Chapter 9

Bifurcation in billiards

Chapter 5 described bifurcations and forbidden orbits in billiard systems by introducing a pruning front. An orbit was forbidden if its symbolic value was in the forbidden region and the orbit bifurcated if its symbolic value was on the pruning front. We will in this chapter investigate the bifurcation process in billiard systems; the structure in the phase space and how the bifurcations are organized in families. This will enable us to connect bifurcations in a hard billiard system with the bifurcations in a soft Hamiltonian system even if it is difficult to obtain a pruning front for a smooth potential.

The bifurcations in billiard systems have received very little attention in the literature. It has even been claimed that there are no bifurcation structure in billiards; "...the E-τ plots [phase space as function of parameter] for this problem [anisotropic Kepler] has no interesting structure and shows no branching. The same is true of the various 'Billiards' problems." [23]. I disagrees with the statement on the billiard systems. The lack of interest in bifurcations in billiards should not be because these billiards are too artificial, because the billiards are very popular to use in e.g. quantum chaos calculations. It may be that the problems with symbolic dynamics have discouraged studies of bifurcations in billiards, but that is unlikely since bifurcations in the more complicated smooth potentials are much studied. Anyway I find these problems an interesting and a not too complicated exercise.

9.1 Tent map revisited

The best way to understand bifurcations in billiards is first to study the one dimensional tent map. In chapter 1 we made some remarks on bifurcations in the tent map. In the one parameter tent map a family of orbits is born at one critical parameter value. In the one dimensional phase space x the orbit at the bifurcation

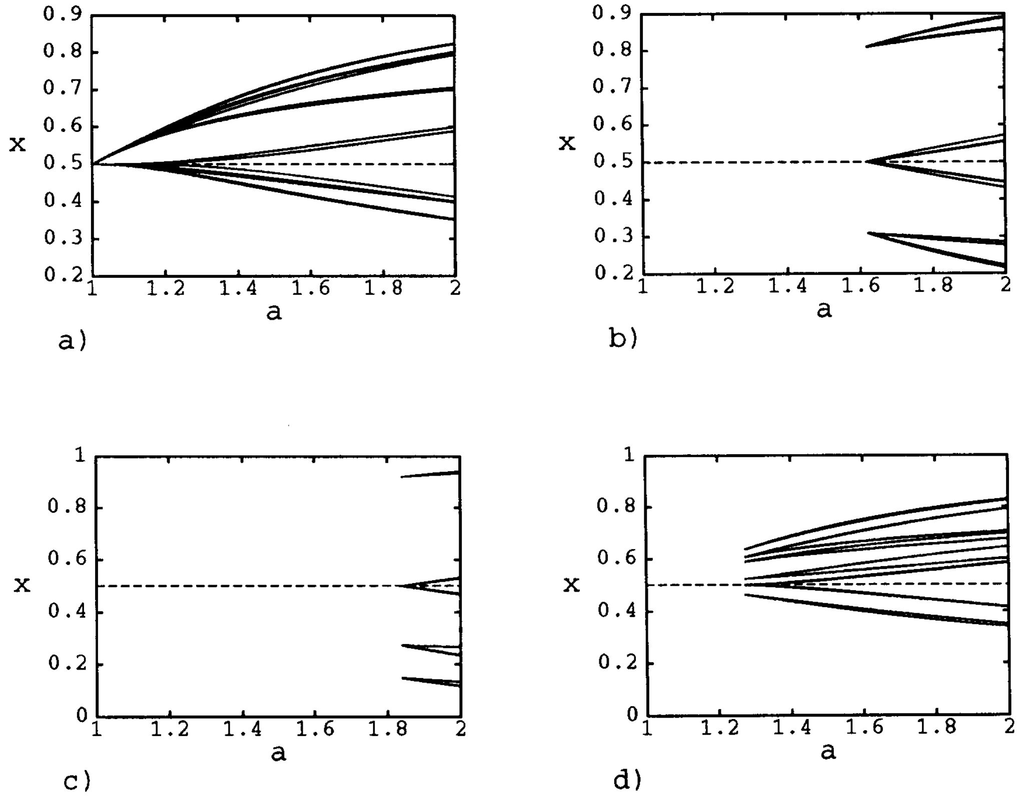

Figure 9.1: Bifurcation of families of periodic orbits in the tent map. a) $\overline{1}$, b) $\overline{100}$, c) $\overline{1000}$, d) $\overline{101111}$.

has one point at $x_c = 1/2$ and moves away as the parameter a increases. Some examples of bifurcating orbits are given in figure 9.1. We define the bifurcation family of orbits to be the period doubling family. This is all orbits of the form

$$\begin{array}{l} \overline{S(1-\epsilon)} \\ \overline{S\epsilon} \\ \overline{S\epsilon S(1-\epsilon)} \\ \overline{S\epsilon S(1-\epsilon)S\epsilon S\epsilon} \\ \overline{S\epsilon S(1-\epsilon)S\epsilon S\epsilon S\epsilon S(1-\epsilon)S\epsilon S(1-\epsilon)} \\ \vdots \end{array} \tag{9.1}$$

with $S = s_1 s_2 s_3 \ldots s_{n-1}$ and $s_i \in \{0,1\}$, $\epsilon_i \in \{0,1\}$, the number of symbol 1's in $S\epsilon$ is odd, $S\epsilon$ can not be written as $S'(1-\epsilon)S'\epsilon$ and finally $\overline{S\epsilon}$ has to be the cyclic permutation giving $\tau^{\max}$. This corresponds to all harmonics of an orbit in the MSS terminology [147].

With this definition is it only the critical parameter $r_c = 1$ in eq. (1.3) that gives a bifurcation of only one family. This is the family of the fixed point $\overline{1}$ where the

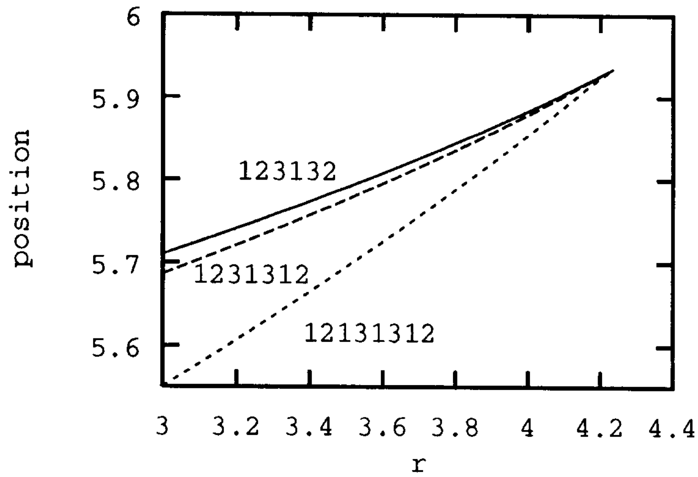

Figure 9.2: The position of one point of the orbits $\overline{123132}$, $\overline{12131312}$ and $\overline{1231312}$ as a function of the parameter $\tilde{r}$ close to the bifurcation in the non-symmetric 3 disk system.

string S consists of no symbols (here $\overline{S(1-\epsilon)} = \overline{0}$ does not bifurcate together with the family). All the other critical parameter values give the bifurcation of several families. The topological entropy increases linearly with the parameter and the map is called *not full* since not all kneading sequences can be obtained. In a more general map with a none-smooth critical point and no stable orbits, the different families may split up and bifurcate at different parameter values, while the different orbits belonging to the same family (9.1) always bifurcate at the same parameter value. The different families bifurcate in the MSS order but with critical parameter values where many orbits are created simultaneously.

In the tent map we find that the period 3 orbit family $\overline{S\epsilon} = \overline{100}$ bifurcates together with all other families with $\tau(\overline{101}) < \tau(\overline{S'\epsilon}) < \tau(100\overline{101}) = 0.111\overline{010}$ which is all orbits in the resonance of the logistic map. In general the orbits $\overline{S'}$ which bifurcate together with the primary family $\overline{S\epsilon}$ have $\tau(\overline{S(1-\epsilon)}) < \tau(\overline{S'}) < \tau(S\epsilon\overline{S(1-\epsilon)})$

The shortest orbits in the families $\overline{1}$, $\overline{100}$, $\overline{1000}$ and $\overline{101111}$ are drawn in figure 9.1 as a function of the parameter a.

9.2 Dispersing billiards

The bifurcation of a whole family of orbits at one parameter value is also happening in the billiards, but an important difference is that the billiards have a one

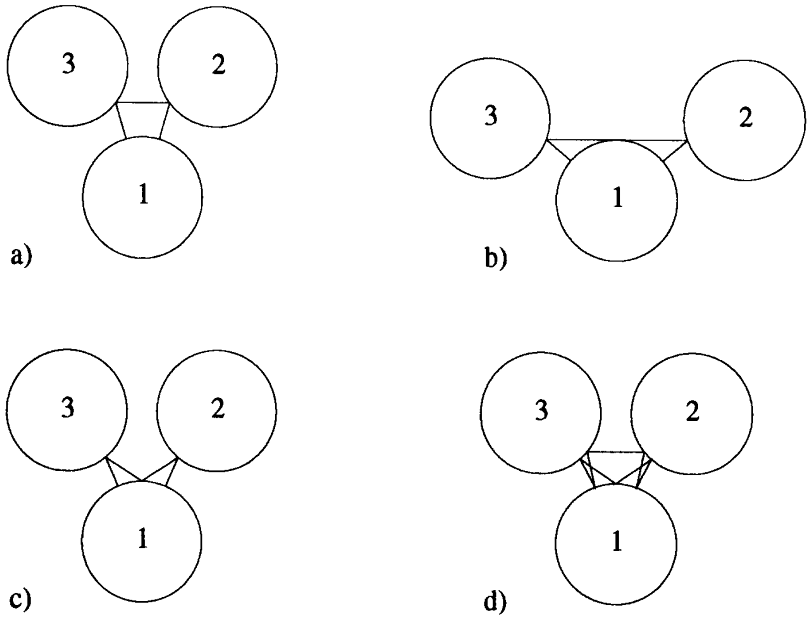

Figure 9.3: The orbits: a) and b) $\overline{123132}$, c) $\overline{12131312}$ and d) $\overline{1213132}$ for the parameter values: a), c) and d) $\tilde{r} = 2.5$ and b) $\tilde{r}_c = 4.212\ldots$.

dimensional family of critical orbits while the tent map only has one critical orbit.

9.2.1 The bifurcation family

Figure 9.2 shows a point in some orbits as a function of the parameter $\tilde{r}$ in a 3 disk system with the center-center distances $d_{12} = d_{13} = 2.5$, $d_{23} = \tilde{r}$, and with radius equal to 1. These orbits are the orbits in the family

$$\ldots 313 s_{-1} 212 s_0 313 s_1 212 s_2 313 \ldots \tag{9.2}$$

with s_i either empty or the symbol 1. An equivalent definition of this family is that it consists of the orbits constructed by using the alphabet

$$\hat{s}_i \in \{313, 3131, 212, 2121\} \tag{9.3}$$

We see that this family has more members than the period doubling family of the one dimensional map.

The reason why this is the correct symbolic description of the family is understood by the description of the singular orbits in figure 9.3. We know that an orbit in a dispersing billiard without corners bifurcates; that is changes between admissible and not admissible, because either

1) a free flight of the particle becomes tangential to the border

or

2) a bounce off the wall has the outgoing angle $\phi = \pi/2$.

In the configuration space at the bifurcation parameter r_c, these two cases look the same. From the figure 9.3 b) it is not possible to tell if the straight line between disk 2 and 3 is case 1) or 2). The parameter value r_c therefore has to be the bifurcation value of both the orbit where this straight line does not bounce and have no symbol, and for the orbit where it bounces and has the symbol 1. The orbit is infinite in future and in past and each time it passes the tangent point it may have a bounce or not. The descriptions (9.2) and (9.3) are exactly the descriptions of these orbits using symbols.

If we study an orbit which is tangent at one point but never returns to this point tangentially, there are only two orbits that bifurcate together for this parameter value. This is the case for hetroclinic orbits.

The argument for why the orbits bifurcate at the same parameter value does not depend on the details of how the billiard changes with a parameter. The only necessary knowledge is which straight line that becomes tangential to the wall, or which angle that becomes $\pi/2$. Figure 9.4 shows the same orbits in a 3-disk system as a function of a parameter $\hat{r}$ when we choose different radiuses of the disks

$$\text{radius(disk 1)} = 2, \quad \text{radius(disk 2)} = 1/2, \quad \text{radius(disk 3)} = 1$$

with the center-center distances

$$d_{12} = d_{13} = d_{23} = \hat{r}$$

The positions and the parameter change from the previous example but the same orbits (9.2) belong to the bifurcation family. The ordering along $\hat{r}$ for when the different families bifurcate may however change. This ordering is not fixed here as it is for the unimodal map (the MSS ordering).

An other example of a dispersing billiard is the symmetric 4 disk system and in figure 9.5 the position of one bounce of some long orbits is drawn as a function of the parameter r. The orbits are drawn in figure 9.6 and this family of orbits are described by the string

$$\ldots 1 s_{-1} (32)^4 t_{-1} 4 t_0 (23)^4 s_0 1 s_1 (32)^4 t_1 4 t_2 (23)^4 s_2 \ldots \tag{9.4}$$

with s_i either 2 or no symbol, and t_i either 3 or no symbol. Because of the symmetry of this family there is a bifurcation two places in the orbit simultaneously. The critical parameter value is $r_c = 2.0312\ldots$.

We will return to this example later when we discuss the smooth potentials.

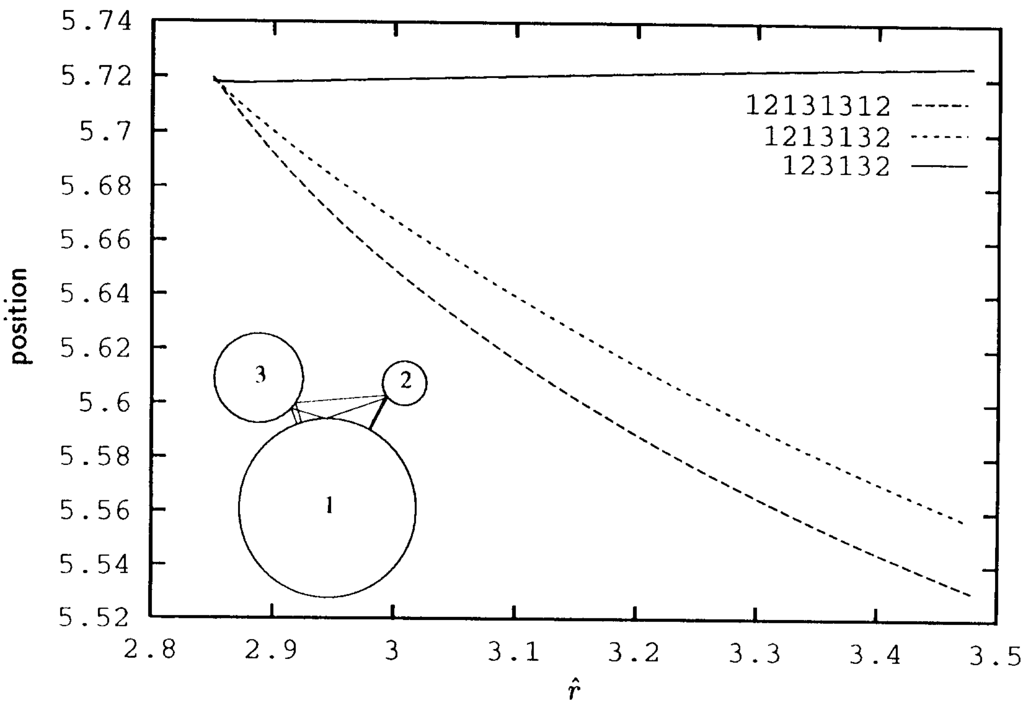

Figure 9.4: Bifurcation of the orbit $\overline{123132}$ and its family in a not symmetric 3-disk system. The position of the bounce on disk 3 as a function of $\hat{r}$.

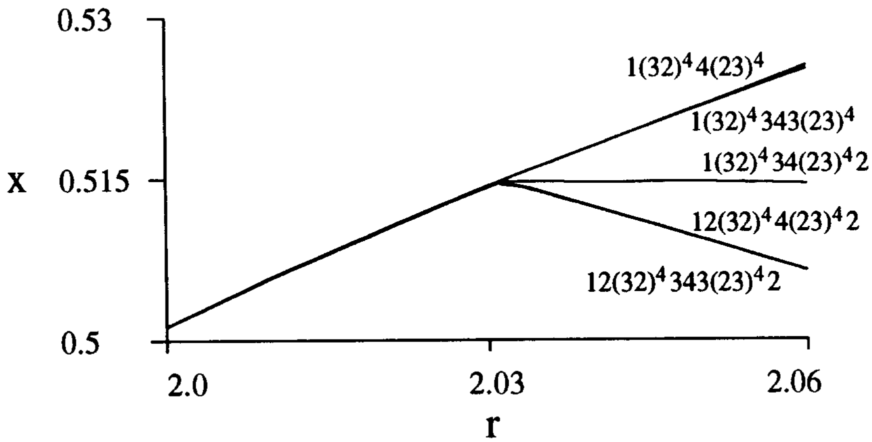

Figure 9.5: Bifurcation of the orbit $\overline{1(32)^4 4(23)^4}$ and its family in the 4-disk system. The position of the bounce on disk 1 as a function of r.

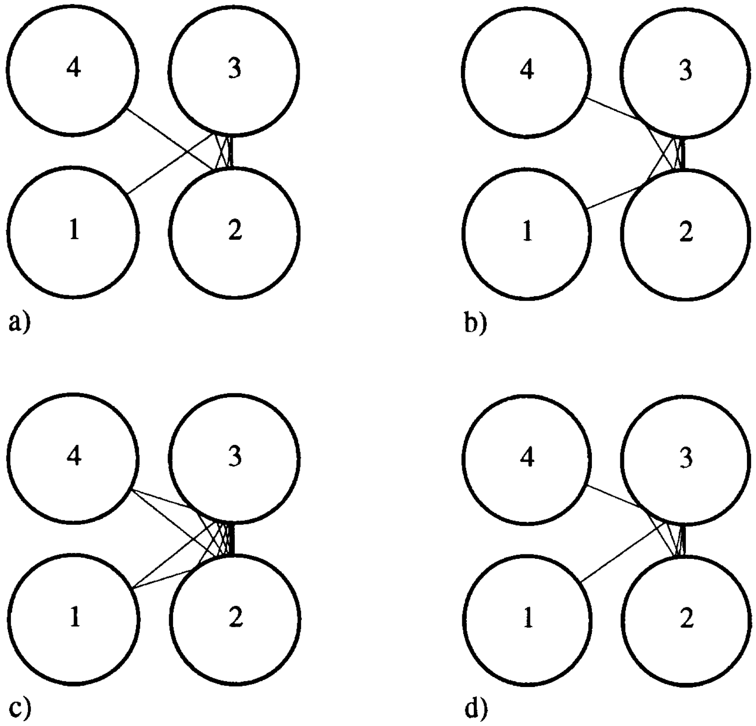

Figure 9.6: The orbits a) $\overline{1(32)^4 4(23)^4}$, b) $\overline{12(32)^4 343(23)^4 2}$, c) $\overline{1(32)^4 34(23)^4 2}$ and d) $\overline{1(32)^4 343(23)^4}$ in the 4 disk system for $r = 2.5$.

9.2.2 The parameter space

In the tent map each bifurcating orbit has one point equal to the critical point $x_c = 1/2$. In the dispersing billiards the critical points are a function of one parameter x which is the position of the tangent bounce. We call the orbit tangential to the border at x for $x_c(x)$ if this orbit is in the non-wandering set of the system. If the dispersing billiard is closed then $x_c(x)$ is continuous in the phase space. In an open billiard $x_c(x)$ is a point set, possibly a Cantor set, or it is empty.

The different families bifurcate at different positions on $x_c(x)$ and if we choose two different ways to parameterize the billiard with two parameters r_1 and r_2 then the different families with different $x_c(x)$ are not necessarily ordered the same way in contrast to the unimodal map. The only ordering between the families follows from the requirement that the pruning front is monotone.

The number of parameters necessary for describing all possible ways the system may bifurcate is infinite. We can deform a small part of the wall without destroying the dispersive properties. This will change the orbits that bounce in this part of the wall but not the other orbits. By making this deformation we can change the bifurcation point of one orbit without changing the bifurcation point of another.

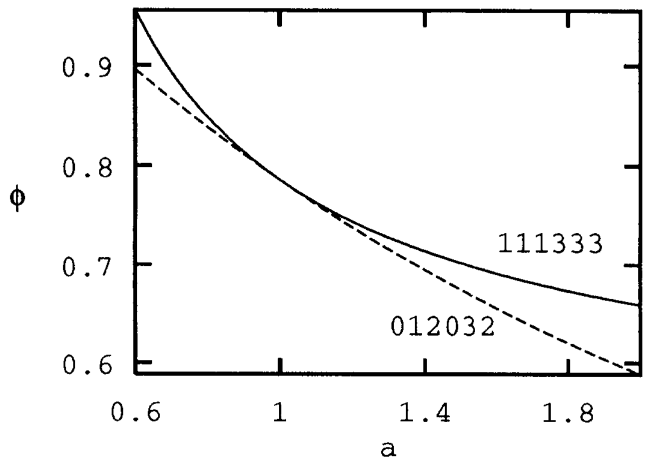

Figure 9.7: The angle of a bounce in the orbits $\overline{111333}$ and $\overline{012032}$ as a function of a for the stadium billiard.

These different local deformations may be considered as the different parameters in the system. Another point of view on the parameters is to understand each point of the pruning front as one parameter. This also gives an infinite number of parameters equivalent to the discussion above. In the folding maps of the Hénon type the pruning front has large steps, and we found a natural hierarchic structure of the infinite parameter space which gave a good way of describing the bifurcations of the map. We have not been able to find a similar ordering into more and less important parameters for the billiards because the pruning front does not have any large steps but is rather smooth.

9.3 Stadium billiard

The focusing stadium billiard also has the same kind of singular bifurcations of families as the dispersing billiards. Figure 7.26 shows some orbits for different parameter values. The outgoing angle ϕ of one bounce of the orbits as a function of the half length of the straight line, a, is plotted in figure 9.7. The structure of the singular bifurcations is similar to the dispersive billiards where all orbits belonging to one bifurcation family bifurcate at one parameter point. The family for the example in figures 7.26 and 9.7 is given by the symbol strings in symbols s^a

$$\ldots c_0 1 d_0 e_0 3 f_0 c_1 1 d_1 e_1 3 f_1 \ldots \tag{9.5}$$

with

$$c_i \in \{0,1\}, \quad d_i \in \{1,2\}, \quad e_i \in \{2,3\} \text{ and } f_i \in \{0,3\}.$$

An orbit in the stadium billiard becomes not admissible because either

1) The point where the particle bounces in the semi-circle moves to the end of the semi-circle

or

2) The point where the particle bounces in the straight line moves to the end of the straight line.

Assume an orbit bounces exactly off the singular point on the border where the straight line and the semi-circle join. In the configuration space is it not possible to decide whether this orbit is bouncing in the semi-circle or in the straight line. The symbol of this bounce is then given by either the semi-circle symbol or the symbol for the straight line. If the orbit is periodic then the orbit bounces off the singular point every n-th bounce and therefore a whole family bifurcates at this parameter value. The family of orbits is described by an alphabet

$$s_i = S\epsilon \tag{9.6}$$

where S is a fixed symbol string and ϵ is either a semi-circle or a straight line symbol. If there are symmetries of the orbit such that it bounces several times in a singular point before it closes, then the alphabet may be more complicated as the example above shows.

9.4 Corner bifurcations

We have a corner bifurcation in the wedge billiard where the singularity is the tip between the planes and in the corners of the overlapping disk systems. An orbit becomes illegal because a bouncing point on the wall moves from bouncing legally outside the corner until until it hits the corner at the bifurcation parameter. The only other orbit with a point that hits the corner for the same parameter value is the orbit which bounces off the other wall in a symmetric system. Because of the symmetry this is the orbit that is a mirror image of the first orbit or it is the same orbit if this orbit also is symmetric. In a fundamental domain is it only one orbit bifurcating. The bifurcation family is only the trivial family consisting of the orbit, its reflection and the time reversed orbit.

One exception is the orbits bifurcating for $\theta = 60°$ in the wedge billiard. As observed by Smilansky [185, 189] there are several orbits bifurcation simultaneously for this parameter value. One may expect this for some special parameter values but generically it does not seem to be true. Typically will the size of the family depend crucially on the smoothness of the singularity in the system.