Part V

Quantum Chaos and Zeta Functions

An important application of the theory of symbolic dynamics and pruning discussed in the chapters above is the semi-classical quantization of classical chaotic systems. A control of the geometrical structure of the classical system is essential for controlling the convergence of the semi-classical expansions, as showed in several examples by Cvitanović [42, 43, 46, 48], Artuso, Aurell and Cvitanović [10, 11, 14], Ezra, Richter, Tanner and Wintgen [70] and others. Gutzwiller [103] states: "Finding the appropriate code seems the most important task when facing a dynamical system with hard chaos".

Chapter 11

Quantum Chaos

11.1 Semi-classical methods

The semi-classical Bohr-Sommerfeld theory, or the first quantum theory described successfully the quantum spectrum of hydrogen. However, this method failed in describing even the ground level of the helium atom, the calculation which was the first triumph triumph of the new quantum mechanics [120]. Before the introduction of the quantum mechanics of Heisenberg, Born, Jordan, Dirac, Pauli and Schrödinger, Einstein noticed that the Bohr-Sommerfeld quantization rested on the construction of action-angle variables and would fail for systems which are not integrable [69]. Understanding of the geometrical phase factors Morse [156] and Maslov indices [145, 6, 37] came much later and the ground level of helium was calculated with semi-classical methods by Percival and Leopold as late as 1980 [132].

Lately there has been much interest in applying semi-classical methods to determine spectra of systems whose classical dynamics is chaotic, both because semi-classical methods are a useful tool for obtaining numerical results, and because they offer a classical intuitive picture of the quantum system. Fundamental work was done by Gutzwiller around 1970 [102, 103], with the Gutzwiller trace formula which connects a sum over periodic orbits in a completely chaotic classical system to the eigenvalues of the corresponding quantum mechanical system

$$g_c(E) = \frac{1}{i\hbar} \sum_{\text{p.o.}} \frac{T_p}{|\det(M_p - I)|^{\frac{1}{2}}} e^{\frac{i}{\hbar} S_p(E) - i\pi\sigma_p/2} \tag{11.1}$$

where $g_c(E)$ is the trace of the semi-classical Green's function. M_p is the monodromy matrix, T_p the time of the primitive periodic orbit p, S_p the classical action along the periodic orbit and σ the Maslov index for the orbit. The poles in $g_c(E)$ give eigenvalues of the quantum system; energy levels, resonances, decay times, correlations, etc. There are several different ways to formulate this result; the zeta-function formulation from thermodynamic theory, see below, gives a slightly different formula.

The classical dynamical zeta function in thermodynamics was introduced by Ruelle [170, 171, 172] and applied to chaotic quantum systems by Cvitanović [42] and others. The classical dynamical zeta function is given by

$$1/\zeta = \prod_p (1 - t_p) \quad ; \quad t_p = \frac{z^{T_p \gamma}}{|\Lambda_p|} \tag{11.2}$$

and the corresponding quantum zeta function can be written as

$$1/\zeta = \prod_p (1 - t_p) \quad ; \quad t_p = \frac{1}{\sqrt{\Lambda_p}} e^{\frac{i}{\hbar} S_p(E) + i\pi\sigma_p/2} \tag{11.3}$$

where Λ the leading eigenvalue of the Jacobian matrix. The zeros of the zeta functions corresponds to the semiclassical eigenvalues of the system. (11.3) is a truncation of the Gutzwiller-Voros zeta function [196, 195]

$$Z_{qm} = \prod_p \prod_{k=0}^{\infty} \left(1 - \frac{e^{\frac{i}{\hbar} S_p(E) + i\pi\sigma_p/2}}{|\Lambda_p|^{1/2} \Lambda_p^k} \right) \tag{11.4}$$

or the recently introduced "quantum Fredholm determinant" of Cvitanović and Rosenqvist [55]

$$Z_{qm} = \prod_p \prod_{k=0}^{\infty} \left(1 - \frac{e^{\frac{i}{\hbar} S_p(E) + i\pi\sigma_p/2}}{|\Lambda_p|^{1/2} \Lambda_p^k} \right)^{k+1} \tag{11.5}$$

and even more recent determinants constructed suggested by Vattay et.al. [194].

These different formulations are expected to give the same leading eigenvalues, but they differs in the domain of analyticity and the speed of convergence. Formally the sums or products in such formulas are divergent, and only a "clever" expansion will yield a good result. The trace formula (11.1) will usually give very few eigenvalues, while the quantum Fredholm determinant is claimed to have the largest domain of analyticity, and yelds most eigenvalues [55, 56]. The classical Fredholm determinant is entire for an axiom A system [173, 16, 17], and this fact motivates the belief that quantum Fredholm also has good analytic properties.

A fast convergence for these formulas depends on a good expansion, usually ordered according to the length of the periodic orbits. If we have a complete binary symbolic description the expansion can be done according to the symbolic description.

The dynamical zeta function is formally given by the sum

$$\begin{aligned} 1/\zeta = \prod_p (1 - t_p) &= 1 - \sum_{p_1 p_2 \ldots p_k} t_{p_1 + p_2 + \cdots + p_k} \\ t_{p_1 + p_2 + \cdots + p_k} &= (-1)^{k+1} t_{p_1} t_{p_2} \ldots t_{p_k} \end{aligned} \tag{11.6}$$

where the product and sum is over all distinct non-repeating combinations of prime periodic orbits. If the orbits are given by a complete binary symbolic description, we can reorder of the terms as follows:

$$\begin{aligned} 1/\zeta &= (1-t_1)(1-t_0)(1-t_{10})(1-t_{100})(1-t_{101})(1-t_{1000}) \\ &\quad (1-t_{1001})(1-t_{1011})(1-t_{10000})(1-t_{10001}) \\ &\quad (1-t_{10010})(1-t_{10011})(1-t_{10101})(1-t_{10111})\ldots \\ &= 1-t_1-t_0-[t_{10}-t_1t_0]-[(t_{100}-t_{10}t_0)+t_{101}-t_{10}t_1] \\ &\quad -[(t_{1000}-t_{100}t_0)+(t_{1110}-t_1t_{110}) \\ &\quad +(t_{1001}-t_{100}t_1-t_{101}t_0+t_{10}t_0t_1)]\ldots \end{aligned} \tag{11.7}$$

The terms in square brackets are called the n-th curvature correction c_n by Cvitanović [43], and the first part of the expansion is called the fundamental part. If all orbits with the same symbolic description have approximately the same weight the terms in the curvatures almost cancel each other, and the convergeness of the expansion is fast. This near cancelationcan be understood as a shadowing effect, as shown numerically for 1-dimensional repellors and the well-separated 3-disk system [50].

The weight of the term t_p may be different for some orbits and the simple shadowing might fail. One example is the unimodal Farey map

$$T = \begin{cases} x/(1-x) & \text{if } x < 1/2 \\ (1-x)/x & \text{if } x > 1/2 \end{cases} \tag{11.8}$$

discussed by Artuso, Aurell and Cvitanović [10]. Here the fixed point $\overline{0}$ is marginally stable while all other orbits are unstable. The term t_0 cannot shadow any of the other orbits but Artuso et.al. found that one can resum the unstable terms in such a way that different infinite sums shadow each other, with the fundamental part of the zeta-function given by a geometrical series

$$\begin{aligned} 1/\zeta &= 1-(t_1+t_{10}+t_{100}+t_{1000}+\cdots) \\ &\quad -[(t_{110}+t_{1100}+t_{11000}+\cdots)-t_1(t_{10}+t_{100}+t_{1000}+\cdots)] \\ &\quad -[(t_{1110}+t_{11100}+\cdots)-t_1(t_{110}+t_{1100}+\cdots)] \\ &\quad -[(t_{10100}+t_{101000}+\cdots)-t_{10}(t_{100}+t_{1000}+\cdots)] \\ &\quad -\cdots \end{aligned} \tag{11.9}$$

this sum can be written as [10]

$$1/\zeta = 1-\hat{t}_1-[\hat{t}_{12}-t_1\hat{t}_2]-[\hat{t}_{112}-t_1\hat{t}_{12}]-[\hat{t}_{23}-t_2\hat{t}_3]-\cdots \tag{11.10}$$

where the index of t_k for $k > 1$ denotes a string 10^{k-1} and $\hat{t}_{klm\ldots n}$ is the infinite sum starting with $t_{klm\ldots n}$ and increasing the number of 0's in the end of the symbol string; $\hat{t}_{klm...n} = t_{klm...n} + t_{klm...(n+1)} + t_{klm...(n+2)} + \ldots$. We then have to evaluate an infinite sum to obtain the fundamental part of the zeta function. The terms in this sum will in typical examples converge as a power law, and the sum can be estimated from just a few terms.

This kind of orbits seems to be common in chaotic systems. In the stadium billiard an orbit bouncing infinitely many times successively in one semi-circle does not exist, but the whispering gallery orbits bouncing an arbitrary number of times do. The length of these orbits converges to a finite length as the number of bounces goes to infinity, and the fundamental part of a zeta function has to include at least one such infinite sum. In the wedge billiard there are the orbits bouncing n times successively on one tilted plane, denoted 0^n. The length (and action) of these orbits with increasing n converges to a finite length (action), but the fixed point $\overline{0}$ orbit does not exist. In smooth Hamiltonian systems with stable islands we expect this type of orbits to be generic. The orbits inside islands are stable, but there always exist unstable orbits wandering arbitrarily close to the outermost KAM torus. These orbits have to be included in the zeta function expansions as infinite sums.

11.2 Markov diagrams

Given a finite Markov diagram for the admissible orbits, one can easily read off the terms in the fundamental part of the zeta function. As we did when finding the topological entropy in section 1.3 we fidentify all non-self-intersecting loops and non-intersecting combinations of these loops. We record the symbol string corresponding to each such loop in the diagram and this is the index for each fundamental term t_k. Combinations of loops with no common node give products of terms $t_k t_l \cdots t_m$, with the indices corresponding to the different loops. The self-intersecting loops, combinations of these, and intersecting combinations of non-intersecting loops give the curvature terms of the zeta-function. A few examples of getting the terms from a diagram illustrate the procedure.

The loops in the binary graph 1.11 gives t_0 and t_1, no combination of loops, and the zeta function is

$$1/\zeta = 1 - t_0 - t_1 + (\text{curvatures}).$$

The graph in figure 1.17 b) describing the repellor when the period 3 orbit of the unimodal map is stable gives the loops t_1 and t_{10}

$$1/\zeta = 1 - t_1 - t_{10} + (\text{curvatures}). \qquad (11.11)$$

An example of the zeta function from a graph describing the bimodal map is given in figure 2.12. The zeta function has the fundamental orbits

$$\{\overline{1}\}, \{\overline{01}\}, \{\overline{20}\}, \{\overline{200}\} \tag{11.12}$$

and in addition the combination of the orbits $\overline{1}$ and $\overline{200}$ is not a shadow of any orbit in the expansion of the ζ–function. The ζ–function is now expanded and gives

$$\begin{aligned}
1/\zeta &= 1 - t_{\overline{1}} - t_{\overline{10}} - t_{\overline{20}} - t_{\overline{200}} + t_{\overline{1}}t_{\overline{200}} \qquad (11.13)\\
&-[t_{\overline{201}} - t_{\overline{20}}t_{\overline{1}}] - [t_{\overline{101}} - t_{\overline{10}}t_{\overline{1}}]\\
&-[t_{\overline{2010}} - t_{\overline{20}}t_{\overline{10}}] - [t_{\overline{1011}} - t_{\overline{101}}t_{\overline{1}}] - [t_{\overline{2011}} - t_{\overline{201}}t_{\overline{1}}]\\
&-[t_{\overline{10111}} - t_{\overline{1011}}t_{\overline{1}}] - [t_{\overline{10020}} - t_{\overline{10}}t_{\overline{020}}] \qquad (11.14)\\
&-[t_{\overline{20020}} - t_{\overline{200}}t_{\overline{20}}] - [t_{\overline{10101}} - t_{\overline{101}}t_{\overline{01}}]\\
&-[t_{\overline{20101}} + t_{\overline{20110}} - t_{\overline{201}}t_{\overline{01}} - t_{\overline{2010}}t_{\overline{1}} - t_{\overline{20}}t_{\overline{110}} + t_{\overline{20}}t_{\overline{10}}t_{\overline{1}}] - \ldots
\end{aligned}$$

where for smooth flows the shadowing terms become small compared with the fundamental orbits.

If a loop in the Markov diagram corresponds to a forbidden orbit or an orbit isolated from all other orbits then we can find a fundamental part of the zeta function with infinite sums as in the above Farey map example of Artuso, Aurell and Cvitanović [10, 11]. Instead of the forbidden orbit in the diagram we choose the series of non-selfintersection loops in the diagram running n times through the loop of the forbidden orbit. Examples of this are the stadium billiard and the wedge billiard.

In the billiard systems we have made an approximation of the pruning front to obtain the Markov graphs. The zeta functions we obtain from these graphs will then be an approximation, but we expect this zeta function to have good convergence properties since we have an approximation both to the fundamental parts and the shadowing parts of the expansion.