turbulence,
and what to do about it

Predrag Cvitanović, Han Liang & Matt Gudorf

ChaosBook.org/overheads/spatiotemporal
$\rightarrow$ chaotic field theory talks, papers

Niels Bohr International Academy Colloquium, Copenhagen

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Dreams of grand schemes

Navier-Stokes

$$\rho \frac{\partial u_i}{\partial t} + \rho u_j \frac{\partial u_i}{\partial x_j} = \rho X_i - \frac{\partial p}{\partial x_i} + \mu \nabla^2 u_i$$

Einstein

$$R_{ik} - \frac{1}{2} g_{ik} R = \frac{8\pi k}{c^4} T_{ik}$$

$$R^{i}_{klm} = \frac{\partial \Gamma^{i}_{km}}{\partial x^{l}} - \frac{\partial \Gamma^{i}_{kl}}{\partial x^{m}} + \Gamma^{i}_{nl}\Gamma^{n}_{km} - \Gamma^{i}_{nm}\Gamma^{n}_{kl}$$

Yang-Mills

$$\mathcal{L} = -\frac{1}{4} F^{a}_{\mu\nu} F_{a}^{\mu\nu}$$

$$F^{a}_{\mu\nu} = \partial_\mu A^{a}_{\nu} - \partial_\nu A^{a}_{\mu} + g C_{abc} A^{b}_{\mu} A^{c}_{\nu}$$

QUANTUM

Now, go solve the problem of turbulence

the answer is :

new ![3]

spatiotemporal zeta function

$$\zeta[\beta, s] = \prod_p \zeta_p$$

$\Phi_p \quad = \quad$ a prime (non-repeating) multi-periodic state

will explain

[3] P. Cvitanović and H. Liang, *A chaotic lattice field theory in two dimensions*, 2025.

a theory of turbulence ? who needs it ?

have : a turbulent pipe flow

Q : how much power?

to move crude at velocity $\langle v \rangle$?

how much power?

velocity v of a fluid element is an ‘observable’

to evaluate

expectation value

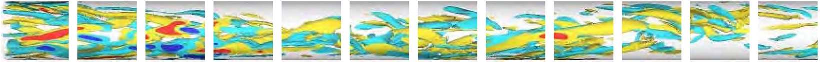
of observable v averaged over all field configurations

$$\langle v \rangle = \int d\Phi \, p[\Phi] \, v[\Phi] \,, \quad d\Phi = \prod_z d\phi_z$$

need to know the probability of every field configuration Φ

have : a theory of turbulence

a turbulent pipe flow[1]



we have a detailed theory of small turbulent fluid cells[2]

can we can we construct the infinite pipe by tiling it by small turbulent configurations Φ?

Q. what would that theory look like ?

A. it's here : this talk

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[1] M. Avila and B. Hof, Phys. Rev. E **87**, 063012 (2013).

[2] J. F. Gibson et al., J. Fluid Mech. **611**, 107–130 (2008).

wisdom of statistical mechanicians

partition function

field configuration Φ occurs with probability

$$p(\Phi) = \frac{1}{Z} e^{-S[\Phi]}, \qquad Z = Z[0]$$

partition function $=$ sum over all field configurations

$$Z[\mathrm{J}] = \int [d\phi]\, e^{-S[\Phi]+\Phi\cdot\mathrm{J}}, \qquad [d\phi] = \prod_z \frac{d\phi_z}{\sqrt{2\pi}}$$

on importance of a configuration

how likely is a configuration Φ?

$$p(\Phi) = \frac{1}{Z} e^{-S[\Phi]}$$

this talk : determine probability of field configuration Φ

semiclassical field theory

sum over all deterministic configurations !

path integral

field configuration Φ occurs with probability amplitude

$$p(\Phi) \;=\; \frac{1}{Z}\, e^{\frac{i}{\hbar} S[\Phi]}\,, \qquad Z = Z[0]$$

partition sum $=$ integral over all configurations

$$Z[\mathrm{J}] \;=\; \int [d\phi]\, e^{\frac{i}{\hbar}(S[\Phi]+\Phi\cdot \mathrm{J})}\,, \qquad [d\phi] = \prod_z \frac{d\phi_z}{\sqrt{2\pi}}$$

evaluate how ?
here : WKB or semiclassical approximation

method of stationary phase

defining equations

$$\frac{\delta S[\Phi_c]}{\delta \phi_z} = 0$$

a global deterministic solution Φ_c satisfies this local extremal condition on every spacetime point z

WKB or semiclassical approximation

$$S[\Phi] = S[\Phi_c] + \frac{1}{2}(\Phi - \Phi_c)^\top \mathcal{J}_c (\Phi - \Phi_c) + \cdots$$

orbit Jacobian operator

$$(\mathcal{J}_c)_{z'z} = \left. \frac{\delta^2 S[\Phi]}{\delta \phi_{z'} \delta \phi_z} \right|_{\Phi=\Phi_c}$$

deterministic solution Φ_c probability amplitude

$$p(\Phi_c) = \frac{1}{Z} \frac{e^{iS[\Phi_c]+im_c}}{|\mathrm{Det}\, \mathcal{J}_c|^{1/2}}, \qquad Z = Z[0]$$

partition sum : support on deterministic solutions

$$Z[\mathsf{J}] = \sum_c \frac{e^{i(S[\Phi_c]+m_c+\Phi_c\cdot \mathsf{J})}}{|\mathrm{Det}\, \mathcal{J}_c|^{1/2}}$$

example : Gutzwiller trace formula[4]

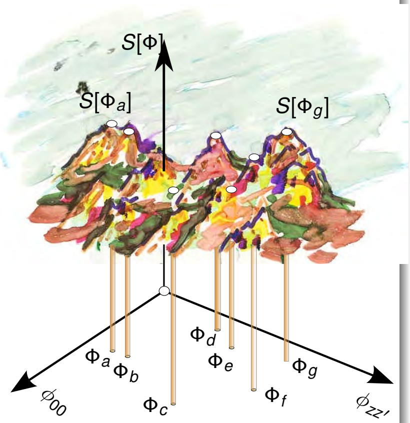
$$\int [d\Phi]\, A[\Phi]\, e^{iS[\Phi]} \approx \sum_c A[\Phi_c] \frac{e^{iS[\Phi_c]+im_c}}{|\mathrm{Det}\, \mathcal{J}_c|^{1/2}}$$

$1D$ time evolution quantum mechanics, so not field theory

[4] M. C. Gutzwiller, J. Math. Phys. **8**, 1979–2000 (1967).

bird's eye view

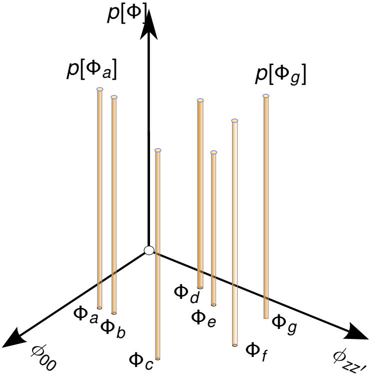
semiclassical field theory



$$\langle v \rangle \approx \sum_c v[\Phi_c] \frac{e^{\frac{i}{\hbar}S[\Phi_c]+im_c}}{|\mathrm{Det}\, \mathcal{J}_c|^{1/2}}$$

deterministic field theory

is its WKB backbone



$$\langle v \rangle = \sum_c v[\Phi_c] \frac{1}{|\mathrm{Det}\, \mathcal{J}_c|}$$

fluid turbulence is described by

new !

deterministic field theory

deterministic partition function :
sum over the deterministic solutions

first : determine "all"

periodic states

think globally, act locally

definition : periodic state is

a global deterministic solution

$$\Phi_c = \{\phi_{c,z}\} = \text{set of field values}$$

periodic along each translationally invariant direction

that satisfies the

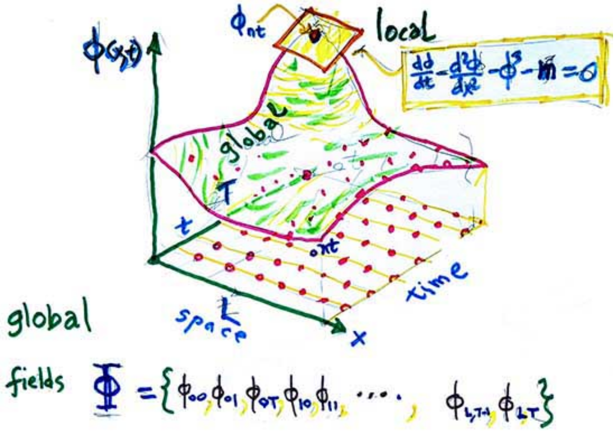
local condition : defining equations

$$F[\Phi_c]_z = 0$$

on every spacetime point z of multi-periodic primitive cell $\mathbb{A}$

think globally, act locally

a global deterministic solution Φ_{nt}



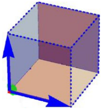
satisfies the local defining equations
everywhere all at once

computing periodic states

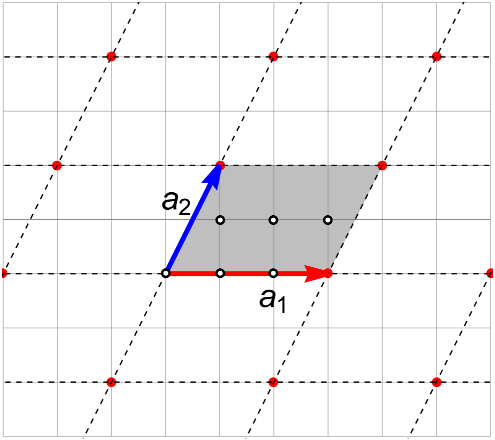
search for zeros of defining equations

$$F[\Phi_c]_z = 0$$

the entire global periodic state Φ_c over primitive cell $\mathbb{A}$ is a single point $(\phi_1, \phi_2, \cdots, \phi_n)$

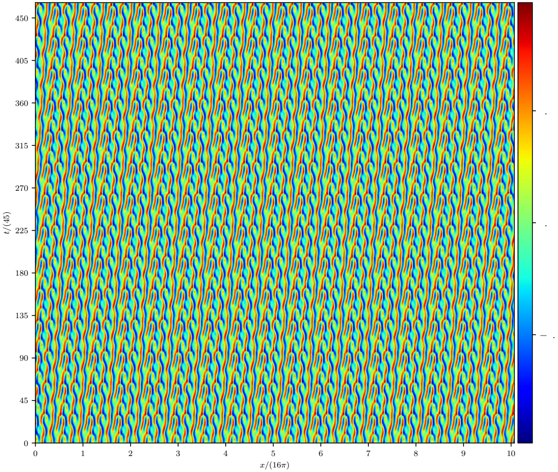
in the V_c-dimensional state space, $\Phi_c \in$ 

periodic state's primitive cell



primitive cell $\mathbb{A} = [3\times 2]_1$ that tiles a relative-periodic state

an example : spacetime tiled by a larger tile



Kuramoto-Sivashinsky tiling by
relative-periodic state $(L, T) = (33.73, 35)$

Gudorf 2018

UNLEARN: 3-d VISUALIZATION	THINK: ∞-d PHASE SPACE

instant in turbulent evolution:

a 3-d video frame,

each pixel a 3-d velocity field

instant in turbulent evolution:

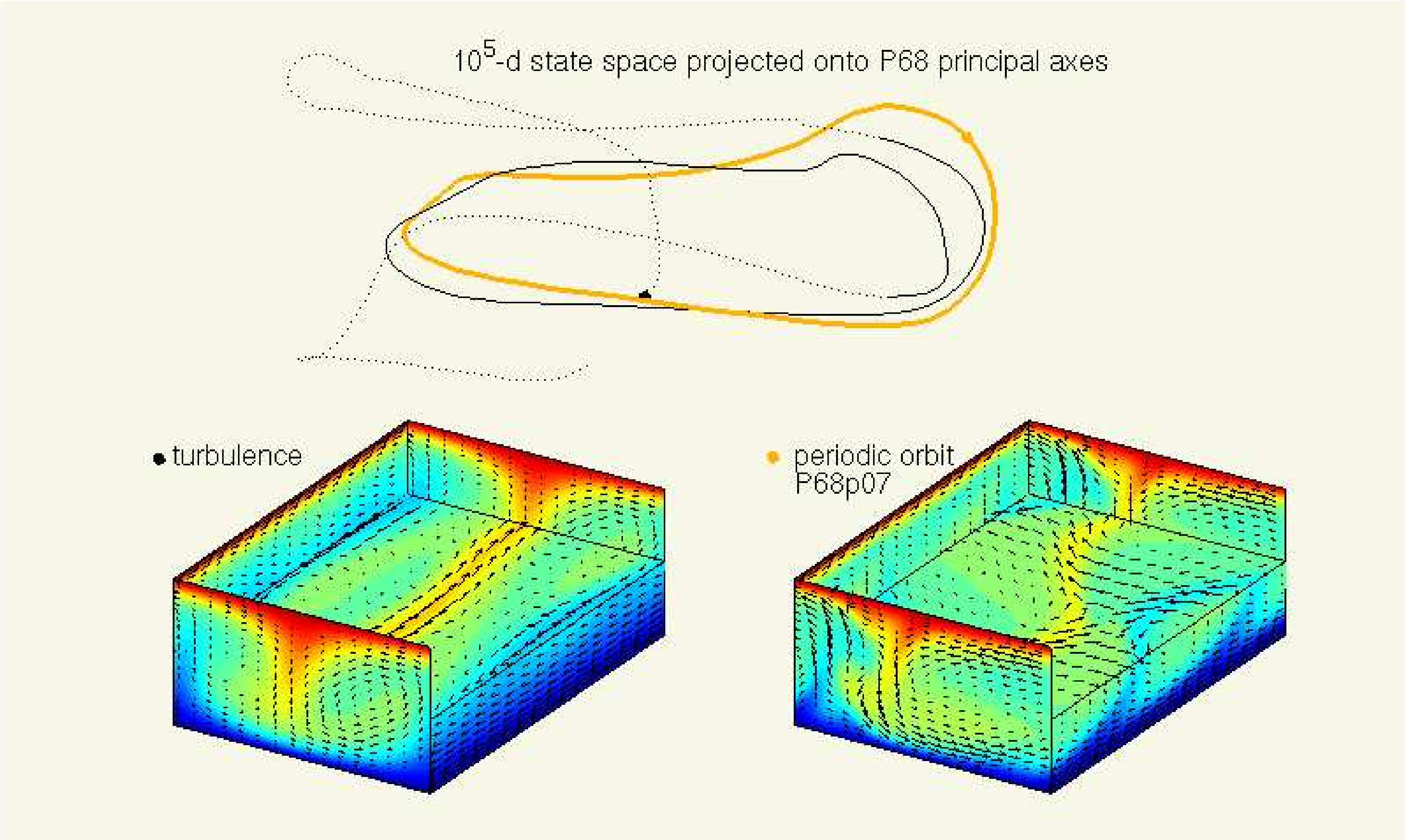
a unique point

theory of turbulence =

geometry of the state space

[E. Hopf 1948]

Periodic orbits shadow turbulence



A turbulent trajectory making a close pass to a periodic orbit.

click here to see the online video

ChaosBook.org/tutorials

can hierarchically compute ‘all’ solutions

orbitHunter

optimization of rough initial guesses converges

no exponential instabilities

stability : compute on reciprocal lattice

gitHub code[5]

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[5] M. N. Gudorf, *Orbithunter: Framework for Nonlinear Dynamics and Chaos*, tech. rep. (School of Physics, Georgia Inst. of Technology, 2021).

In theory there is no difference between theory and practice. In practice there is. (Anonymous)

Pattern Formation and Control Lab

the team[1,2,3,4,5]

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- Mike F. Schatz
- Balachandra Suri
- Jeffrey Tithof
- Wesley Toler

[1] B. Suri et al., Phys. Rev. E **98**, 023105 (2018).

[2] B. Suri et al., Phys. Rev. Lett. **125**, 064501 (2020).

[3] M. C. Krygier et al., J. Fluid Mech. **923**, A7 (2021).

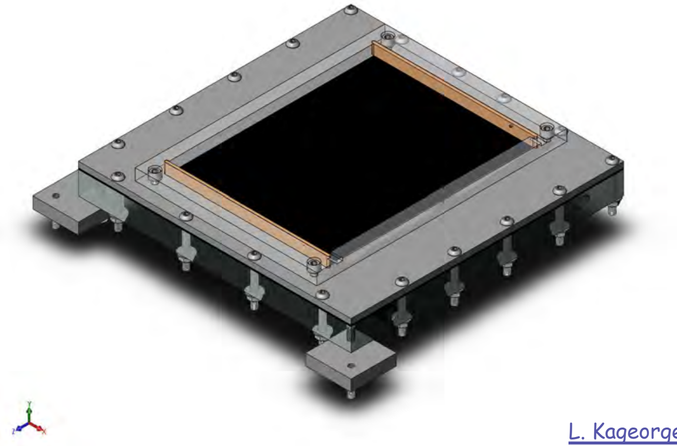
[4] C. J. Crowley et al., *Observing a dynamical skeleton of turbulence in Taylor-Couette flow experiments*, 2022.

[5] C. J. Crowley et al., Proc. Natl. Acad. Sci. **119**, 120665119 (2022).

3rd millennium experiment 1

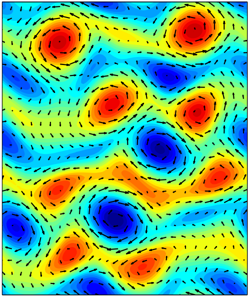
The Kolmogorov flow apparatus.

turbulence in 2 dimensions



L. Kageorge PhD Thesis

turbulence in 2D : 2-d video & state-space visualizations



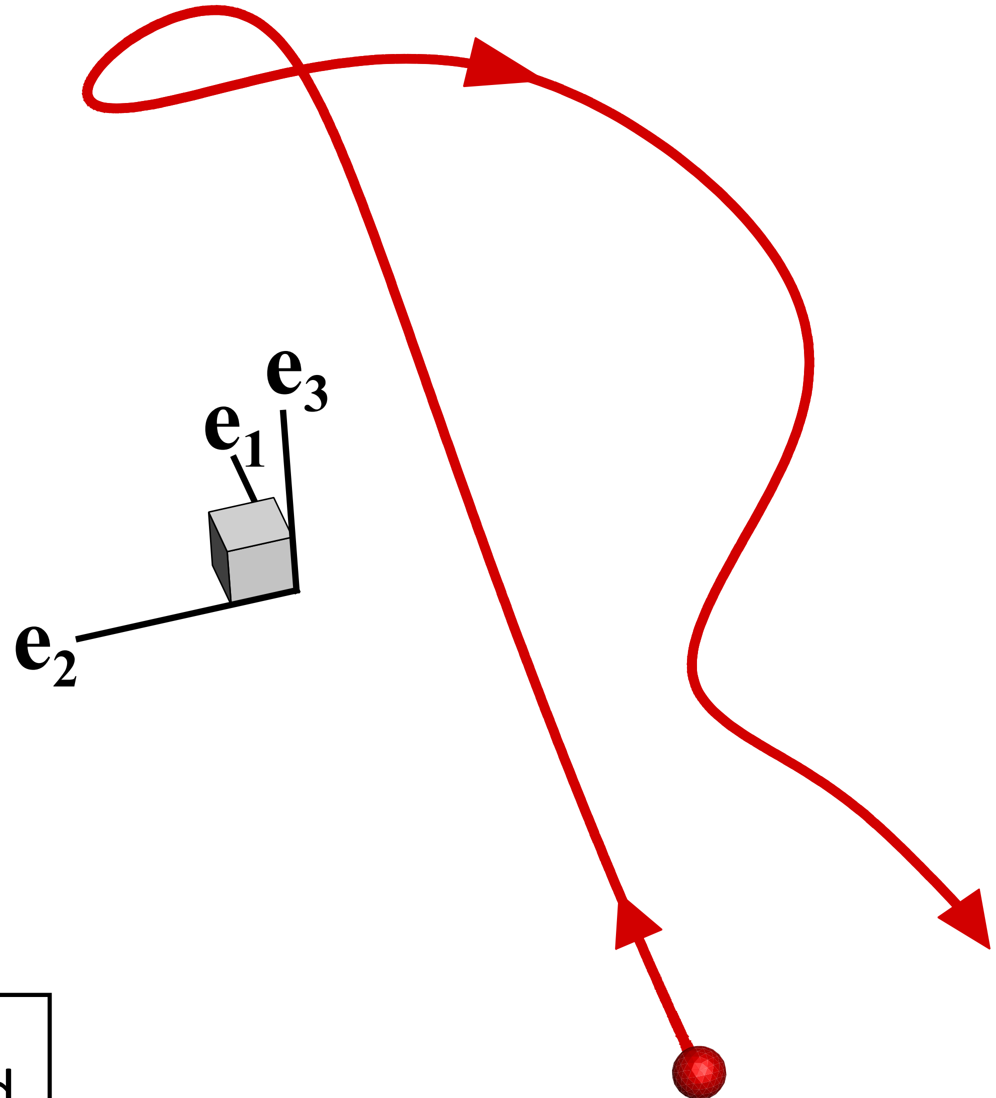
$$\mathbf{u} = \begin{bmatrix} u_{(1,1)x} \\ u_{(2,1)x} \\ u_{(3,1)x} \\ \vdots \\ u_{(n,m)x} \\ u_{(1,1)y} \\ u_{(2,1)y} \\ u_{(3,1)y} \\ \vdots \\ u_{(n,m)y} \end{bmatrix}$$

(a) (b)

Figure 1.1: (a) An illustrative example of a 2D velocity field. The heat map represents the vorticity of the velocity field, and helps guide the eye to the structure of the flow. (b) An example of how such a vector field can be converted into a state space vector by concatenating vector components.

click here to see the online video

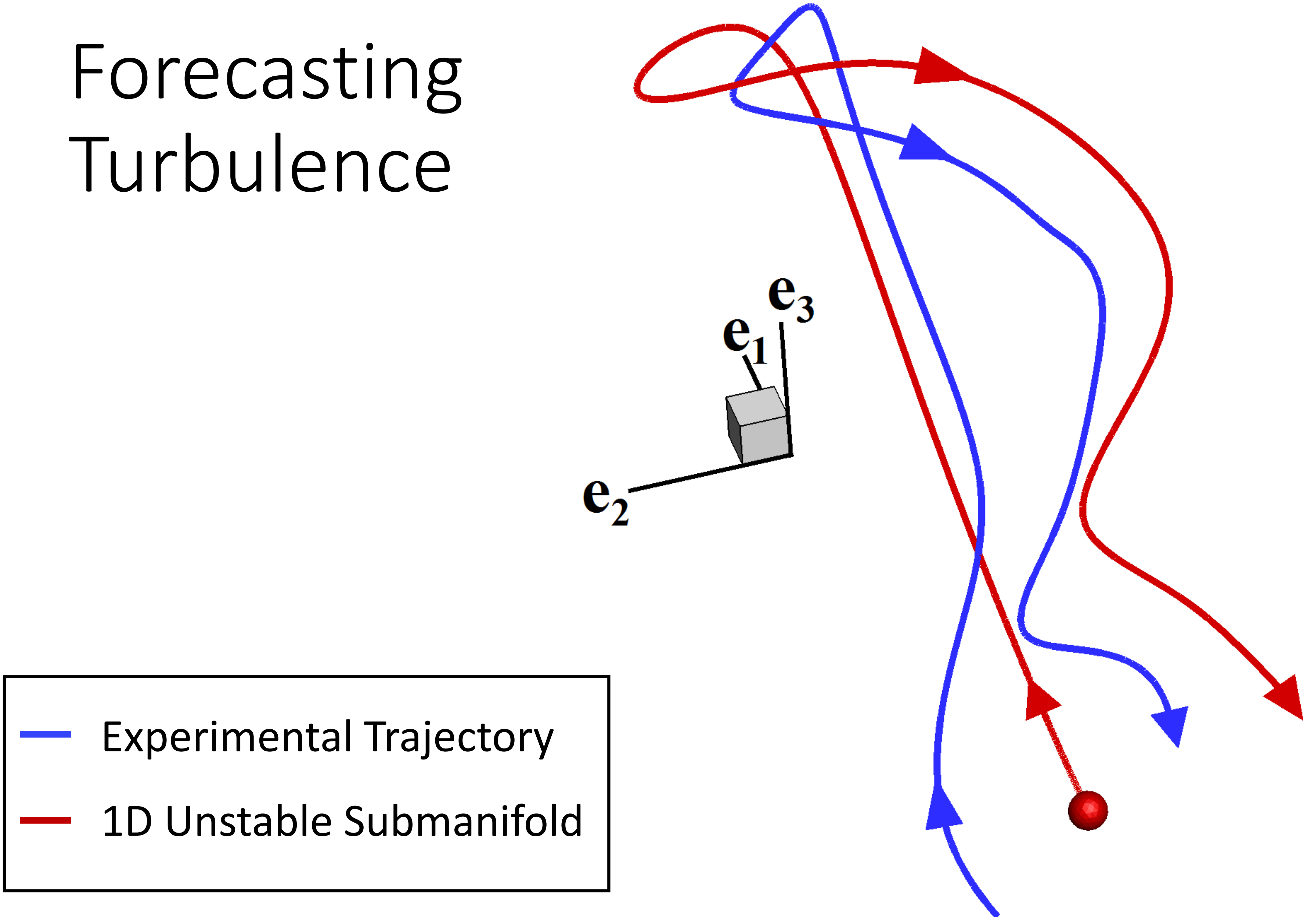
Forecasting Turbulence



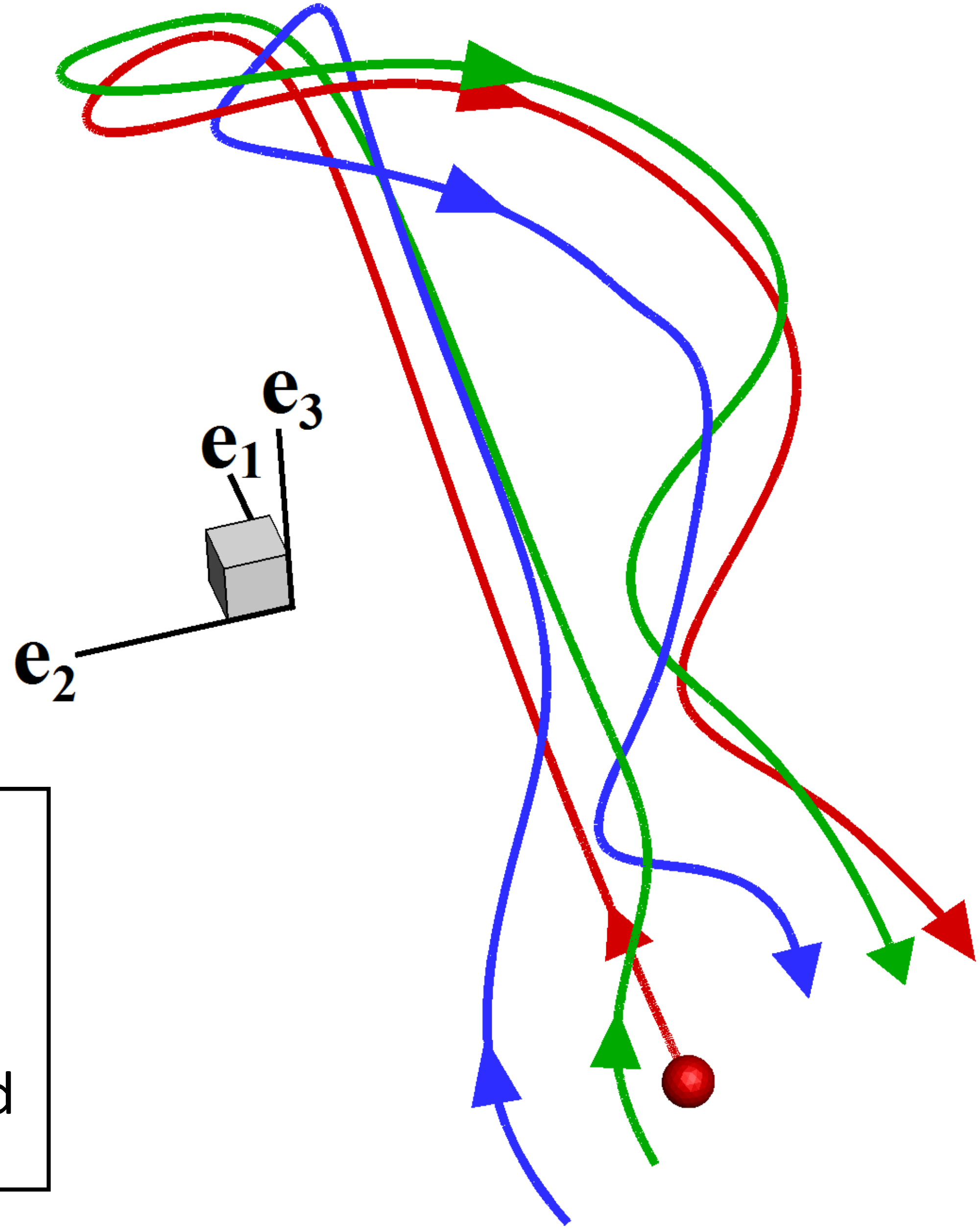
1D Unstable Submanifold

B Suri PhD Thesis

Forecasting Turbulence



Forecasting Turbulence



—	Numerical Trajectory
—	Experimental Trajectory
—	1D Unstable Submanifold

click here to see the online video

the big deal

the first experimental confirmation

1. of a Navier-Stokes predicted unstable manifold

turbulence in 2D : RPOs embedded in invariant measure

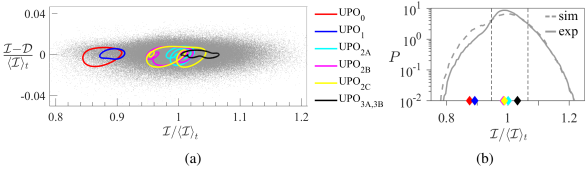


Figure 3.10: (a) Energy input rate $\mathcal{I}$ versus the difference between input and dissipation rates $(\mathcal{I} - \mathcal{D})$ for turbulent time series in experiment (scatter plot) and UPOs (closed loops). (b) Probability density function of $\mathcal{I}(t)$ for turbulent flow in experiment (solid gray) and DNS (dashed gray). Colored symbols show the mean values of $\mathcal{I}$ for each of the seven UPOs and the dashed black lines represent the range of $\mathcal{I}$ for $UPO_{2A\text{-}C}$ and $UPO_{3A,B}$.

L. Kageorge PhD Thesis

next: 3rd millennium mathematics

In the seminal 1948 paper, E. Hopf presciently noted that "The geometrical picture of the phase flow is, however, not the most important problem of the theory of turbulence. Of greater importance is the determination of the probability distributions associated with the phase flow".

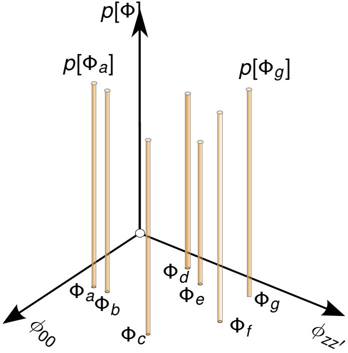
Hopf's call for understanding probability distributions associated with the phase flow has indeed proven to be a key challenge, one in which dynamical systems theory has made the greatest progress.

see seminars on ChaosBook.org/overheads/spatiotemporal

stability exponents

deterministic field theory : bird's eye view

Dirac porcupine



$$\langle v \rangle = \sum_c v[\Phi_c] \frac{1}{|\mathrm{Det}\,\mathcal{J}_c|}$$

The Importance of Being Φ_c

Φ_c is an exact, deterministic solution, so its

probability density

is V_c-dimensional Dirac delta function (!!! determinism !!!)

$$p_c(\Phi) = \frac{1}{Z}\,\delta(F[\Phi])$$

and

probability weight of deterministic solution Φ_c

$$\int_{\mathcal{M}_c} [d\Phi]\,\delta(F[\Phi]) = \frac{1}{|\mathrm{Det}\,\mathcal{J}_c|}$$

$\mathcal{M}_c$ = small neighborhood of periodic state Φ_c

our task : evaluate $\mathrm{Det}\,\mathcal{J}_c$

the most important thing : understand perturbations

- find a deterministic solution

$$F[\Phi_c]_z = 0 \quad \text{fixed point condition}$$

- then evaluate $\mathrm{Det}\,\mathcal{J}_c$ of

orbit Jacobian operator

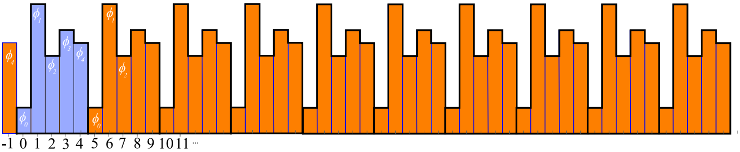
$$(\mathcal{J}_c)_{z'z} = \frac{\delta F[\Phi_c]_{z'}}{\delta \phi_z}$$

what does this global orbit Jacobian operator do?

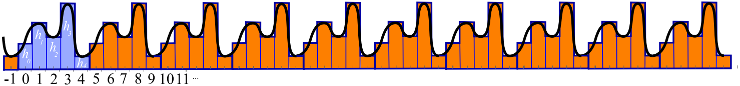
global stability

of periodic state Φ_c, perturbed everywhere

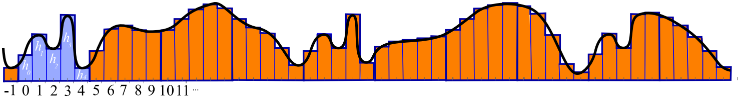
perturbations are into full state space



repeats of a period-5 periodic state Φ_c



an internal perturbation h_z, periodicity of Φ_c, has discrete spectrum, evaluated over Φ_c's primitive cell



a transverse perturbation h_z has continuous spectrum, evaluated over Φ_c's Brillouin zone[6]

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[6] A. S. Pikovsky, Phys. Lett. A **137**, 121–127 (1989).

the most critical thing

new ! functional 'fluctuation' determinant

$$\mathrm{Det}\,\mathcal{J}_c$$

must be computed on the

infinite Bravais lattice

stability exponent of periodic state Φ_c

new ! assign to each periodic state c
stability exponent λ_c per unit-spacetime-volume

exact deterministic weight

$$\frac{1}{|\mathrm{Det}\,\mathcal{J}_c|} = e^{-V_c \lambda_c}$$

in any spacetime dimension

- λ_c : stability exponent
- V_c : Φ_c Bravais lattice volume, the number of lattice sites in the primitive cell

vastly preferable to the
dynamical systems forward-in-time formulation

wisdom of solid state physicists

exact stability exponent

is given by bands over the Brillouin zone

traditional periodic orbit theory[7,8,9]

alles falsch :(

is not smart :
finite periodic states orbit Jacobians are only approximations

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[7] M. C. Gutzwiller, J. Math. Phys. **8**, 1979–2000 (1967).

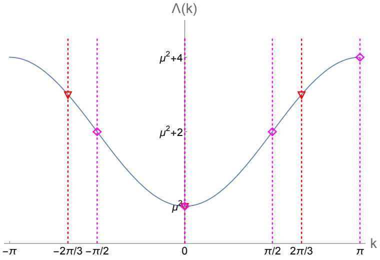
[8] D. Ruelle, Bull. Amer. Math. Soc **82**, 153–157 (1976).

[9] P. Cvitanović et al., *Chaos: Classical and Quantum*, (Niels Bohr Inst., Copenhagen, 2025).

temporal lattice orbit Jacobian operator spectra $\Lambda(k)$

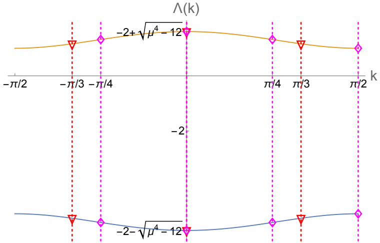
smooth curves : Brillouin zone bands[10]
discrete points : orbit Jacobian matrix spectrum consists of n eigenvalues embedded into $\Lambda(k)$

1D compact boson



period 3 (triangles)
period 4 (diamonds)

1D ϕ^3 theory



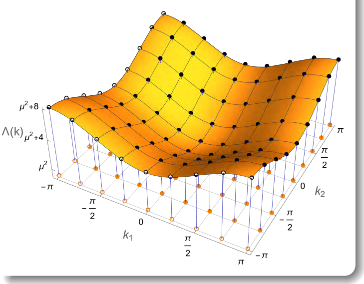
period 2 $\Phi_{LR} = \{\phi_L, \phi_R\}$
period 6 (triangles)
period 8 (diamonds)

[10] H. Liang and P. Cvitanović, J. Phys. A **55**, 304002 (2022).

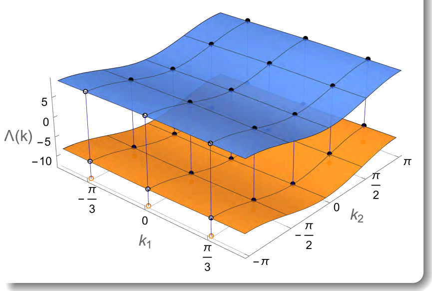
spatiotemporal lattice orbit Jacobian operator spectra (k_1, k_2)

smooth surfaces : Brillouin zone bands

massive compact boson



ϕ^4 theory in 2D



black dots : orbit Jacobian matrix eigenvalues,
finite volume primitive cells

[left] primitive cell periodicity $[8 \times 8]_0$
[right] primitive cell tiled by repeats of $[2 \times 1]_0$ periodic state

wisdom of solid state physicists

in 2D spacetime, the stability exponent

$$\lambda_c = \frac{1}{V_c} \ln \mathrm{Det}\, \mathcal{J}_c$$

is given by the band integral over the Brillouin zone

exact stability exponent of a periodic state c

$$\lambda_c = \frac{1}{(2\pi)^2} \int_{-\pi/L_c}^{\pi/L_c} \int_{-\pi/T_c}^{\pi/T_c} dk_1\, dk_2\, \ln\left(p(k_1)^2 + p(k_2)^2 + \mu_c^2\right),$$

'lattice momentum' $p = 2\sin\frac{k}{2}$

can you do it analytically ?

recap : fluid turbulence is described by

deterministic field theory

deterministic partition function :
sum over the deterministic solutions

definition : deterministic field theory

deterministic partition function has support only on the solutions Φ_c to saddle-point condition

$$F[\Phi_c]_z = \frac{\delta S[\Phi_c]}{\delta \phi_z} = 0$$

which we refer to as

defining equations

$$F[\Phi_c]_z = 0$$

note : works both for dissipative and Hamiltonian systems

deterministic partition sum

Φ_c is an exact, deterministic solution, so its

probability density

is V_c-dimensional Dirac delta function (determinism !!!)

$$p_c(\Phi) \,=\, \frac{1}{Z}\,\delta(F[\Phi])$$

and[11]

deterministic partition sum

$$Z[\mathsf{J}] = \sum_c \int_{\mathcal{M}_c} [d\Phi]\,\delta(F[\Phi]) e^{\Phi\cdot\mathsf{J}} = \sum_c \frac{e^{\Phi_c\cdot\mathsf{J}}}{|\mathrm{Det}\,\mathcal{J}_c|}$$

$\mathcal{M}_c$ = small neighborhood of periodic state Φ_c

sum over probabilities of all periodic states over primitive cell $\mathbb{A}$

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[11] P. Cvitanović and H. Liang, *A chaotic lattice field theory in two dimensions*, 2025.

new !

deterministic partition sum is a-mazing !

literally the sum over all periodic states c

$$Z[\beta, s] = \sum_c t_c$$
$$t_c = \left(e^{\beta \cdot v_c - \lambda_c - s}\right)^{V_c}$$

- t_c : weight of periodic state c
- λ_c : stability exponent
- v_c : Birkhoff average of observable v over periodic state Φ_c
- V_c : Bravais lattice volume
- s : ‘entropy’ parameter

field theorist's chaos

definition : chaos is

expanding	stability exponents	λ_c
exponential ♯	periodic states	$N_{\mathbb{A}}$

the precise sense in which
a field theory is deterministically chaotic

note : there is no 'time' in this definition

3rd millennium experiment 2

turbulence in 3 dimensions : Taylor-Couette duct

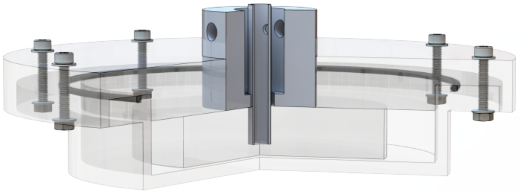


Figure 3.1: CAD model of the TCF cell. The cell is made of transparent PMMA allowing for unobstructed, optical access to the entire flow domain.

C.J. Crowley PhD Thesis

Taylor-Couette duct : full 3D flow visualization

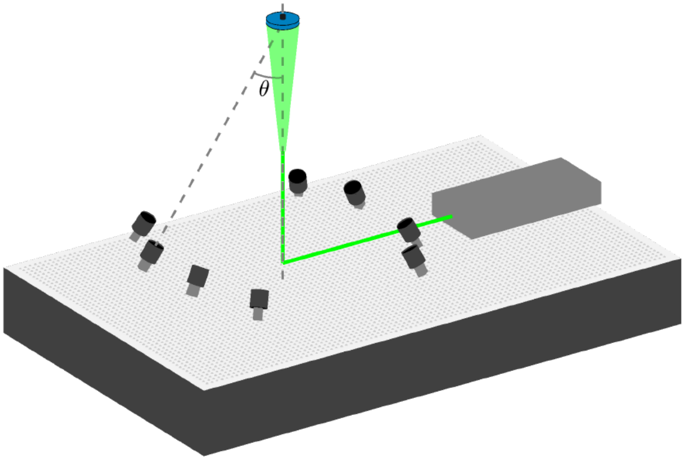


Figure 3.9: Camera configuration for 3D-3C measurements. The viewing angle, θ, is the angle between the camera and the z-axis of the TCF cell.

C.J. Crowley PhD Thesis

Taylor-Couette duct : a turbulent snapshot

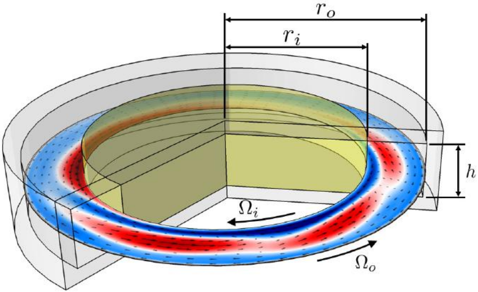
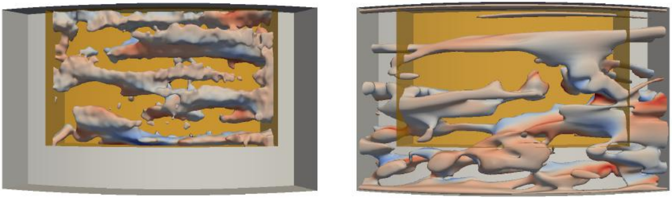


Figure 5.1: Turbulence is visualized in a laboratory flow between concentric, independently-rotating cylinders with radii r_i, r_o and corresponding angular velocities Ω_i, Ω_o. Fluid is confined between the cylinders and bounded axially by end caps co-rotating with the outer cylinder. The red-white-blue colors indicate the fluid's deviation from the mean azimuthal velocity component.

C.J. Crowley PhD Thesis

Taylor-Couette duct : experiment /DNS velocity isosurfaces



(a) (b)

Figure 4.5: A snapshot of a turbulent flow in experiment (a) and DNS (b). Each image shows a single isosurface of the perturbation field, $\tilde{\mathbf{v}}_\theta$, for $Re_\mathrm{i} = 650$ and $Re_\mathrm{o} = -1000$ inside a cylindrical subvolume. The color indicates the corresponding azimuthal velocity component. Red (blue) indicates flow in the same direction as the inner (outer) cylinder rotation.

C.J. Crowley PhD Thesis

Taylor-Couette duct : three state-space visualizations

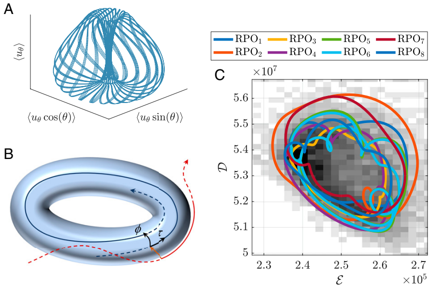
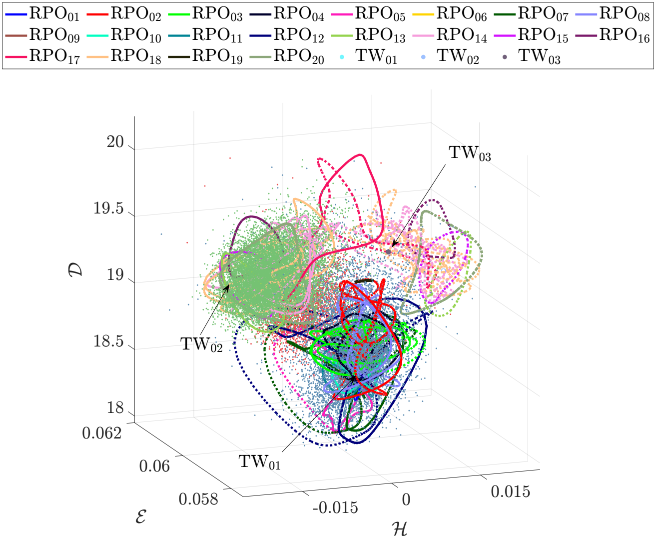


Fig. 2. Low-dimensional projections suggest that RPOs, i.e., solutions to the governing equations that recur indefinitely in time, are relevant to turbulence. (*A*) To demonstrate that RPOs are truly two-tori when rotational symmetry is not reduced, RPO_2 is plotted over 80 periods using the coordinates shown, where u_θ represents the azimuthal component of the flow velocity and $\langle \cdot \rangle$ indicates a spatial average. (*B*) Cartoon depicting how a portion of a turbulent trajectory (solid red curve) shadows, i.e., follows, an RPO (light blue surface) for a period of time. Shown in dark blue is the trajectory belonging to the RPO, which is most similar to the turbulent trajectory. The orange arrow relates a point on the turbulent trajectory to the point closest to it on the torus. (*C*) Using energy $\mathcal{E}$ and energy dissipation rate $\mathcal{D}$ of the flow as projection coordinates, eight RPOs are represented by closed trajectories (shown in color). The chaotic behavior of turbulence is indicated by the distribution (shown in gray) of visits to particular regions of the projection (darker regions have higher likelihood of visitation).

Taylor-Couette duct : RPOs embedded in invariant measure



Taylor-Couette duct : experiment trolls theory

examples of the experimental turbulent flow visiting (shadowing) relative periodic orbits (RPOs)

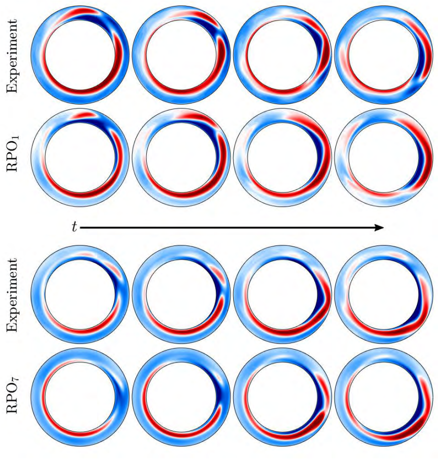


Figure 5.9: Experimental evidence that turbulence and RPOs, i.e., solutions to the governing equations that reccur indefinitely in time, co-evolve when the ‘shadowing’ criteria are met. Turbulence closely follows RPO_1 (top) and, during a different time interval, tracks RPO_7 (bottom).

experiment trolls theory : a movie of the experimental turbulent flow visiting (shadowing) a relative periodic orbit

click here to see the online video

walk!

the first experimental measurements

1. of a Navier-Stokes predicted unstable manifold
2. of shadowing by a Navier-Stokes predicted relative periodic orbit

Conclusions: geometry of Navier Stokes

dual ODE / CFD representations of Navier–Stokes

State space portraits

Computed eigenvalues, eigenfunctions of equilibrium states

Heteroclinic connections between equilibria

Turbulent dynamics

you can do this at home : channelflow.org

openpipeflow.org

orbithunter

periodic orbit theory

then this happens

the 3rd theorist says : wait ! if you have a symmetry,

you must use it !

replace the partition sum by the zeta function :

$$Z[\beta, s] = \frac{d}{ds} \ln \zeta[\beta, s]$$

the two solid state guys get up, go to another bar

what's up with $\zeta[\beta, s]$?

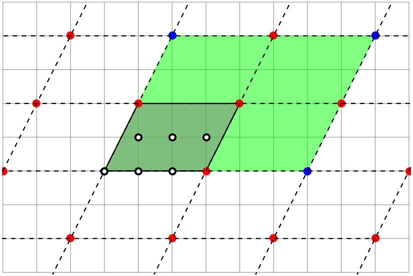
new !

deterministic zeta function

prime periodic state (not obvious)

Φ_c is either prime,
or a repeat of a prime periodic state Φ_p

Bravais lattice



Bravais lattice $[6\times4]_2$, blue dots, is a sublattice of $[3\times2]_1$, blue and red dots

prime periodic state : primitive cell is a $[3\times2]_1$ (gray)
4^{th}-repeat of a prime : primitive cell is $[6\times4]_2$ (green)

wisdom of mathematicians

for every translational symmetry, replace the partition sum over periodic states Φ_c

'Selberg' trace formula

$$Z[\beta, s] = \sum_c t_c$$

by sum over prime periodic states Φ_p,

deterministic zeta function

a product over all prime orbits[a]

$$1/\zeta = \prod_p 1/\zeta_p \,, \qquad 1/\zeta_p = \prod_{n=1}^{\infty}(1 - t_p^n)$$

2 spatiotemporal dim : Euler function (1741)

[a] J. Bell, *Euler and the pentagonal number theorem*, 2005.

what is all this good for ?

a theory of turbulence ? you need it !

have : a turbulent pipe flow

Q : how much power?

to move crude at velocity $\langle v \rangle$?

zeta function predicts

expectation value of any observable 'v'

$$\langle v \rangle = \left. \frac{\frac{\partial \zeta[\beta, s]}{\partial \beta}}{\frac{\partial \zeta[\beta, s]}{\partial s}} \right|_{\beta=0, s=s_0} = \frac{[\text{observable}]}{[\text{lattice volume}]}$$

where one needs to[12]

average observable over each prime orbit

$$\langle v \rangle_p = \frac{1}{V_{\mathbb{A}}} \sum_{z}^{\mathbb{A}} v(\Phi_p)_z$$

· · · details · · · · · ·

[12] P. Cvitanović et al., *Chaos: Classical and Quantum*, (Niels Bohr Inst., Copenhagen, 2025).

bye bye, dynamics

1. Q. : describe states of turbulence in infinite spatiatemporal domains
2. A. : determine, weigh all prime spatiotemporal periodic states

there is no more time

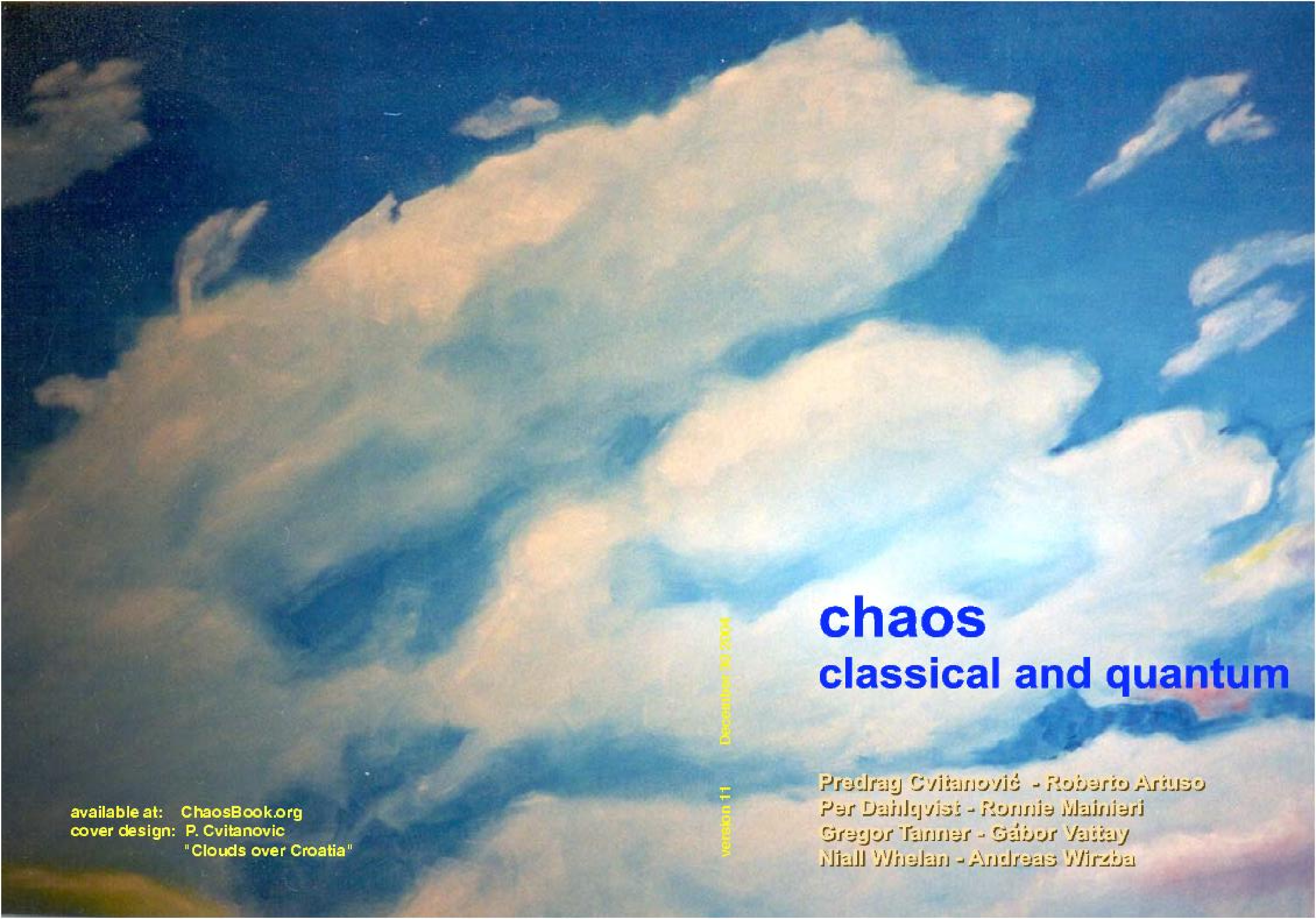
there is only determination of
admissible spacetime periodic states

for a deep dive

chaotic field theory talks, papers $\Rightarrow$

ChaosBook.org/overheads/spatiotemporal

Future looks bright


chaos
classical and quantum
Predrag Cvitanović - Roberto Artuso
Per Dahlqvist - Ronnie Mainieri
Gregor Tanner - Gábor Vattay
Niall Whelan - Andreas Wirzba
version 11 December 30 2004
available at: ChaosBook.org
cover design: P. Cvitanovic
"Clouds over Croatia"

if anyone asks : extra slides

how is deterministic field theory different from other theories?

- we always work in the ‘broken-symmetry’ regime, as almost every ‘turbulent’ (spatiotemporally chaotic) solution breaks all symmetries
- we work ‘beyond perturbation theory’, in the anti-integrable, strong coupling regime. This are not weak coupling expansions around a ground state
- our ‘far from equilibrium’ field theory is not driven by external noise. All chaoticity is due to the intrinsic deterministic instabilities
- our formulas are exact, not merely saddle points approximations to the exact theory

ODEs, PDEs linear operators wisdom

Hill's 1886 formula[13]

Gel'fand-Yaglom 1960 theorem[14]

orbit Jacobian operator $\mathcal{J}$ is fundamental

temporal evolution Jacobian matrix J is merely
one of the methods to compute it

all of dynamical systems theory is subsumed in spatiotemporal field-theoretic formulation

[13] G. W. Hill, Acta Math. **8**, 1–36 (1886).

[14] I. M. Gel'fand and A. M. Yaglom, J. Math. Phys. **1**, 48–69 (1960).

orbit stability vs. temporal stability

orbit Jacobian matrix

$\mathcal{J}_{z'z} = \frac{\delta F[\Phi]_{z'}}{\delta \phi_z}$ stability under global perturbation of the whole orbit

for n large, a huge $[dn \times dn]$ matrix

temporal Jacobian matrix

J propagates initial perturbation n time steps

small $[d \times d]$ matrix

J and $\mathcal{J}$ are related by[15]

Hill's 1886 remarkable formula

$$|\mathrm{Det}\, \mathcal{J}_\mathrm{M}| = |\det(\mathbf{1} - J_\mathrm{M})|$$

$\mathcal{J}$ is huge, even ∞-dimensional matrix
J is tiny, few degrees of freedom matrix

[15] G. W. Hill, Acta Math. **8**, 1–36 (1886).

wisdom of solid state physicists

in 1D temporal lattice, the stability exponent

$$\lambda_c = \frac{1}{n_c} \ln \mathrm{Det}\, \mathcal{J}_c$$

is given by the band integral over the Brillouin zone

exact stability exponent of a periodic state c

$$\begin{aligned} \lambda_c &= \frac{1}{2\pi} \int_{-\pi/n_c}^{\pi/n_c} dk \ln \left[4 \sin^2 \frac{n_c k}{2} + \mu_c^2 \right] \\ &= \ln \mu_c^2 + 2 \ln \frac{1 + \sqrt{1 + 4/\mu_c^2}}{2} \end{aligned}$$

expectation value of an observable

deterministic partition sum

sum over all deterministic solutions c

$$Z[\beta, s] = \sum_c t_c$$
$$t_c = \left(e^{\beta \cdot v_c - \lambda_c - s}\right)^{V_c}$$

- λ_c : stability exponent
- v_c : Birkhoff average, observable v over periodic state Φ_c
- V_c : Φ_c Bravais lattice volume

observables

for a deterministic solution Φ_c, the *Birkhoff average* of observable v is

$$v[\Phi]_c = \frac{1}{V_c} \sum_{z \in \mathbb{A}} v_z$$

for example, if observable $v_z = \phi_z$, the Birkhoff average is the average 'height' ϕ_z