external forcing, equilibrium of a fluid is reached when isopycnals and isobars are parallel to each other.*

If this condition is not met, transient motions would be generated to extract the excess potential energy, and leave the system in its lower energy state.

The term responsible for the instability is quantified mathematically as the curl between the density and pressure gradients, that is:

$$\nabla\rho \times \nabla p\,, \tag{1}$$

as we will show in sect. 3. Obviously, this definition agrees with the intuitively notion we just developed.

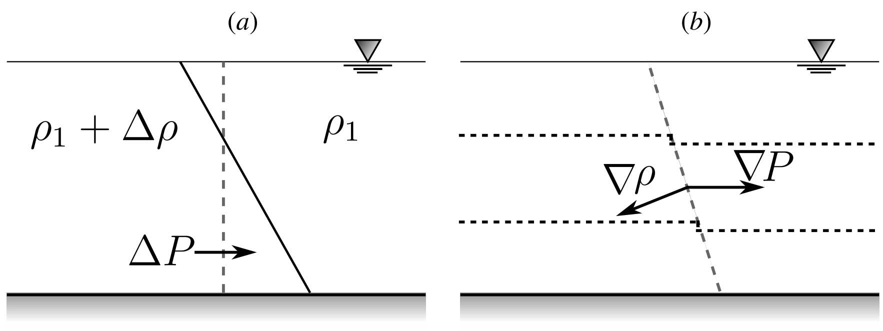

Figure 1: (a) Adjustment of a fluid subject to a horizontal difference in density in a non-rotating frame. (b) Isopycnals and isobars (dashed lines), and the pressure and density gradient for a unstable condition.

Consider now a further complication, and think about what would happen if the frame of reference were to be rotating in our previous experiment. In that case we would have to consider the apparent forces that develop on the individual fluid parcels of our flow, one proportional to the position (the centrifugal acceleration) and other proportional to the velocity (the Coriolis force) of each parcel. However, in geophysical applications the former one is often unimportant[1] [5], and only the effects of the Coriolis Force needs to be consider. Thus, if we repeat our experiment from a initial state as before, the motion of the fluid would be determined by the effects of the pressure gradient and the Coriolis force.

In this new setup, each individual parcel starts moving due to the pressure gradient force but the presence of rotation makes them deviate from a straight trajectory; as Coriolis force acts at right angles of the parcel velocity. Once these transient motions disappear, an new balanced state is reached, where the velocity becomes perpendicular to the pressure gradient force (see figure 2). This balance state is referred commonly as geostrophic balance, and it differs from the previous example in that in this new balance the pressure gradient does not vanishes, but rather is balanced by the Coriolis force. That is, from the inviscid, unforced Navier-Stokes equations one would get:

$$\mathbf{u}_g = -\frac{1}{f\rho}\mathbf{k} \times \nabla_z p \,, \tag{2}$$

where $\mathbf{u}_g$ is the horizontal component of the velocity, f is the Coriolis parameter, ρ is the density of the flow, p is the pressure and g the acceleration due to gravity. In addition, the use of the hydrostatic relation implies the following relation for the change

[1] Due to the departure from sphericity of the earth

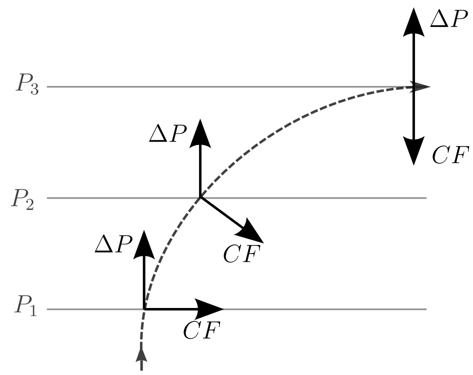

Figure 2: Process of geostrophic adjustment for an initially motionless fluid parcel.

in the vertical; which is commonly referred as the thermal wind equation [6]:

$$\begin{aligned}\frac{\partial \rho u_g}{\partial p} &= \frac{1}{f}\left(\frac{\partial \alpha}{\partial y}\right)\\ \frac{\partial \rho u_g}{\partial p} &= -\frac{1}{f}\left(\frac{\partial \alpha}{\partial x}\right),\end{aligned} \tag{3}$$

where α is the specific volume of the fluid.

From the last relations, it is concluded that there is a equilibrium state, in the presence of rotation, where the isopycnals and isobars are not necessarily parallel to each other. Nonetheless, instability occurs when the isopycnal slope exceeds a critical value; this type of phenomena is referred as baroclinic instability (see ref. [6]).

But which are the stability properties of this equilibrium, and the motions that generate afterwards?. The first of this questions has been studied widely, and it uses linear approximations to develop the relations between flow properties that need to be meet in order for it to remain stable, or become unstable. The study of the motions that develops after the initial instability is a more challenging problem as nonlinear terms now become important. The hopes are that nonlinear theory allow us to extract important knowledge about this phenomena even in high dimensional simulations.

3 Vorticity Equation

What is clear from Marsigli's experiment, is that the impact of the solenoidal term is to induce vorticity in the fluid[2]. This variable is a fundamental property of the fluid, and studying it has the power of greatly simplify the analysis of a given flow. In fact, the first satisfactory attempts to model the atmosphere where done by Charney considering the vorticity of quasigeostrophic flows.

[2]The reader is referred to ref. [7] for an introductory, yet rigorous treatment of vorticity. However, be aware that in Kundu's derivation of the vorticity equation the density is not treated as a function of both pressure and temperature, but only of pressure; so that the solenoidal term is not considered.

Vorticity arises naturally from Navier-Stokes equations, and is just a measure of the rotational speed of the fluid parcel. To make the derivation explicitly, let's consider the inviscid equations of motion in a rotating frame of reference[3], that is:

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} + 2\Omega \times \mathbf{u} = -\frac{1}{\rho}\nabla p\,. \tag{4}$$

From vector calculus we note that:

$$(\mathbf{u} \cdot \nabla)\mathbf{u} = \frac{\nabla(\mathbf{u} \cdot \mathbf{u})}{2} - \mathbf{u} \times \omega \tag{5}$$

where ω is the vorticity. So that we can rewrite Navier-Stokes equation in the following way:

$$\frac{\partial \mathbf{u}}{\partial t} + (\omega + 2\Omega) \times \mathbf{u} = -\frac{1}{\rho}\nabla p - \frac{\mathbf{u} \cdot \mathbf{u}}{2} \tag{6}$$

Taking the curl of this equation (noting that $\omega = \nabla \times \mathbf{u}$) gives:

$$\frac{\partial \omega}{\partial t} + \nabla \times ((\omega + 2\Omega) \times \mathbf{u}) = -\nabla \times \left(\frac{1}{\rho}\nabla p\right) - \nabla \times \frac{\mathbf{u} \cdot \mathbf{u}}{2} \tag{7}$$

Finally using the identity $\nabla \times (A \times B) = A(\nabla \cdot B) - B(\nabla \cdot A) + (B \cdot \nabla)A - (A \cdot \nabla)B$, the chain rule for the pressure $\frac{1}{\rho}\nabla p = \nabla(p/\rho) - p\nabla(1/\rho)$, and the fact that the curl of a gradient is zero, the desired expression for the vorticity it is obtained from (6) (see ref. [6]). That is:

$$\frac{d}{dt}\left(\frac{\omega_a}{\rho}\right) = \left[\left(\frac{\omega_a}{\rho}\right) \cdot \nabla\right]\mathbf{u} + \nabla p \times \nabla\left(\frac{1}{\rho}\right) \tag{8}$$

where $\omega_a = \omega + 2\Omega$ is the absolute vorticity. It is then evident from this equations how the baroclinic term ($\nabla p \times \nabla(1/\rho)$) can induce vorticity in the fluid by means of the instabilities previously discussed.

Equation (8) can also be written for a single layer shallow water system as (see ref. [8]):

$$\frac{d}{dt}\left(\frac{\eta + f}{h}\right) = 0\,, \tag{9}$$

where η is the vertical component of the vorticity, h the instantaneous depth of the fluid, and the quantity conserved is referred as potential vorticity ($q = (\eta + f)/h$). A generalization to two or more layers is straight forward (see refs. [8, 9]). Simplified form of this expression, filtering unimportant motions for the large scale circulations is developed next for a two layer system. However, the equations would be rich in dynamics as advection terms are retained for the horizontal (so that a lot of interesting nonlinearities remains). Up to the date there still much interesting features of this system to be studied.

[3] As explained in Section 1, only the Coriolis' force ($\Omega \times \mathbf{u}$) is considered in the equations, as the centripetal acceleration due to earths rotation can be ignored.

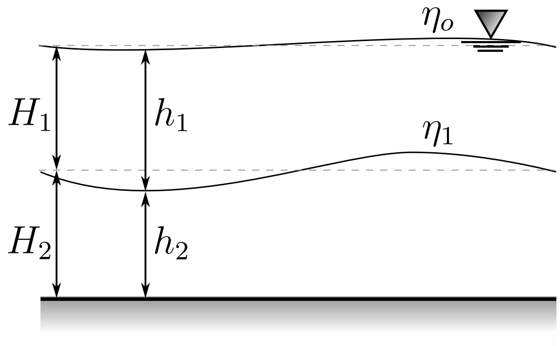

Figure 3: Two layer model considered in the simulations.

4 2 Layer QG-Equations

If we intend to study baroclinic instability, we need to be able to capture two basic properties of the flow (as shown in sect. 2): the nearly geostrophic motions of the fluid and its vertical structure as given by (3). One of the simplest mathematical systems that would capture this type of behavior ref. [9], is a quasigeostrophic set of equations discretized for two vertical layers. A quick derivation is given here[4]. For in-depth derivations the reader is referred to refs. [8, 5, 6, 9].

Consider the situation in figure 3, where we have two layer of different densities, equal depths H, and the Rossby number is small enough so that we can use the quasi-geostrophic approximation. The potential vorticity of each layer can then be written as:

$$q_i = \frac{\eta_i + f}{h_i}\,, \tag{10}$$

where η_i is the relative vorticity of each layer, f the planetary vorticity and h_i distance between layers[5]. As we learned, this property must be conserved by each layer, then:

$$\frac{dq_i}{dt} = \frac{d}{dt}\left(\frac{\eta_i + f}{h_i}\right) \simeq \frac{d}{dt}\left(f + \eta_i - f\frac{h_i'}{H_i}\right) = 0\,, \tag{11}$$

where the approximation is only valid when Rossby number is small. H_i is the unperturbed layer thickness, and h_i' is the thickness perturbation for layer i. Now let us introduce the geostrophic equations to approximate the wind and the vorticity.

First note that from (2) that the velocities only depend on the pressure gradient, and that the fluid can be regarded as incompressible if we assume that the vertical gradient of w is small. Then the pressure will depend only depend on the height of the interfaces. An it can in turn be related to h_i. That is, the height of the surfaces is given by:

$$\eta_o = h_1 + h_2 = h_1' + h_2' + 2H \tag{12}$$

[4] We follow a intuitively approach for the sake of simplicity. However a much more systematic derivation can be made with asymptotic methods and the reader is advise to explore them (see for instance refs. [8, 10, 11]), as they expose how different approximations relate to each other.

[5] We follow the derivation and the notation of ref. [8]

$$\eta_1 = h_2 = h_2' + H \tag{13}$$

so that the pressure in the first layer is given by:

$$p = g\rho_1(\eta_o - z)\,, \tag{14}$$

and the horizontal gradient is:

$$\nabla p = g\rho_1 \nabla \eta_o = g\rho_1(\nabla h_1' + \nabla h_2') \tag{15}$$

With the same procedure it is found for the second layer:

$$\nabla p = g\rho_1(\nabla h_1' + \nabla h_2') + g'\rho_1 \nabla h_2'\,, \tag{16}$$

where $g' = (\rho_2 - \rho_1)/\rho_1$ is the reduced gravity. The geostrophic condition then gives for the top layer:

$$u_g = -\frac{g}{f}\frac{\partial}{\partial y}(h_1' + h_2') \tag{17}$$

$$v_g = \frac{g}{f}\frac{\partial}{\partial x}(h_1' + h_2')\,, \tag{18}$$

and for the bottom layer:

$$u_g = -\frac{g}{f}\frac{\partial}{\partial y}(h_1' + h_2') - \frac{g'}{f}\frac{\partial h_2'}{\partial y} \tag{19}$$

$$v_g = \frac{g}{f}\frac{\partial}{\partial x}(h_1' + h_2') + \frac{g'}{f}\frac{\partial h_2'}{\partial x} \tag{20}$$

From the equations above, it is clear that the equations can be written in terms of a stream function for each layer, such that $u_i = -\partial\psi_i/\partial y$, $v_i = \partial\psi_i/\partial x$ and the vorticity is simply written as $\eta_i = \nabla^2\psi$. The stream functions being:

$$\psi_1 = \frac{g}{f}(h_1' + h_2') \tag{21}$$

$$\psi_2 = \frac{g}{f}(h_1' + h_2') + \frac{g' h_2'}{f} \tag{22}$$

An expression for the potential vorticity of each layer in terms of the stream functions, is obtained if we use these in (11):

$$q_1 = \beta y + \nabla^2\psi_1 + \frac{f_o^2}{g'H}(\psi_2 - \psi_1) - \frac{f_o^2}{gH}\psi_1 \tag{23}$$

$$q_2 = \beta y + \nabla^2\psi_2 + \frac{f_o^2}{g'H}(\psi_1 - \psi_2)\,, \tag{24}$$

where the term $f_o^2\psi_1/gH$ is much smaller than the others and can be dropped from the equations.

Note, that we now have a closed set of equation to solve. The equations above are only dependent on ψ_1 and ψ_2, and as they are the potential vorticity for each layer, they are exactly conserved ($dq_i/dt = 0$). Nonetheless, it is convenient to nondimensionalize the variables; and we use the same scales as in ref. [12] for this purposes. That is, we use U for the characteristic velocity of the flow, $L_R = (g'H)^{1/2}/f_o$ the internal Rossby deformation radius as the horizontal scale, and L_R/U for the time scale. This way, (23) and (24) transform to:

$$q_1^* = \beta^* y + \nabla^2 \psi_1^* + F(\psi_2^* - \psi_1^*) \tag{25}$$

$$q_2^* = \beta^* y + \nabla^2 \psi_2^* + F(\psi_1^* - \psi_2^*)\,, \tag{26}$$

where $\beta^* = \beta U/L_R^2$ and $F = f_o^2 L_R^2/(g'H)$. The superscript denotes a nondimensional variable and will be dropped hereafter, writing above equations as:

$$q_1 = \beta y + \nabla^2 \psi_1 + F(\psi_2 - \psi_1) \tag{27}$$

$$q_2 = \beta y + \nabla^2 \psi_2 + F(\psi_1 - \psi_2) \tag{28}$$

Lastly, it is also convenient to introduce some sort of dissipation in the system. Newtonian dissipation can be added easily by considering the conservation equations for q_i as in ref. [12], that is:

$$\frac{dq_i}{dt} = \frac{\partial q_i}{\partial t} + J(\psi_i, q_i) = -\nu \nabla^2 \psi_i\,, \tag{29}$$

where $J(\psi, q) = \psi_x q_y - \psi_y q_x$ and ν is a nondimensional viscosity coefficient. Equivalently this three las equations can be rewritten as:

$$\begin{aligned}
&\frac{\partial}{\partial t}\left(\nabla^2\psi_1 + F(\psi_2 - \psi_1)\right) + \mathbf{u}\cdot\nabla\left(\nabla^2\psi_1 + F(\psi_2 - \psi_1)\right) + \beta\frac{\partial \psi_1}{\partial x} = -\nu\nabla^2\psi_1 \\
&\frac{\partial}{\partial t}\left(\nabla^2\psi_2 + F(\psi_1 - \psi_2)\right) + \mathbf{u}\cdot\nabla\left(\nabla^2\psi_2 + F(\psi_1 - \psi_2)\right) + \beta\frac{\partial \psi_2}{\partial x} = -\nu\nabla^2\psi_2
\end{aligned} \tag{30}$$

The last two equations constitute a closed set for ψ_1 and ψ_2, far more convenient than its dimensional form, as all terms are scaled so that its magnitude is $O(1)$. We can now use this model as a simplified version to study baroclinic instability. Nonlinear terms are retained, so that its dynamics still is rich and not fully understand up to date.

5 Stability Theory

As we seen in the previous chapter, geostrophy is an equilibrium state of the Navier-Stokes equations. It is now convenient to ask how stable this state is, that is, how much does the isopycnals need to slope in order for baroclinic instability to develop, or equivalently, how fast the winds must be. For this we assume a background state of the flow, linearize the equations of motion around this state, assume a sinusoidal form for the disturbances and extract stability coefficients from the analysis (i.e. a bifurfication analysis is performed). It is fair to say that this has been the most common approach when dealing with baroclinic instability. And there have been great studies which differ in their assumptions. Here we focus in the particular set of equations derived in the previous sections (commonly called the Philip problem). However, more general forms of this analysis exist, where the equations for a continuously stratified atmosphere are considered (for example in the Eady or the Charney problems).

It is also important to keep in mind the limitations of this kind of analysis. They are highly idealized as they consider linearized relationships around simple background flows. Nevertheless, they are indispensable to understand atmospheric and oceanic circulation

We follow the procedure outlined by Hasha [11] or in a simpler fashion in ref. [6], as it is an elegant systematic approach which borrows notation from quantum mechanics. First, assume a shear flow with mean velocities U in the upper layer and $-U$ in the lower one. This implies that the stream functions can be computed as:

$$\begin{aligned} \psi_1 &= -Uy + \psi_1' \\ \psi_2 &= Uy + \psi_2' \, , \end{aligned} \tag{31}$$

where the primes denote perturbations around the mean state. Replacing this in (30), and ignoring the high order perturbation terms and viscous dissipation , one obtains:

$$\begin{aligned} \left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\left(\nabla^2\psi_1 + F(\psi_2 - \psi_1)\right) + \frac{\partial \psi_1}{\partial x}(\beta + 2FU) = 0 \\ \left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\left(\nabla^2\psi_2 + F(\psi_1 - \psi_2)\right) + \frac{\partial \psi_2}{\partial x}(\beta - 2FU) = 0 \\ , \end{aligned} \tag{32}$$

where we have dropped the primes. This can be written in matrix form as:

$$\left(\frac{\partial}{\partial t}M - L\right)\Psi = 0 \, , \tag{33}$$

where

$$\begin{pmatrix} \nabla^2 - F & F \\ F & \nabla^2 - F \end{pmatrix} \tag{34}$$

$$\begin{pmatrix} -U\frac{\partial}{\partial x}(\nabla^2 - F) - (\beta + 2FU)\frac{\partial}{\partial x} & -UF\frac{\partial}{\partial x} \\ UF\frac{\partial}{\partial x} & U\frac{\partial}{\partial x}(\nabla^2 - F) - (\beta - 2FU)\frac{\partial}{\partial x} \end{pmatrix} \tag{35}$$

Seeking solutions of the form

$$\Psi(x,y,t) = \hat{\Psi}(k,m,U)e^{i(kx+my-\omega t)} \, , \tag{36}$$

where k, m are the horizontal wave numbers and ω the frequency of the perturbation, one obtains new matrixes $\hat{M}$ and $\hat{L}$ as functions of this new variables (see ref. [11]), so that:

$$\left(\hat{L} + i\omega\hat{M}\right)\Psi = 0 \, . \tag{37}$$

In order to have nontrivial solution the determinant of $\left(\hat{L} + i\omega \hat{M}\right)$ must vanish, which gives a relation for ω (i.e. the dispersion relationship for the perturbations) and a critical velocity for the flow to be unstable. However, in our specific case we would set $\beta = 0$ so that the flow would always be unstable.[6] Thinking in terms of state space for the dynamical system [7] (seeref. [13]), what was has been done up to here, is to find an equilibrium point of the system and analyze its bifurcations. This is no small task, and that is the reason of the vast amount of literature published in the subject. Now we are interested in what happens next. And we will try to analyze this by making use of the symmetries of the flow.

6 Nonlinear Theory

6.1 A New Framework

At this point we have a good idea of what baroclinic instability is and how it has been commonly approached. Now we wish to move on and try to explore the dynamics described by our simplified model while retaining all nonlinear contributions. We will need a smart way to solve our equations of motion, and an even smarter way to represent their solutions.

The first step is to think about our equations in a dynamical framework; that is, in state space. Thus, writing the system as follows:

$$\frac{dX_i}{dt} = F(X_i) \tag{38}$$

where each X_i correspond to a coordinate in this space. Clearly, not the case of (6), where $\mathbf{u}$ is a dependent variable of both space and time. Nonetheless, we can formulate it in this form by the use of spectral methods expressing the velocity as:

$$\mathbf{u} = \sum \widehat{u}\phi(\mathbf{x}) \tag{39}$$

where ϕ are basis function defined for the entire domain[8] and $\widehat{u}$ their respective amplitudes. Replacing this into (6), and using and appropriate relationship to account for the pressure, we obtain a system like (38) for the amplitudes of the basis functions. That is:

$$\frac{d\widehat{u}_i}{dt} = G(\widehat{u}_i) \tag{40}$$

system which can be relatively easily integrated. It should be noted that the above is much more than just a convenient method to solve the Navier-Stokes equations; it is a framework to study continuum equations as dynamical systems. In state space, a point represents a physical state of the full system, i.e. the three dimensional velocity field. This implies that Navier-Stokes equations can be thought as an infinite dimensional

[6] Unless the domain is not wide enough for the instability to occur.
[7] That is, decomposing the PDE of the flow into a series of ODE for the amplitude of the harmonics.
[8] This could be any set of orthogonal functions (i.e. Fourier, Legendre or Chebyshev series)

space where the harmonics represent each dimension of it [9]. However, the number of dimension consider in practice is finite and limited either by the computer power available, or the smallest scale one can actually solve. No long time ago considering only a handful of dimensions was possible. Nowadays, computer power is such that DNS[10] are possible and considering 100,000 dimensions or more is common for small domains.

But given the exuberant number of dimensions, how are we to visualize the flows in state space? there is not need to panic about this, although visualizations is challenging the important dynamics are embedded in lower dimensional manifolds. This was first noted by Hopf, who speculated that viscosity must lower the effective dimensions of the system. An example would be a laminar flow which is embedded in a infinite space; which due to its high viscosity is effectively represented by a single point in this space (for instance see ref. [3]). In the case of a turbulent flow invariant solutions (periodic orbits, relative periodic orbits, equilibrium points and traveling waves) are starting point for finding the lower dimensional invariant manifolds to consider. These solutions are usually unstable, which makes imperative the use of a search algorithm in order to find them.

Nonetheless, their role in dynamics is of such importance (see ref. [15]) that these intrinsic difficulties should not discourage us from seeking for them. For instance, invariant solutions can be thought to represent the coherent structures that characterize turbulent dynamics (see refs. [16, 17]). In this view, the system spends some time in a neighbor near one of this solutions, then jumps to the next and spends some time there until it jumps to another, and so on. What it is most striking is that this cycles resemble closely turbulent trajectories, and are hard to tell apart from them. In fact, the measurable quantities of a system depend mostly on these invariant solutions (see ref. [13]), so that finding them gives a better understanding of the system. In this study we give the first steps towards finding these solutions in a baroclinic instability model by showing a plausible way to reduce the symmetries of the system.

Reducing the symmetries greatly simplifies the search of this structures. As solution do not wonder as much in the reduced state space. In this space, traveling waves reduce to equilibrium points, relative periodic orbits to periodic orbits and equilibrium points are conserved. Our tool of choice for this reduction is the method of slices. We are interested in rotating all the solutions to a hyperplane which intersects all symmetry group tangents (see ref. [13] for details). For this we impose the condition:

$$\frac{\partial}{\partial \phi} \parallel a - g\hat{a}' \parallel^2 = 0\,, \tag{41}$$

where a is our vector in state space, ϕ the rotating frame (i.e. the angle by which the solution is rotated) and $\hat{a}'$ a template representative of our local dynamics. Of course, our slice would not be good for the entire manifold, and it is necessary to consider a set of this slices in order to capture the entire dynamics of the system ref. [18]. More details of this will be given in the next section.

[9] This view is due to Hopf, who's insight on turbulence where way ahead of his time; and have proven to be an invaluable tool (see ref. [14])

[10] Direct Numerical Simulations. The approach of this method is to solve Navier-Stoke equations up to the scale where dissipation occurs and kinetic energy is converted to heat.

Once we are done with the slicing, the hope is that this makes it easier to look for this invariant solutions. After all, the "dancing" solutions have been settled to the slice. But first, we need to define them in a quantitative manner so it is clear what we mean by invariant solutions. First, let's think again in terms of physical space (6). Note that we can write the system as:

$$\frac{\partial \mathbf{u}}{\partial t} = F(\mathbf{u})\,, \quad f^t(\mathbf{u}) = \mathbf{u} + \int_0^t F(\mathbf{u}) d\tau\,, \tag{42}$$

thus, we are seeking solutions of the following form (as shown in ref. [3]):

$$\begin{aligned} F(\mathbf{u}_{EQ}) &= 0 \\ F(\mathbf{u}_{TW}) &= -c \cdot \nabla \mathbf{u}_{TW} \\ f^{T_p}(\mathbf{u}_p) &= u_p \\ f^{T_p}(\mathbf{u}_p) &= u_p\,, \end{aligned} \tag{43}$$

where this represent respectively equilibrium points, relative equilibrium points, periodic orbits and relative periodic orbits.

Finally, note that one additionally aspect one should care about is the connection between these different solutions. That is, how is one to get from one recurrent pattern to the other? Heteroclitic connections (i.e. the trajectories which exactly connect invariant solutions; see for instance ref. [19]), are commonly studied for this purposes as these organize how the flow transits from one manifold to the next.

In this project, we limit ourselves to show how to make the slicing, hoping this would encourage more in depth study of the subject. However, we briefly discuss next how to start searching for the recurrences in the system one the slicing is completed.

6.2 Finding invariant solutions

Finding invariant solution of a system with a high number of dimension is not an easy task. And it is even harder if we do not reduce the symmetries first. In doing so one is required to use a numerical method to find them. However, due to the fact that huge Jacobian matrix would have to be computed, implementing the Newton-Raphson standard search method is not an option.

But not all hopes are lost. Algorithms to find this structures have been implemented and used successfully in a variety of simulations (see for instance ref. [17]). This methods depend on initial guesses of this periodic orbits, and there are several approaches one can take to make the right ones. One of this approaches is given in ref. [17], where the rate of energy dissipation and energy input for a system is computed based on the velocity field, and then plotted together. This way the entire multidimensional system is projected into two physical meaningful dimensions and guesses are made based on approximate recurrences of the traced curve.

After reducing the symmetries, the next step would be to evaluate the possibility of a periodic orbit based on the criteria discussed here. However, we proceed one step at a time. So let us go back to reducing the symmetry.

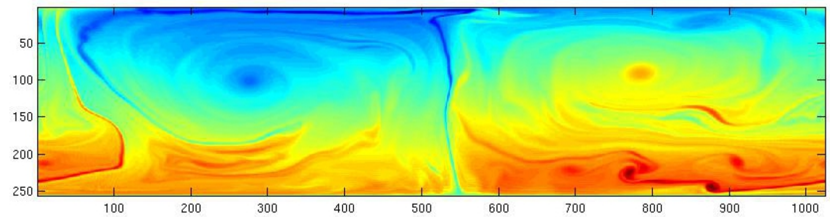

Figure 4: Vorticity field (ξ) for the top layer in the model.

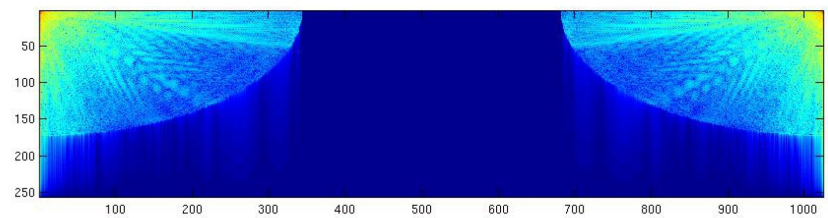

Figure 5: Spectral representation of the vorticity field ($ln(1 + abs(a_{lk}))$). Note that some frequencies are just the complex conjugate of others.

6.3 Symmetry Reduction

Up to here we have gather enough background to start making attempts to reduce the symmetry. The system to consider is that of Section 4. And the code is the same as the one programmed by Bracco in ref. [12][11]. The typical results of the simulation are shown in figure 5.

The code makes uses a pseudo-spectral method to solve the nonlinear PDE given in 32 but with $\beta = 0$ and $\psi_2 = \psi_2'$. For this, Fast Sine Transforms are used along the y direction, and Fast Fourier Transforms along the x direction . This means that the vorticity can be represented as:

$$\xi(x, y) = \sum_{k=0}^{N-1} a_{lk} e^{-2i\pi kx/N} \tag{44}$$

Additionally, the system has periodic boundaries along the x axis, so it has a continuous symmetry along this axis $\xi(x) = \xi(x + L_x)$; where L_x stands for the length of the domain. The one-parameter rotation group acting on this variable, would be then given by (see ref. [20]):

$$g(\phi) = \text{diag}\{e^{-2i\pi k\phi/N}\} \tag{45}$$

When acting on a point in state space ($g(\phi)a$), the effect of g is to translate the solution along x without changing any o the properties of the simulation. Furthermore,

[11] Without bottom topography or beta effect.

in state space the group orbit traces a trajectory which topologically is a circle, and the group tangent of the circle at point a is given by the derivative of g:

$$\mathbf{T} = \mathrm{diag}\{-2i\pi k/N\} \tag{46}$$

Now, to reduce the symmetry, we select a template representative our our physical simulation (the slice a'), and seek for the rotating frame $\phi(t)$ that continually rotates a trajectory back to the slice. This, by making use of condition 41. Care is taken as we are dealing with complex vectors, so that the norm represents the multiplication of a vector with its conjugate transform. That is, starting with:

$$\frac{\partial}{\partial \phi} \| a - g\hat{a}' \|^2 = 0\,, \tag{47}$$

expanding, simplifying and noting that $\mathrm{Re}\,(a^\dagger \mathbf{T} a) = 0$ one arrives at the slicing condition:

$$\mathrm{Re}\,(\hat{a}^\dagger \mathbf{T} \hat{a}') = 0 \tag{48}$$

which is a similar condition to the one obtained for real cases. Translating it into a somewhat more implementable equation gives:

$$\mathrm{Re}\,(\hat{a}^\dagger \mathbf{T} \hat{a}') = \mathrm{Re}\left(\frac{-2\pi i}{N} \sum_{k=0}^{N-1} \sum_{k=0}^{M-1} \overline{a_{lk}} a'_{lk} k e^{\frac{-2\pi i k}{N}\phi} \right) = 0 \tag{49}$$

Now we are all set to seek for this solutions. A Newton-Raphson, or similar, seeking method can be implemented by using $F(\phi) = \mathrm{Re}\,(\hat{a}^\dagger \mathbf{T} \hat{a}')$ and $F'(\phi) = \frac{\partial}{\partial \phi} \mathrm{Re}\,(\hat{a}^\dagger \mathbf{T} \hat{a}')$ for each time step. Effort are being carried right now to implement this algorithm, however for the time being we only provide the methodology.

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