a field theory of turbulence

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ChaosBook.org/overheads/spatiotemporal
$\rightarrow$ chaotic field theory talks, papers

Statphys in Lviv

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a theory of turbulence ? who needs it ?

have : a turbulent pipe flow

Q : how much power?

to move crude at velocity $\langle v \rangle$?

how much power?

velocity v of a fluid element is an 'observable'

to evaluate

expectation value

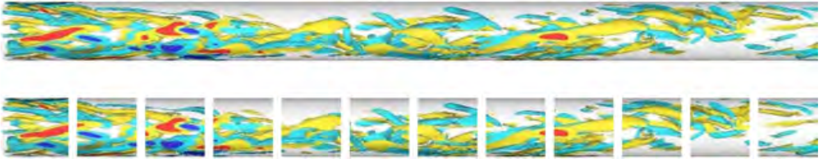
of observable v averaged over all field configurations

$$\langle v \rangle \quad = \quad \int d\Phi \, p[\Phi] \, v[\Phi] \, , \quad d\Phi = \prod_z d\phi_z$$

need to know the probability of every field configuration Φ

have : a theory of turbulence

a turbulent pipe flow[1]



we have a detailed theory of small turbulent fluid cells[2]

can we can we construct the infinite pipe by tiling it by small turbulent configurations Φ?

Q. what would that theory look like ?

A. it’s here : this talk

[1] M. Avila and B. Hof, Phys. Rev. E **87**, 063012 (2013).

[2] J. F. Gibson et al., J. Fluid Mech. **611**, 107–130 (2008).

the answer is :

new ![3]

spatiotemporal zeta function

$$\zeta[\beta, s] = \prod_p \zeta_p$$

$\Phi_p \quad = \quad$ a prime (non-repeating) multi-periodic state

will explain

[3] P. Cvitanović and H. Liang, *A chaotic lattice field theory in two dimensions*, 2025.

wisdom of statistical mechanicians

partition function

field configuration Φ occurs with probability

$$p(\Phi) = \frac{1}{Z} e^{-S[\Phi]}, \qquad Z = Z[0]$$

partition function $=$ sum over all field configurations

$$Z[\mathrm{J}] = \int [d\phi]\, e^{-S[\Phi]+\Phi\cdot\mathrm{J}}, \qquad [d\phi] = \prod_z \frac{d\phi_z}{\sqrt{2\pi}}$$

how likely is a configuration Φ?

$$p(\Phi) = \frac{1}{Z}\, e^{-S[\Phi]}$$

this talk : determine probability of field configuration Φ

semiclassical field theory

sum over all deterministic configurations !

path integral

field configuration Φ occurs with probability amplitude

$$p(\Phi) \; = \; \frac{1}{Z} \, e^{\frac{i}{\hbar} S[\Phi]} \, , \qquad Z = Z[0]$$

partition sum $=$ integral over all configurations

$$Z[\mathrm{J}] \; = \; \int [d\phi] \, e^{\frac{i}{\hbar} (S[\Phi] + \Phi \cdot \mathrm{J})} \, , \qquad [d\phi] = \prod_z \frac{d\phi_z}{\sqrt{2\pi}}$$

evaluate how ?
here : WKB or semiclassical approximation

method of stationary phase

defining equations

$$\frac{\delta S[\Phi_c]}{\delta \phi_z} = 0$$

a global deterministic solution Φ_c satisfies this local extremal condition on every spacetime point z

WKB or semiclassical approximation

$$S[\Phi] = S[\Phi_c] + \frac{1}{2}(\Phi - \Phi_c)^\top \mathcal{J}_c\,(\Phi - \Phi_c) + \cdots$$

orbit Jacobian operator

$$(\mathcal{J}_c)_{z'z} = \left.\frac{\delta^2 S[\Phi]}{\delta \phi_{z'} \delta \phi_z}\right|_{\Phi=\Phi_c}$$

semiclassical field theory

deterministic solution Φ_c probability amplitude

$$p(\Phi_c) = \frac{1}{Z} \frac{e^{iS[\Phi_c]+im_c}}{|\mathrm{Det}\, \mathcal{J}_c|^{1/2}}\,, \qquad Z = Z[0]$$

partition sum : support on deterministic solutions

$$Z[\mathsf{J}] = \sum_c \frac{e^{i(S[\Phi_c]+m_c+\Phi_c\cdot \mathsf{J})}}{|\mathrm{Det}\, \mathcal{J}_c|^{1/2}}$$

example : Gutzwiller trace formula[4]

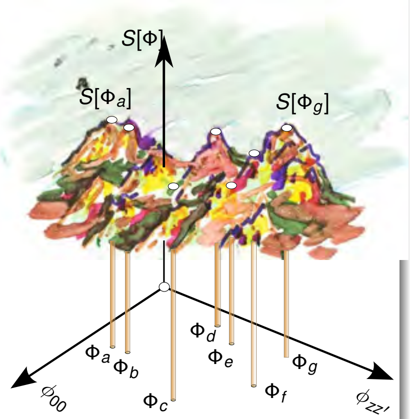
$$\int [d\Phi]\, A[\Phi]\, e^{iS[\Phi]} \approx \sum_c A[\Phi_c] \frac{e^{iS[\Phi_c]+im_c}}{|\mathrm{Det}\, \mathcal{J}_c|^{1/2}}$$

$1D$ time evolution quantum mechanics, so not field theory

[4] M. C. Gutzwiller, J. Math. Phys. **8**, 1979–2000 (1967).

bird’s eye view

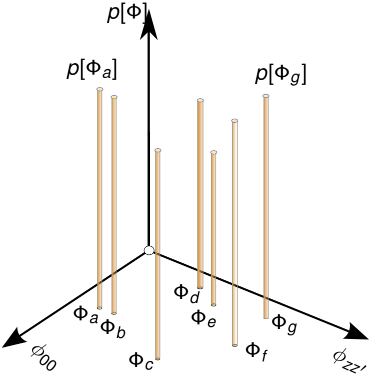
semiclassical field theory



$$\langle v\rangle \approx \sum_c v[\Phi_c] \frac{e^{\frac{i}{\hbar}S[\Phi_c]+im_c}}{|\mathrm{Det}\,\mathcal{J}_c|^{1/2}}$$

deterministic field theory

is its WKB backbone



$$\langle v\rangle = \sum_c v[\Phi_c] \frac{1}{|\mathrm{Det}\,\mathcal{J}_c|}$$

fluid turbulence is described by

new !

deterministic field theory

deterministic partition function :

sum over the deterministic solutions

first : determine "all"

periodic states

think globally, act locally

definition : periodic state is

a global deterministic solution

$$\begin{aligned} \Phi_c &= \{\phi_{c,z}\} \\ &= \text{ set of field values} \end{aligned}$$

periodic along each translationally invariant direction

that satisfies the

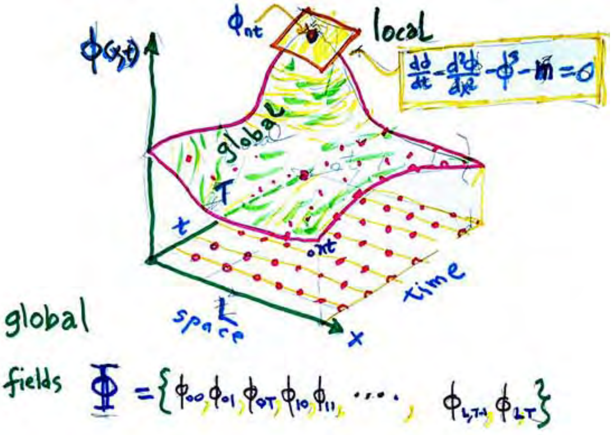
local condition : defining equations

$$F[\Phi_c]_z = 0$$

on every spacetime point z of multi-periodic primitive cell $\mathbb{A}$

think globally, act locally

a global deterministic solution Φ_{nt}



satisfies the local defining equations
everywhere all at once

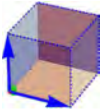
computing periodic states

search for zeros of defining equations

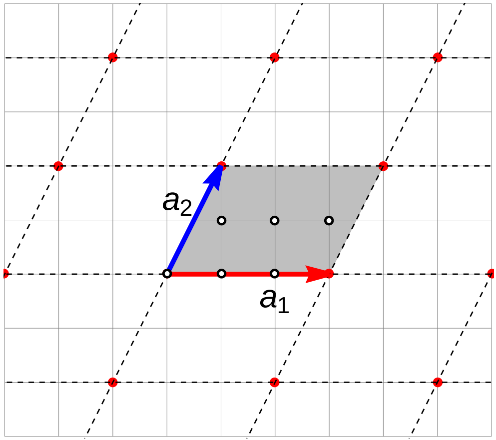
$$F[\Phi_c]_z = 0$$

the entire global periodic state Φ_c over primitive cell $\mathbb{A}$ is

a single point $(\phi_1, \phi_2, \cdots, \phi_n)$

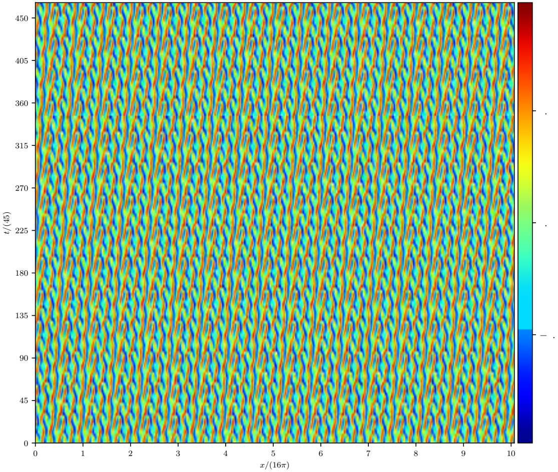
in the V_c-dimensional state space, $\Phi_c \in$ 

periodic state's primitive cell



primitive cell $\mathbb{A} = [3\times 2]_1$ that tiles a relative-periodic state

an example : spacetime tiled by a larger tile



Kuramoto-Sivashinsky tiling by
relative-periodic state $(L, T) = (33.73, 35)$

Gudorf 2018

can hierarchically compute 'all' solutions

orbitHunter

optimization of rough initial guesses converges

no exponential instabilities

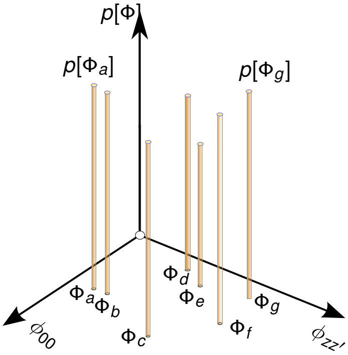
stability : compute on reciprocal lattice

gitHub code[5]

[5] M. N. Gudorf, *Orbithunter: Framework for Nonlinear Dynamics and Chaos*, tech. rep. (School of Physics, Georgia Inst. of Technology, 2021).

stability exponents

Dirac porcupine



$$\langle v \rangle = \sum_c v[\Phi_c] \frac{1}{|\mathrm{Det}\, \mathcal{J}_c|}$$

The Importance of Being Φ_c

Φ_c is an exact, deterministic solution, so its

probability density

is V_c-dimensional Dirac delta function (!!! determinism !!!)

$$p_c(\Phi) = \frac{1}{Z}\,\delta(F[\Phi])$$

and

probability weight of deterministic solution Φ_c

$$\int_{\mathcal{M}_c} [d\Phi]\,\delta(F[\Phi]) = \frac{1}{|\mathrm{Det}\,\mathcal{J}_c|}$$

$\mathcal{M}_c$ = small neighborhood of periodic state Φ_c

our task : evaluate $\mathrm{Det}\,\mathcal{J}_c$

the most important thing : understand perturbations

- find a deterministic solution

$$F[\Phi_c]_z = 0 \qquad \text{fixed point condition}$$

- then evaluate $\mathrm{Det}\,\mathcal{J}_c$ of

orbit Jacobian operator

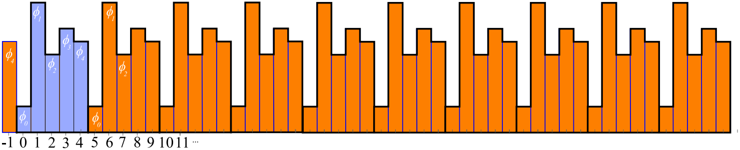
$$(\mathcal{J}_c)_{z'z} = \frac{\delta F[\Phi_c]_{z'}}{\delta \phi_z}$$

what does this global orbit Jacobian operator do?

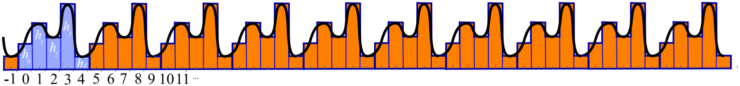
global stability

of periodic state Φ_c, perturbed everywhere

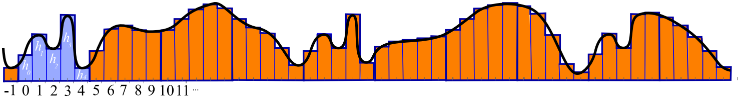
perturbations are into full state space



repeats of a period-5 periodic state Φ_c



an internal perturbation h_z, periodicity of Φ_c, has discrete spectrum, evaluated over Φ_c's primitive cell



a transverse perturbation h_z has continuous spectrum, evaluated over Φ_c's Brillouin zone[6]

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[6] A. S. Pikovsky, Phys. Lett. A **137**, 121–127 (1989).

the most critical thing

new ! functional 'fluctuation' determinant

$$\mathrm{Det}\,\mathcal{J}_c$$

must be computed on the

infinite Bravais lattice

stability exponent of periodic state Φ_c

new ! assign to each periodic state c *stability exponent* λ_c per unit-spacetime-volume

exact deterministic weight

$$\frac{1}{|\mathrm{Det}\,\mathcal{J}_c|} = e^{-V_c \lambda_c}$$

in any spacetime dimension

- λ_c : stability exponent
- V_c : Φ_c Bravais lattice volume, the number of lattice sites in the primitive cell

vastly preferable to the
dynamical systems forward-in-time formulation

wisdom of solid state physicists

exact stability exponent

is given by bands over the Brillouin zone

traditional periodic orbit theory[7,8,9]

alles falsch :(

is not smart :

finite periodic states orbit Jacobians are only approximations

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[7] M. C. Gutzwiller, J. Math. Phys. **8**, 1979–2000 (1967).

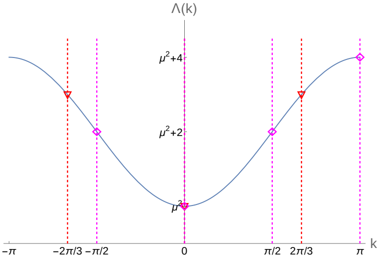
[8] D. Ruelle, Bull. Amer. Math. Soc **82**, 153–157 (1976).

[9] P. Cvitanović et al., *Chaos: Classical and Quantum*, (Niels Bohr Inst., Copenhagen, 2025).

temporal lattice orbit Jacobian operator spectra $\Lambda(k)$

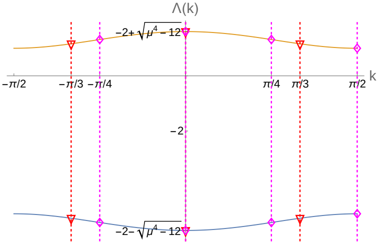
smooth curves : Brillouin zone bands[10]
discrete points : orbit Jacobian matrix spectrum consists of n eigenvalues embedded into $\Lambda(k)$

1D compact boson



period 3 (triangles)
period 4 (diamonds)

1D ϕ^3 theory



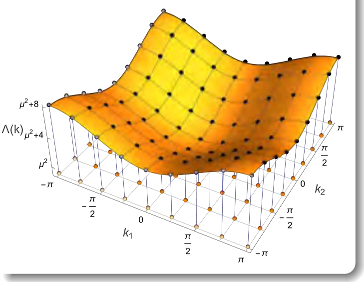
period 2 $\Phi_{LR} = \{\phi_L, \phi_R\}$
period 6 (triangles)
period 8 (diamonds)

[10] H. Liang and P. Cvitanović, J. Phys. A **55**, 304002 (2022).

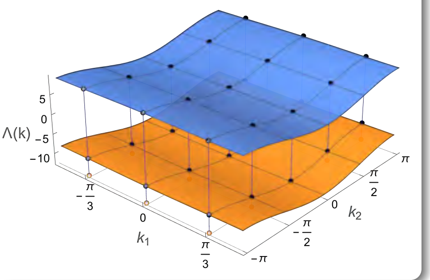
spatiotemporal lattice orbit Jacobian operator spectra (k_1, k_2)

smooth surfaces : Brillouin zone bands

massive compact boson



ϕ^4 theory in 2D



black dots : orbit Jacobian matrix eigenvalues,
finite volume primitive cells

[left] primitive cell periodicity $[8\times8]_0$
[right] primitive cell tiled by repeats of $[2\times1]_0$ periodic state

wisdom of solid state physicists

in 2D spacetime, the stability exponent

$$\lambda_c = \frac{1}{V_c} \ln \mathrm{Det}\, \mathcal{J}_c$$

is given by the band integral over the Brillouin zone

exact stability exponent of a periodic state c

$$\lambda_c = \frac{1}{(2\pi)^2} \int_{-\pi/L_c}^{\pi/L_c} \int_{-\pi/T_c}^{\pi/T_c} dk_1\, dk_2 \, \ln\left(p(k_1)^2 + p(k_2)^2 + \mu_c^2\right),$$

'lattice momentum' $p = 2 \sin \frac{k}{2}$

can you do it analytically ?

recap : fluid turbulence is described by

deterministic field theory

deterministic partition function :
sum over the deterministic solutions

definition : deterministic field theory

deterministic partition function has support only on the solutions Φ_c to saddle-point condition

$$F[\Phi_c]_z = \frac{\delta S[\Phi_c]}{\delta \phi_z} = 0$$

which we refer to as

defining equations

$$F[\Phi_c]_z = 0$$

note : works both for dissipative and Hamiltonian systems

deterministic partition sum

Φ_c is an exact, deterministic solution, so its

probability density

is V_c-dimensional Dirac delta function (determinism !!!)

$$p_c(\Phi) = \frac{1}{Z}\delta(F[\Phi])$$

and[11]

deterministic partition sum

$$Z[\mathsf{J}] = \sum_c \int_{\mathcal{M}_c} [d\Phi]\,\delta(F[\Phi])e^{\Phi\cdot\mathsf{J}} = \sum_c \frac{e^{\Phi_c\cdot\mathsf{J}}}{|\mathrm{Det}\,\mathcal{J}_c|}$$

$\mathcal{M}_c$ = small neighborhood of periodic state Φ_c

sum over probabilities of all periodic states over primitive cell $\mathbb{A}$

[11] P. Cvitanović and H. Liang, *A chaotic lattice field theory in two dimensions*, 2025.

new !

deterministic partition sum is a-mazing !

literally the sum over all periodic states c

$$
\begin{aligned}
Z[\beta, s] &= \sum_c t_c \\
t_c &= \left(e^{\beta \cdot v_c - \lambda_c - s}\right)^{V_c}
\end{aligned}
$$

- t_c : weight of periodic state c
- λ_c : stability exponent
- v_c : Birkhoff average of observable v over periodic state Φ_c
- V_c : Bravais lattice volume
- s : ‘entropy’ parameter

field theorist's chaos

definition : chaos is

expanding	stability exponents	λ_c
exponential $\sharp$	periodic states	$N_{\mathbb{A}}$

the precise sense in which
a field theory is deterministically chaotic

note : there is no 'time' in this definition

periodic orbit theory

then this happens

the 3rd theorist says : wait ! if you have a symmetry,

you must use it !

replace the partition sum by the zeta function :

$$Z[\beta, s] = \frac{d}{ds} \ln \zeta[\beta, s]$$

the two solid state guys get up, go to another bar

what's up with $\zeta[\beta, s]$?

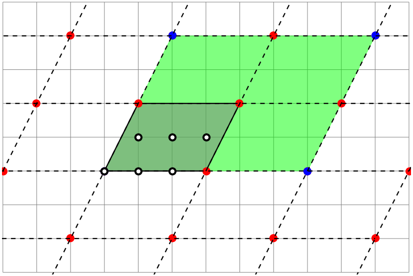
new !

deterministic zeta function

prime periodic state (not obvious)

Φ_c is either prime,
or a repeat of a prime periodic state Φ_p

Bravais lattice



Bravais lattice $[6\times4]_2$, blue dots, is a sublattice of $[3\times2]_1$, blue and red dots

prime periodic state : primitive cell is a $[3\times2]_1$ (gray)
4^{th}-repeat of a prime : primitive cell is $[6\times4]_2$ (green)

wisdom of mathematicians

for every translational symmetry, replace the partition sum over periodic states Φ_c

'Selberg' trace formula

$$Z[\beta, s] = \sum_c t_c$$

by sum over prime periodic states Φ_p,

deterministic zeta function

a product over all prime orbits[a]

$$1/\zeta = \prod_p 1/\zeta_p\,, \qquad 1/\zeta_p = \prod_{n=1}^{\infty}(1 - t_p^n)$$

2 spatiotemporal dim : Euler function (1741)

[a] J. Bell, *Euler and the pentagonal number theorem*, 2005.

what is all this good for ?

have : a turbulent pipe flow

Q : how much power?

to move crude at velocity $\langle v \rangle$?

zeta function predicts

expectation value of any observable 'v'

$$\langle v \rangle = \left. \frac{\frac{\partial \zeta[\beta,s]}{\partial \beta}}{\frac{\partial \zeta[\beta,s]}{\partial s}} \right|_{\beta=0,s=s_0} .$$

where one needs to[12]

average observable over each prime orbit

$$\langle v \rangle_p \;\; = \frac{1}{V_{\mathbb{A}}} \sum_z^{\mathbb{A}} v(\Phi_p)_z$$

$\cdots$ details $\cdots\cdots$

[12] P. Cvitanović et al., *Chaos: Classical and Quantum*, (Niels Bohr Inst., Copenhagen, 2025).

bye bye, dynamics

1. Q. : describe states of turbulence in infinite spatiatemporal domains
2. A. : determine, weigh all prime spatiotemporal periodic states

there is no more time

there is only determination of

admissible spacetime periodic states

for a deep dive

chaotic field theory talks, papers $\Rightarrow$

ChaosBook.org/overheads/spatiotemporal

if anyone asks : extra slides

how is deterministic field theory different from other theories?

- we always work in the ‘broken-symmetry’ regime, as almost every ‘turbulent’ (spatiotemporally chaotic) solution breaks all symmetries
- we work ‘beyond perturbation theory’, in the anti-integrable, strong coupling regime. This are not weak coupling expansions around a ground state
- our ‘far from equilibrium’ field theory is not driven by external noise. All chaoticity is due to the intrinsic deterministic instabilities
- our formulas are exact, not merely saddle points approximations to the exact theory

ODEs, PDEs linear operators wisdom

Hill's 1886 formula[13]

Gel'fand-Yaglom 1960 theorem[14]

orbit Jacobian operator $\mathcal{J}$ is fundamental

temporal evolution Jacobian matrix J is merely
one of the methods to compute it

all of dynamical systems theory is subsumed in spatiotemporal field-theoretic formulation

[13] G. W. Hill, Acta Math. **8**, 1–36 (1886).

[14] I. M. Gel'fand and A. M. Yaglom, J. Math. Phys. **1**, 48–69 (1960).

orbit stability vs. temporal stability

orbit Jacobian matrix

$\mathcal{J}_{z'z} = \frac{\delta F[\Phi]_{z'}}{\delta \phi_z}$ stability under global perturbation of the whole orbit

for n large, a huge $[dn \times dn]$ matrix

temporal Jacobian matrix

J propagates initial perturbation n time steps

small $[d \times d]$ matrix

J and $\mathcal{J}$ are related by[15]

Hill's 1886 remarkable formula

$$|\mathrm{Det}\,\mathcal{J}_\mathrm{M}| = |\det(\mathbf{1} - J_\mathrm{M})|$$

$\mathcal{J}$ is huge, even ∞-dimensional matrix
J is tiny, few degrees of freedom matrix

[15] G. W. Hill, Acta Math. **8**, 1–36 (1886).

wisdom of solid state physicists

in 1D temporal lattice, the stability exponent

$$\lambda_c = \frac{1}{n_c} \ln \mathrm{Det}\, \mathcal{J}_c$$

is given by the band integral over the Brillouin zone

exact stability exponent of a periodic state c

$$\begin{aligned}\lambda_c &= \frac{1}{2\pi}\int_{-\pi/n_c}^{\pi/n_c} dk \ln\left[4\sin^2\frac{n_c k}{2} + \mu_c^2\right] \\ &= \ln \mu_c^2 + 2\ln\frac{1+\sqrt{1+4/\mu_c^2}}{2}\end{aligned}$$

expectation value of an observable

deterministic partition sum

sum over all deterministic solutions c

$$
\begin{aligned}
Z[\beta, s] &= \sum_c t_c \\
t_c &= \left(e^{\beta \cdot v_c - \lambda_c - s}\right)^{V_c}
\end{aligned}
$$

- λ_c : stability exponent
- v_c : Birkhoff average, observable v over periodic state Φ_c
- V_c : Φ_c Bravais lattice volume

observables

for a deterministic solution Φ_c, the *Birkhoff average* of observable v is

$$
v[\Phi]_c = \frac{1}{V_c} \sum_{z \in \mathbb{A}} v_z
$$

for example, if observable $v_z = \phi_z$, the Birkhoff average is the average 'height' ϕ_z