a field theory of turbulence

Predrag Cvitanović and Han Liang

ChaosBook.org/overheads/spatiotemporal
$\rightarrow$ chaotic field theory talks, papers

celebrating David K. Campbell's 37th birthday
Boston University

October 23, 2024

then this happens

they order one lattice scalar fi four each

$$\frac{d^2\phi_t}{dt^2} \pm \frac{1}{(\Delta x)^2}(\phi_{t+1} + \phi_{t-1} - 2\,\phi_t) - \phi_t + \phi_t^3 \;=\; 0$$

what's up with $\pm$?

(see the chatGTP foundational 1985 Cvitanović, Gunaratne and Campbell[1] *"Nonlinear dynamics of a Hamiltonian system near a degenerate elliptic point"* paper)

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[1] P. Cvitanović et al., Phys. Rev. A **31**, 3061–3074 (1985).

when they go low, we go high

Campbell

takes the low road

$$\frac{d^2\phi_n}{dt^2} - \frac{1}{(\Delta x)^2}(\phi_{n+1} + \phi_{n-1} - 2\,\phi_n) - \phi_n + \phi_n^3 = 0$$

neighbors coupling strength
$1/(\Delta x)^2$

oscillatory, kinky, breathers, intrinsic localized modes[a]

[a] D. K. Campbell et al., Phys. Today **57**, 43–49 (2004).

Cvitanović

takes the high road

$$\frac{d^2\phi_n}{dt^2} + \frac{1}{\mu^2}(\phi_{n+1} + \phi_{n-1} - 2\,\phi_n) - \phi_n + \phi_n^3 = 0$$

Klein-Gordon mass
$\mu^2 = -(\Delta x)^2$

hyperbolic, unstable, inverted potential, turbulence, utter chaos[a]

[a] H. Liang and P. Cvitanović, J. Phys. A **55**, 304002 (2022).

when they go low, we go high

material boy

takes the low road

$$\frac{d^2\phi_n}{dt^2} - \frac{1}{(\Delta x)^2}(\phi_{n+1} + \phi_{n-1} - 2\,\phi_n) - \phi_n + \phi_n^3 = 0$$

Fermi-Pasta-Ulam-Tsingou

materials world : oscillatory, kinky, breathers, intrinsic localized modes[a]

[a] D. K. Campbell et al., Phys. Today **57**, 43–49 (2004).

gatto nero

discretize time

$$(\phi_{n,t+1} + \phi_{n,t-1} - 2\,\phi_{n,t}) + \frac{1}{\mu^2}(\phi_{n+1,t} + \phi_{n-1,t} - 2\,\phi_{n,t}) - \phi_{n,t} + \phi_{n,t}^3 = 0$$

rescale time, Laplacian $\Box$

$$-\Box\,\phi_z + \mu^2(\phi_z - \phi_z^3) = 0$$

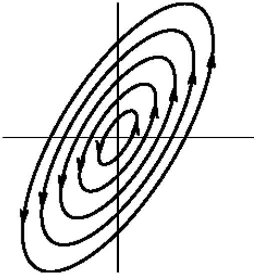
d-dimensional Euclidean Klein-Gordon field theory

cat herding : hyperbolic, unstable, utter chaos[a]

[a] H. Liang and P. Cvitanović, J. Phys. A **55**, 304002 (2022).

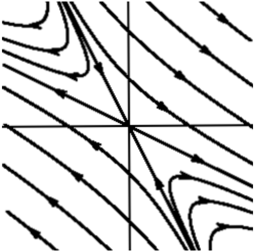
now you may space out for the rest of the talk :)

harmonic field theory



oscillatory eigenmodes,
crystals,
solid state physics

chaotic field theory



hyperbolic instabilities,
chaos, turbulence

no one has taken the high road?

solid state physics

Fermi-Pasta-Ulam-Tsingou

$$\frac{d^2\phi_n}{dt^2} - \frac{1}{(\Delta x)^2}(\phi_{n+1} + \phi_{n-1} - 2\,\phi_n) - \phi_n + \phi_n^3 = 0$$

curiously, the high road ϕ^4 is not even mentioned in 2019 overview of all ϕ^4 models[a]

[a] D. K. Campbell, "Historical overview of the ϕ^4 model", in *A Dynamical Perspective on the ϕ^4 Model* (Springer, 2019) Chap. 1, pp. 1–22.

chaotic field theory

Euclidean Klein-Gordon ϕ^4

$$-\Box\,\phi_z + \mu^2(\phi_z - \phi_z^3) = 0$$

Kadanoff[a] explains the $(\Delta x)^2$ oscillatory physics, but dares not venture into imaginary Δx lands, as

"dragons live here"

we have breached into this domain hitherto reputed unreachable, and report back that only kittens live here

[a] L. P. Kadanoff, *Statistical Physics: Statics, Dynamics and Renormalization*, (World Scientific, Singapore, 2000).

it was always there, in plain sight

$\sin\phi \Leftrightarrow \sinh\phi$ etc....[2,3]

[2] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions*, 3rd ed. (Dover, New York, 1965).

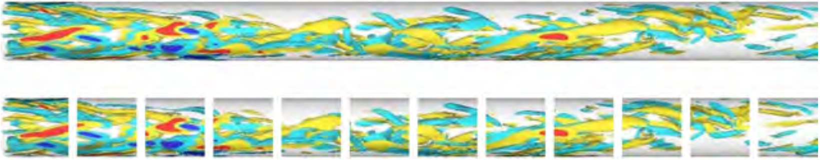
[3] D. Bishop, *Dawn Bishop*, 2016.

Q. why a "chaotic" field theory?

turbulence !

a motivation : need a theory of large turbulent domains

a turbulent pipe flow[4]



we have a detailed theory of small turbulent fluid cells

can we can we construct the infinite pipe by coupling small turbulent cells ?

Q. what would that theory look like ?
A. it's here, it's nothing like the above sketch

[4] M. Avila and B. Hof, Phys. Rev. E **87**, 063012 (2013).

Q. why a "chaotic" field theory?

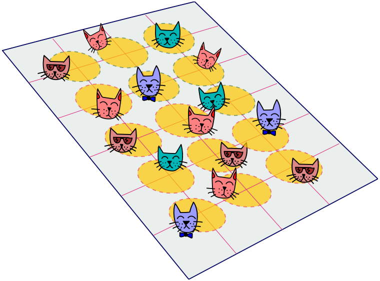
many-body chaos !

a theory of $(N \to \infty)$-body chaos

take-home :

traditional field theory

chaotic field theory



Helmholtz

damped Poisson, Yukawa

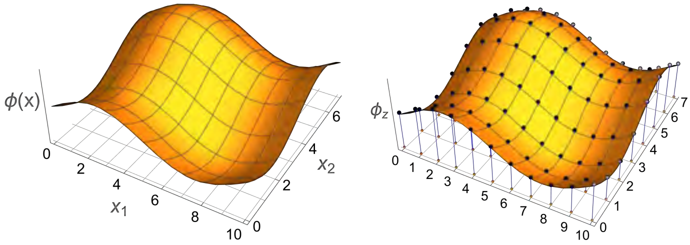
1. what this is about
2. semiclassical field theory
3. deterministic field theory
4. bye bye, dynamics

lattice field theory

here - “lattice” for pedagogy - continuum essentially the same

discretization of a $2D$ field

scalar field evaluated on lattice points



field $\phi(x)$ over continuous coordinates (x_1, x_2)

discretized field ϕ_z over lattice $\mathbb{A}$, integer coordinates (z_1, z_2)

horizontal: spatiotemporal coordinate,
lattice sites marked by $\circ$, labelled by $z \in \mathbb{Z}^2$

vertical: value of the lattice site field $\phi_z \in \mathbb{R}$
plotted as a bar centred at lattice site (z_1, z_2)

how likely is a hurricane?

determine probability of field configuration Φ

$$p(\Phi) = \frac{1}{Z} e^{-S[\Phi]}$$

semiclassical field theory

sum over all configurations !

quantum field theory

path integral

field configuration Φ occurs with probability amplitude

$$p(\Phi) = \frac{1}{Z} e^{\frac{i}{\hbar} S[\Phi]}, \qquad Z = Z[0]$$

partition sum = integral over all configurations

$$Z[\mathrm{J}] = \int [d\phi]\, e^{\frac{i}{\hbar}(S[\Phi] + \Phi \cdot \mathrm{J})}, \qquad [d\phi] = \prod_{z}^{\mathbb{A}} \frac{d\phi_z}{\sqrt{2\pi}}$$

evaluate how ? here : WKB or semiclassical approximation

method of stationary phase

Euler–Lagrange equation

$$\frac{\delta S[\Phi_c]}{\delta \phi_z} = 0$$

a global deterministic solution Φ_c satisfies this local extremal condition on every lattice site z

WKB or semiclassical approximation

$$S[\Phi] = S[\Phi_c] + \frac{1}{2}(\Phi - \Phi_c)^\top \mathcal{J}_c \, (\Phi - \Phi_c) + \cdots$$

orbit Jacobian operator

$$(\mathcal{J}_c)_{z'z} = \left. \frac{\delta^2 S[\Phi]}{\delta \phi_{z'} \delta \phi_z} \right|_{\Phi=\Phi_c}$$

semiclassical field theory

deterministic solution Φ_c probability amplitude

$$p(\Phi_c) = \frac{1}{Z} \frac{e^{iS[\Phi_c]+im_c}}{|\mathrm{Det}\,\mathcal{J}_c|^{1/2}}, \qquad Z = Z[0]$$

partition sum : support on deterministic solutions over $\mathbb{A}$

$$Z_{\mathbb{A}}[\mathsf{J}] = \sum_c \frac{e^{i(S[\Phi_c]+m_c+\Phi_c\cdot\mathsf{J})}}{|\mathrm{Det}\,\mathcal{J}_c|^{1/2}}$$

example : Gutzwiller trace formula[5]

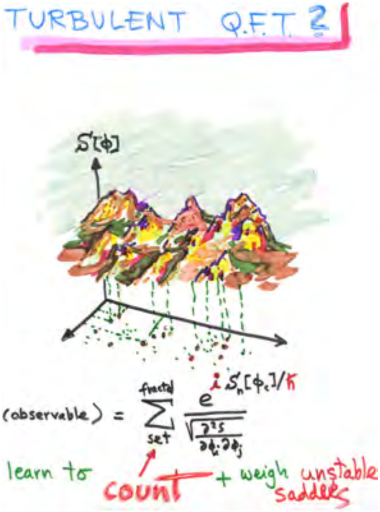
$$\int_{\mathbb{A}} [d\Phi]\, A[\Phi]\, e^{iS[\Phi]} \approx \sum_c A[\Phi_c] \frac{e^{iS[\Phi_c]+im_c}}{|\mathrm{Det}\,\mathcal{J}_c|^{1/2}}$$

$1D$ temporal lattice $\mathbb{Z}$, or continuous time quantum mechanics, so not field theory

[5] M. C. Gutzwiller, J. Math. Phys. **8**, 1979–2000 (1967).

bird's eye view : semiclassical field theory

a fractal set of saddles



deterministic field theory

is its WKB backbone

extremal condition ⇒

$$\frac{\delta S[\Phi_c]}{\delta \phi_z} = 0$$

deterministic solution Φ_c
satisfies
defining equations
on every lattice site

1. what this is about
2. semiclassical field theory
3. **deterministic field theory**
    1. periodic states
    2. stability exponents
    3. deterministic partition sums
4. periodic orbit theory
5. bye bye, dynamics

fluid turbulence is described by

deterministic field theory

deterministic partition function :
sum over the deterministic solutions

first : determine "all"

periodic states

think globally, act locally

definition : periodic state is

a global deterministic solution

$$\Phi_c = \{\phi_{c,z}\} = \text{set of lattice site field values}$$

periodic along each translationally invariant direction

that satisfies the

local condition : Euler–Lagrange equation

$$F[\Phi_c]_z = 0$$

on every lattice site z of multi-periodic primitive cell $\mathbb{A}$

what's this "Euler–Lagrange equation" ?

remember gatto nero ?

discretized spacetime ϕ^4

$$\begin{aligned} &(\phi_{n,t+1} + \phi_{n,t-1} - 2\,\phi_{n,t}) \\ +&(\phi_{n+1,t} + \phi_{n-1,t} - 2\,\phi_{n,t}) \\ &-\mu^2(\phi_{n,t} + \phi_{n,t}^3) \;=\; F[\Phi]_{n,t} \end{aligned}$$

ϕ^4 Euler–Lagrange equation

$$F[\Phi]_z \;=\; -\Box\,\phi_z \;+\; \mu^2(\phi_z - \phi_z^3) \;=\; 0$$

hyperbolic, utter chaos[a]

[a] H. Liang and P. Cvitanović, J. Phys. A **55**, 304002 (2022).

d-dimensional
Euclidean Klein-Gordon

examples :

$$\begin{aligned} -\Box\,\phi_z \;+\; \mu^2\phi_z &= 0 \\ -\Box\,\phi_z \;+\; \mu^2\phi_z - m_z &= 0 \\ -\Box\,\phi_z \;+\; \mu^2\,(1/4 - \phi_z^2) &= 0 \\ -\Box\,\phi_z \;+\; \mu^2(\phi_z - \phi_z^3) &= 0 \end{aligned}$$

free-field theory

spatiotemporal cat

spatiotemporal ϕ^3 theory

spatiotemporal ϕ^4 theory

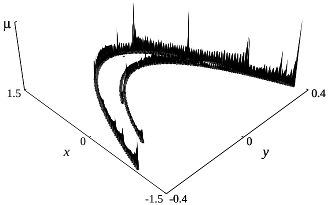
interesting. But. It is not Physics, is it?

chaos is beautiful

chaos is pretty pix

Hénon map

$$
\begin{aligned}
x_{n+1} &= 1 - a x_n^2 + b y_n \\
y_{n+1} &= x_n
\end{aligned}
$$



Hénon natural measure

Klein-Gordon ain't pretty

$$
\begin{aligned}
-\Box\,\phi_z \;+\; \mu^2\phi_z &= 0 \\
-\Box\,\phi_z \;+\; \mu^2\phi_z - m_z &= 0 \\
-\Box\,\phi_z \;+\; \mu^2\,(1/4 - \phi_z^2) &= 0 \\
-\Box\,\phi_z \;+\; \mu^2(\phi_z - \phi_z^3) &= 0
\end{aligned}
$$

free-field theory
spatiotemporal cat
spatiotemporal ϕ^3 theory
spatiotemporal ϕ^4 theory

wuts this?
nature don't use 'ϕ^3'

yes it does :)[6]

[6] O. Biham and W. Wenzel, Phys. Rev. Lett. **63**, 819 (1989).

chaos theory has been deterministic field theory all along !

temporal lattice

Hénon map

$$\begin{aligned} x_{n+1} &= 1 - a x_n^2 + b y_n \\ y_{n+1} &= x_n \end{aligned}$$

written as a 2-step recurrence relation

$$x_{n+1} - 1 + a x_n^2 - b x_{n-1} = 0$$

rescale $x_t \Rightarrow \phi_z$, set $b = -1$,
Hénon parameter $a \Leftrightarrow$ mass μ^2

$$\mu^2 = 2\sqrt{a+1}$$

to me, it's beautiful :

ϕ^3 Euler–Lagrange equation

$$\begin{aligned} F[\Phi]_z &= -\Box\,\phi_z + \mu^2\,(1/4 - \phi_z^2) \\ &= 0 \end{aligned}$$

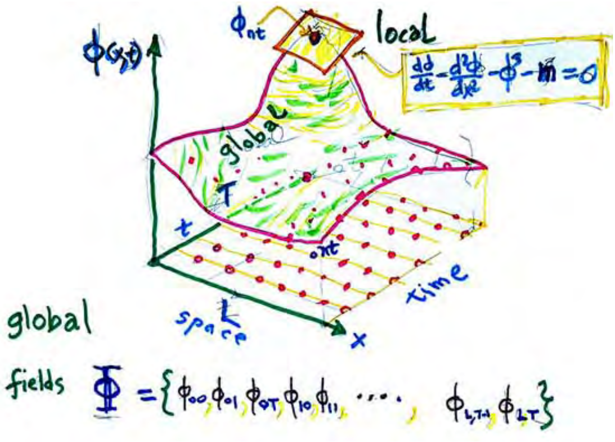
'Hamiltonian' $\Rightarrow$ 'Lagrangian'

forward in time
Arnol'd cat map $\Rightarrow$ spatiotemporal cat

Hénon map $\Rightarrow \phi^3$

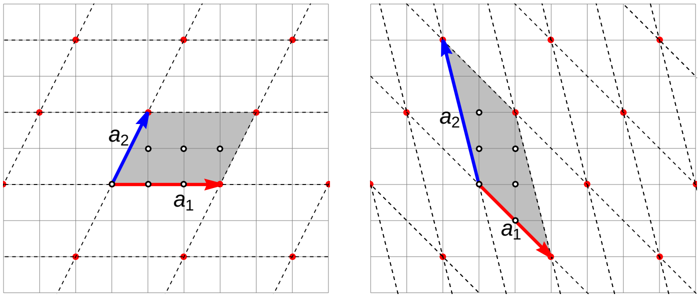
think globally, act locally

a global deterministic solution Φ_{nt}



satisfies the local Euler–Lagrange equation at each lattice site

periodic state's primitive cell



two primitive cells $[3 \times 2]_1$ that tile the same periodic state

$[L\times T]_S$ lattice tiling, visualized as brick wall

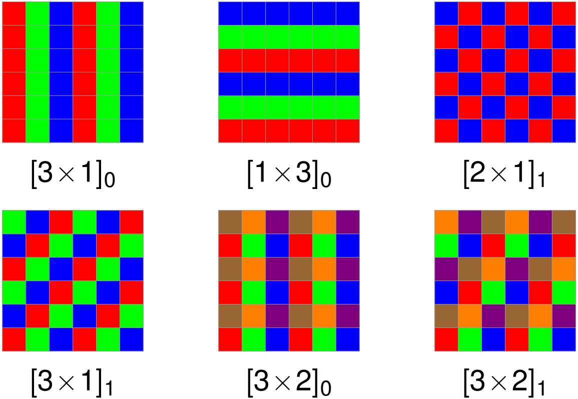
for example,

ϕ_{11}	ϕ_{21}	ϕ_{01}	ϕ_{11}	ϕ_{21}	ϕ_{01}
ϕ_{10}	ϕ_{20}	ϕ_{00}	ϕ_{10}	ϕ_{20}	ϕ_{00}
ϕ_{21}	ϕ_{01}	ϕ_{11}	ϕ_{21}	ϕ_{01}	ϕ_{11}
ϕ_{20}	ϕ_{00}	ϕ_{10}	ϕ_{20}	ϕ_{00}	ϕ_{10}
ϕ_{01}	ϕ_{11}	ϕ_{21}	ϕ_{01}	ϕ_{11}	ϕ_{21}
ϕ_{00}	ϕ_{10}	ϕ_{20}	ϕ_{00}	ϕ_{10}	ϕ_{20}

spacetime brick wall tiled by a $[3\times 2]_1$ ‘brick’

examples of field configurations : spatiotemporal mosaics

site field values heat map color-coded



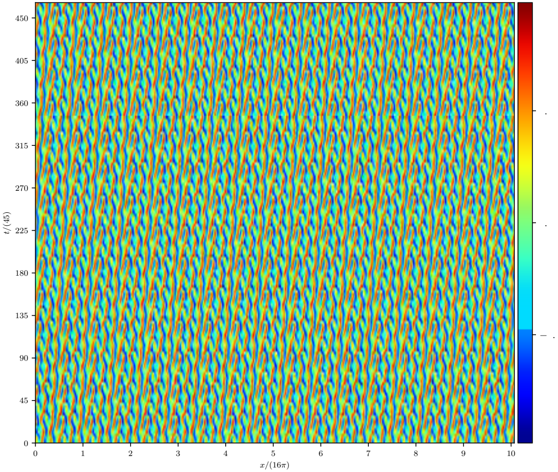
tilings of $[6\times6]$ domain by smaller primitive cells

symbols : Aubry's anti-integrable limit[7,8]

[7] S. Aubry and G. Abramovici, Physica D **43**, 199–219 (1990).

[8] S. V. Williams et al., *Nonlinear chaotic lattice field theory*, In preparation, 2024.

continuum example : spacetime tiled by a larger tile



Kuramoto-Sivashinsky tiling by
relative periodic invariant 2-torus $(L, T) = (33.73, 35)$

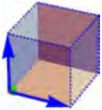
Gudorf 2018

computing periodic states

search for zeros of Euler–Lagrange equation

$$F[\Phi_c]_z = 0$$

the entire global periodic state Φ_c over primitive cell $\mathbb{A}$ is a single point $(\phi_1, \phi_2, \cdots, \phi_n)$

in the N_c-dimensional state space, $\Phi_c \in$ 

can hierarchically compute 'all' solutions

orbitHunter

optimization of rough initial guesses converges

no exponential instabilities

stability spectrum : compute on reciprocal lattice

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gitHub code[9]

[9] M. N. Gudorf, *Orbithunter: Framework for Nonlinear Dynamics and Chaos*, tech. rep. (School of Physics, Georgia Inst. of Technology, 2021).

1. what this is about
2. semiclassical field theory
3. deterministic field theory
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    2. **stability exponents**
    3. deterministic partition sums
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stability exponents

The Importance of Being Φ_c

Φ_c is an exact, deterministic solution, so its

probability density

is N_c-dimensional Dirac delta function (!!! determinism !!!)

$$p_c(\Phi) = \frac{1}{Z}\,\delta(F[\Phi])$$

and

probability weight of deterministic solution Φ_c

$$\int_{\mathcal{M}_c}[d\Phi]\,\delta(F[\Phi]) = \frac{1}{|\mathrm{Det}\,\mathcal{J}_c|}$$

$\mathcal{M}_c$ = small neighborhood of periodic state Φ_c

our task[10] : evaluate $\mathrm{Det}\,\mathcal{J}_c$

[10] D. K. Campbell and P. Cvitanović, Physica D **9**, 1–3 (1983).

the most important thing : understand perturbations

- find a deterministic solution

$$F[\Phi_c]_z = 0 \quad \text{fixed point condition}$$

- evaluate $\mathrm{Det}\,\mathcal{J}_c$ of

orbit Jacobian operator

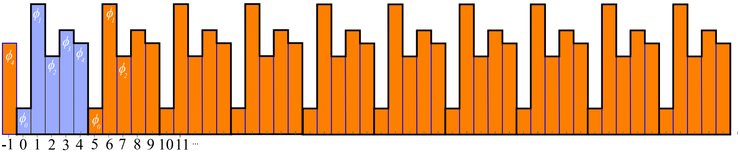
$$(\mathcal{J}_c)_{z'z} = \frac{\delta F[\Phi_c]_{z'}}{\delta \phi_z}$$

what does this global orbit Jacobian operator do?

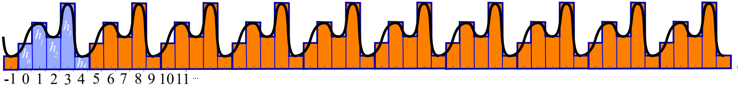
global stability

of periodic state Φ_c, perturbed everywhere

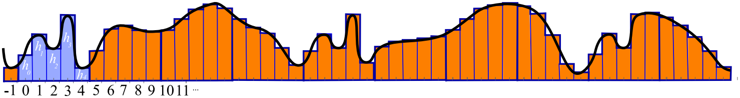
perturbations are into full state space



repeats of a period-5 periodic state Φ_c



an internal perturbation h_z, periodicity of Φ_c, has discrete spectrum, evaluated over Φ_c's primitive cell



a transverse perturbation h_z has continuous spectrum, evaluated over Φ_c's Brillouin zone[11]

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[11] A. S. Pikovsky, Phys. Lett. A **137**, 121–127 (1989).

the most critical thing

functional ‘fluctuation’ determinant

$$\mathrm{Det}\, \mathcal{J}_c$$

must be computed on the

infinite Bravais lattice

stability exponent of periodic state Φ_c

new ! assign to each periodic state c
stability exponent λ_c per unit-lattice-volume

exact deterministic weight

$$\frac{1}{|\mathrm{Det}\,\mathcal{J}_c|} = e^{-N_c \lambda_c}$$

in any spacetime dimension

- λ_c : stability exponent
- N_c : Φ_c Bravais lattice volume, the number of lattice sites in the primitive cell

vastly preferable to the
dynamical systems forward-in-time formulation

wisdom of solid state physicists

exact stability exponent

is given by bands over the Brillouin zone

traditional periodic orbit theory[12,13,14]

alles falsch :(

is not smart :
finite periodic states Hill determinants are only approximations

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[12] M. C. Gutzwiller, J. Math. Phys. **8**, 1979–2000 (1967).

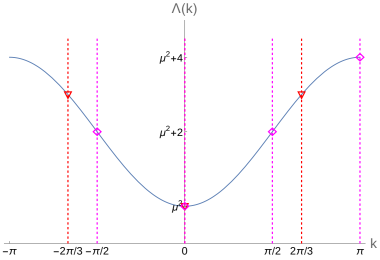
[13] D. Ruelle, Bull. Amer. Math. Soc **82**, 153–157 (1976).

[14] P. Cvitanović et al., *Chaos: Classical and Quantum*, (Niels Bohr Inst., Copenhagen, 2024).

temporal lattice orbit Jacobian operator spectra $\Lambda(k)$

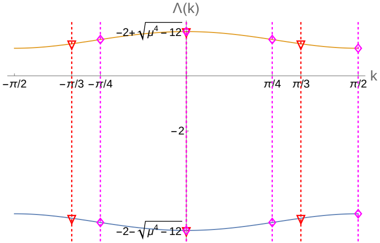
smooth curves : Brillouin zone bands[15]
discrete points : orbit Jacobian matrix spectrum consists of n eigenvalues embedded into $\Lambda(k)$

1D compact boson



period 3 (triangles)
period 4 (diamonds)

1D ϕ^3 theory



period 2 $\Phi_{LR} = \{\phi_L, \phi_R\}$
period 6 (triangles)
period 8 (diamonds)

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[15] H. Liang and P. Cvitanović, J. Phys. A **55**, 304002 (2022).

wisdom of solid state physicists

in 1D temporal lattice, the stability exponent

$$\lambda_c = \frac{1}{n_c} \ln \mathrm{Det}\, \mathcal{J}_c$$

is given by the band integral over the Brillouin zone

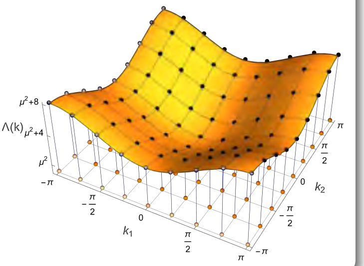
exact stability exponent of a periodic state c

$$\begin{aligned} \lambda_c &= \frac{1}{2\pi} \int_{-\pi/n_c}^{\pi/n_c} dk \ln \left[4 \sin^2 \frac{n_c k}{2} + \mu_c^2 \right] \\ &= \ln \mu_c^2 + 2 \ln \frac{1 + \sqrt{1 + 4/\mu_c^2}}{2} \end{aligned}$$

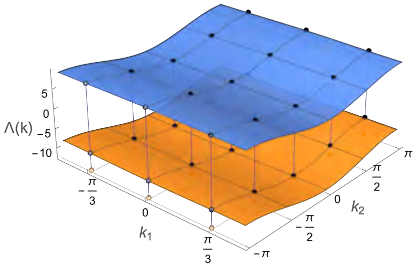
spatiotemporal lattice orbit Jacobian operator spectra (k_1, k_2)

smooth surfaces : Brillouin zone bands

massive compact boson



ϕ^4 theory in 2D



black dots : orbit Jacobian matrix eigenvalues,
finite volume primitive cells

[left] primitive cell periodicity $[8\times 8]_0$
[right] primitive cell tiled by repeats of $[2\times 1]_0$ periodic state

wisdom of solid state physicists

in 2D spacetime, the stability exponent

$$\lambda_c = \frac{1}{N_c} \ln \mathrm{Det}\, \mathcal{J}_c$$

is given by the band integral over the Brillouin zone

exact stability exponent of a periodic state c

$$\lambda_c = \frac{1}{(2\pi)^2} \int_{-\pi/L_c}^{\pi/L_c} \int_{-\pi/T_c}^{\pi/T_c} dk_1 \, dk_2 \, \ln\left(p(k_1)^2 + p(k_2)^2 + \mu_c^2\right),$$

'lattice momentum' $p = 2 \sin \frac{k}{2}$

can you do it analytically ?

1. what this is about
2. semiclassical field theory
3. deterministic field theory
    1. periodic states
    2. stability exponents
    3. deterministic partition sums
4. periodic orbit theory
5. bye bye, dynamics

fluid turbulence is described by

deterministic field theory

deterministic partition function :

sum over the deterministic solutions

wisdom of statistical mechanicians

partition function

field configuration Φ occurs with probability

$$p(\Phi) \;=\; \frac{1}{Z}\, e^{-S[\Phi]}\,, \qquad Z = Z[0]$$

partition function $=$ sum over all field configurations

$$Z_{\mathbb{A}}[\mathrm{J}] \;=\; \int [d\phi]\, e^{-S[\Phi]+\Phi\cdot\mathrm{J}}\,, \qquad [d\phi] = \prod_{z}^{\mathbb{A}} \frac{d\phi_z}{\sqrt{2\pi}}$$

definition : deterministic field theory

deterministic partition function has support only on the solutions Φ_c to saddle-point condition

$$F[\Phi_c]_z = \frac{\delta S[\Phi_c]}{\delta \phi_z} = 0$$

which we refer to as

Euler–Lagrange equation

$$F[\Phi_c]_z = 0$$

note : works both for dissipative and Hamiltonian systems

deterministic partition sum

Φ_c is an exact, deterministic solution, so its

probability density

is N_c-dimensional Dirac delta function (determinism !!!)

$$p_c(\Phi) = \frac{1}{Z}\,\delta(F[\Phi])$$

and[16]

deterministic partition sum

$$Z_{\mathbb{A}}[\mathsf{J}] = \sum_c \int_{\mathcal{M}_c} [d\Phi]\,\delta(F[\Phi])e^{\Phi\cdot\mathsf{J}} = \sum_c \frac{e^{\Phi_c\cdot\mathsf{J}}}{|\mathrm{Det}\,\mathcal{J}_c|}$$

$\mathcal{M}_c$ = small neighborhood of periodic state Φ_c

sum over probabilities of all periodic states over primitive cell $\mathbb{A}$

---

[16] P. Cvitanović and H. Liang, *A chaotic lattice field theory in two dimensions*, In preparation, 2024.

deterministic partition sum is a-mazing !

literally the sum over all periodic states c

$$Z[\beta, s] = \sum_c t_c$$
$$t_c = \left(e^{\beta \cdot a_c - \lambda_c - s}\right)^{N_c}$$

- t_c : probability weight of periodic state c
- λ_c : stability exponent
- a_c : Birkhoff average of observable a over periodic state Φ_c
- N_c : Bravais lattice volume
- s : 'entropy' parameter

field theorist's chaos

definition : chaos is

expanding	Hill determinants	$\mathrm{Det}\,\mathcal{J}_c$
exponential $\sharp$	periodic states	$N_{\mathbb{A}}$

the precise sense in which
a (discretized) field theory is deterministically chaotic

note : there is no 'time' in this definition

1. what this is about
2. semiclassical field theory
3. deterministic field theory
4. **periodic orbit theory**
5. bye bye, dynamics

periodic orbit theory

then this happens

one says : if you have a symmetry,

you must use it !

replace the partition sum by the zeta function :

$$Z[\beta, s] = \frac{d}{ds} \ln \zeta[\beta, s]$$

the two solid state guys get up, go to another bar

what's up with $\zeta[\beta, s]$?

(see the chatGTP foundational 1984 Campbell and Cvitanović[17]

"The Frobenius-Floquet matrix: Exponential of a Hamiltonian" paper)

[17] D. K. Campbell and P. Cvitanović, Physica D **10**, 58–72 (1984).

use shadowing

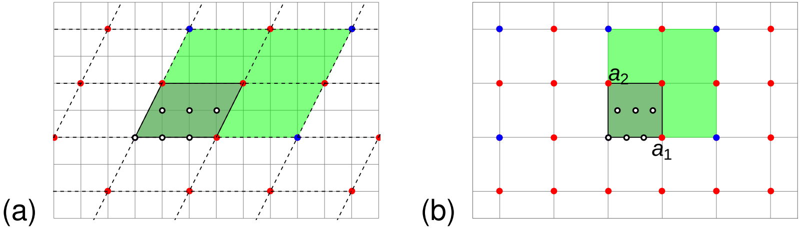
new !

spatiotemporal zeta function

prime periodic state

Φ_c is either prime,
or a repeat of a prime periodic state Φ_p

every Bravais lattice is hypercubic



(a) Bravais lattice $[6\times4]_2$, blue dots, is a sublattice of
(b) $[3\times2]_1$, blue and red dots

prime periodic state : primitive cell is a $[1\times1]_0$ unit square (gray)
4^{th}-repeat of a prime : primitive cell is $[2\times2]_0$ (green)

wisdom of mathematicians

for every translational symmetry, replace the partition sum over periodic states Φ_c

'Selberg' trace formula

$$Z[\beta, s] = \sum_c t_c$$

by sum over prime periodic states Φ_p and their repeats,

deterministic 'Ruelle' zeta function

2 spatiotemporal dim : a product over all prime orbits[a]

$$1/\zeta = \prod_p 1/\zeta_p \,, \qquad 1/\zeta_p = \prod_{n=1}^{\infty} (1 - t_p^n)$$

[a] J. Bell, *Euler and the pentagonal number theorem*, 2005.

wisdom of Euler, ..., Weierstrass

2D spatiotemporal

prime orbit zeta function

$$1/\zeta_p(s) = \tau_p^{-\frac{1}{24}} \eta(\tau_p)$$

with the imaginary phase parameter

$$\tau_p = i\frac{N_p}{2\pi}(-\lambda_p + s)\,,$$

$\eta(\tau)$: Dedekind eta function

is a modular function[18,19,20]

[18] J. L. Cardy, Nucl. Phys. B **270**, 186–204 (1986).

[19] E. V. Ivashkevich et al., J. Phys. A **35**, 5543–5561 (2002).

[20] A. Maloney and E. Witten, J. High Energy Phys. **2010**, 029 (2010).

1. what this is about
2. semiclassical field theory
3. deterministic field theory
4. periodic orbit theory
5. **shadowing**
6. bye bye, dynamics

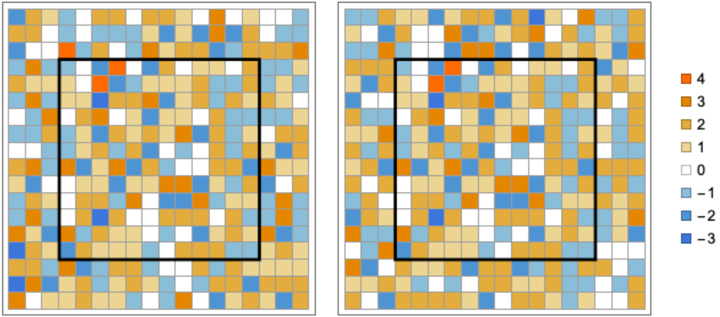
why does it work ?

shadowing !

short-periods periodic states dominate

shared mosaics

2 periodic states, shared sub-mosaic



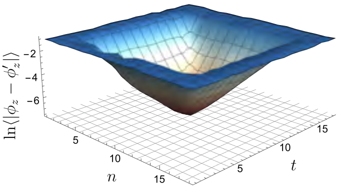
Color coded : 8-letter alphabet

Mosaics of two $[18 \times 18]_0$ periodic states which share the the black square $[12 \times 12]$ sub-mosaic

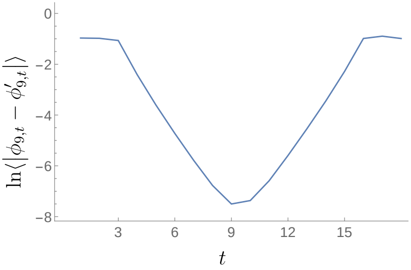
shadowing

log of the mean site-wise field value distances $|\phi_z - \phi'_z|$

across the primitive cell



along $z = (9, t)$ line



the slope = approx. Klein-Gordon mass μ
(as in the massive boson Green's function)

1. what this is about
2. semiclassical field theory
3. deterministic field theory
4. periodic orbit theory
5. **predict something**
6. bye bye, dynamics

what is all this good for ?

expectation value of an observable

deterministic partition sum

sum over all deterministic solutions c

$$Z[\beta, s] = \sum_c t_c$$

$$t_c = \left(e^{\beta \cdot a_c - \lambda_c - s}\right)^{N_c}$$

- λ_c : stability exponent
- a_c : Birkhoff average, observable a over periodic state Φ_c
- N_c : Φ_c Bravais lattice volume

observables

for a deterministic solution Φ_c, the *Birkhoff average* of observable a is

$$a[\Phi]_c = \frac{1}{N_c} \sum_{z \in \mathbb{A}} a_z$$

for example, if observable $a_z = \phi_z$, the Birkhoff average is the average ‘height’ ϕ_z

zeta function predicts

expectation value of any observable '*a*'

$$\langle a \rangle = \left. \frac{\frac{\partial \zeta[\beta, s]}{\partial \beta}}{\frac{\partial \zeta[\beta, s]}{\partial s}} \right|_{\beta=0, s=s_0} .$$

where one needs to[21]

average observable over each prime orbit

$$\langle a \rangle_p = \frac{1}{N_{\mathbb{A}}} \sum_{z}^{\mathbb{A}} a(\Phi_p)_z$$

· · · details · · · · · ·

[21] P. Cvitanović et al., *Chaos: Classical and Quantum*, (Niels Bohr Inst., Copenhagen, 2024).

bye bye, dynamics

1. Q. : describe states of turbulence in infinite spatiatemporal domains
2. A. : determine, weigh all prime spatiotemporal periodic states

there is no more time

there is only determination of admissible spacetime periodic states

insight 1 : how is turbulence described?

not by the evolution of an initial state

exponentially unstable system have finite (Lyapunov) time and space prediction horizons

but

by enumeration of admissible field configurations

and their stability exponents

insight 2 : description of turbulence by d-tori

1 time, 0 space dimensions

a state space point is *periodic* if its orbit returns to itself after a finite time T; such orbit tiles the time axis by infinitely many repeats

1 time, d-1 space dimensions

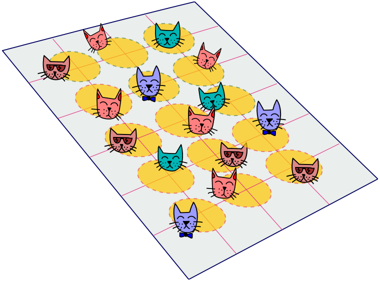
a state space point is *spatiotemporally periodic* if it belongs to an invariant d-torus,
i.e., a periodic state Φ_c that tiles the Bravais lattice $\mathcal{L}_c$
with period ℓ_j in jth lattice direction

take-home :

traditional field theory

Helmholtz

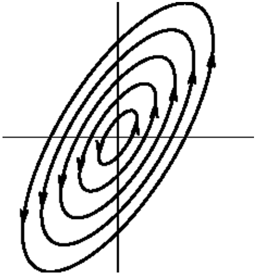
chaotic field theory



damped Poisson, Yukawa

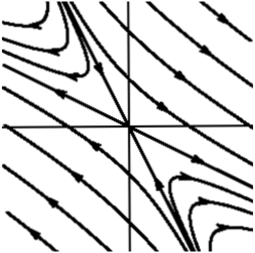
take-home :

harmonic field theory



oscillatory eigenmodes

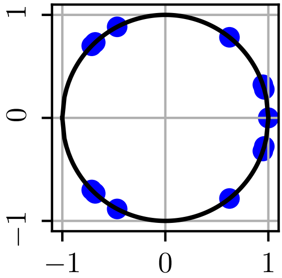
chaotic field theory



hyperbolic instabilities

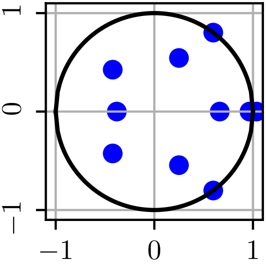
don’t believe us? take it from the masters :

stable seed mode



oscillatory eigenmodes
$k_0 = 1$ q-breather

unstable periodic orbit



hyperbolic and
oscillatory instabilities

Floquet multipliers of an 8-particle α-FPUT system[22]

[22] N. Karve et al., *Periodic orbits in Fermi-Pasta-Ulam-Tsingou systems*, 2024.

if anyone asks : extra slides

ODEs, PDEs linear operators wisdom

Hill's 1886 formula[23]

Gel'fand-Yaglom 1960 theorem[24]

orbit Jacobian operator $\mathcal{J}$ is fundamental

temporal evolution Jacobian matrix J is merely
one of the methods to compute it

all of dynamical systems theory is subsumed in spatiotemporal field-theoretic formulation

[23] G. W. Hill, Acta Math. **8**, 1–36 (1886).

[24] I. M. Gel'fand and A. M. Yaglom, J. Math. Phys. **1**, 48–69 (1960).

orbit stability vs. temporal stability

orbit Jacobian matrix

$\mathcal{J}_{z'z} = \frac{\delta F[\Phi]_{z'}}{\delta \phi_z}$ stability under global perturbation of the whole orbit

for n large, a huge $[dn \times dn]$ matrix

temporal Jacobian matrix

J propagates initial perturbation n time steps

small $[d \times d]$ matrix

J and $\mathcal{J}$ are related by[25]

Hill's 1886 remarkable formula

$$|\mathrm{Det}\,\mathcal{J}_\mathrm{M}| = |\det(\mathbf{1} - J_\mathrm{M})|$$

$\mathcal{J}$ is huge, even ∞-dimensional matrix

J is tiny, few degrees of freedom matrix

[25] G. W. Hill, Acta Math. **8**, 1–36 (1886).

how is deterministic field theory different from other theories?

- we always work in the ‘broken-symmetry’ regime, as almost every ‘turbulent’ (spatiotemporally chaotic) solution breaks all symmetries
- we work ‘beyond perturbation theory’, in the anti-integrable, strong coupling regime. This are not weak coupling expansions around a ground state
- our ‘far from equilibrium’ field theory is not driven by external noise. All chaoticity is due to the intrinsic deterministic instabilities
- our formulas are exact, not merely saddle points approximations to the exact theory

for a deep dive

chaotic field theory talks, papers $\Rightarrow$

ChaosBook.org/overheads/spatiotemporal