Cycle expansions

ChaosBook Chapter 23

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the big picture

Local v.s. **global** way of thinking

key idea (1): replace local time average over an ergodic trajectory by a global average over all periodic orbits

any dynamical average can be extracted from an evolution operator's leading eigenvalue

key idea (2): as long cycles are shadowed by short ones, short cycles give exponentially accurate dynamical averages

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Dynamical averaging

why?

Detailed prediction impossible in chaotic dynamics

Any initial condition will fill whole state space after finite Lyapunov time

Hence we cannot follow them for a long time

Examples of averages:

- transport coefficients: escape rates, mean drifts, diffusion rates
- entropies
- power spectra
- Lyapunov exponents

observable

observable: function $a(x)$ that associates to each point in state space a number, a vector or a tensor

observables report on a property of the dynamical system

integrated observable:

$$A(x_0, t) = \int_0^t \mathrm{d}\tau\, a(x(\tau)), \qquad x(t) = f^t(x_0) \tag{1}$$

if dynamics is given by an iterated mapping the integrated observable after *n* iterations is given by:

$$A(x_0, n) = \sum_{k=0}^{n-1} a(x_k), \qquad x_k = f^k(x_0) \tag{2}$$

periodic orbits

Define

$$A_p = \begin{cases} a_p T_p = \int_0^{T_p} \mathrm{d}\tau \, a(x(\tau)) & \text{for a flow} \\ a_p n_p = \sum_{i=1}^{n_p} a(x_i) & \text{for a map} \end{cases} \tag{3}$$

A_p is an integral / sum of the observable along a single traversal of the prime cycle p

$\overline{a(x_0)}$ is a wild function of x_0 e.g. for a hyperbolic system it takes a different value on (almost) every periodic orbit

exponential generating functions

consider the spatial average

$$\left\langle e^{\beta \cdot A} \right\rangle = \frac{1}{|\mathcal{M}|} \int_{\mathcal{M}} \mathrm{d}x \, e^{\beta \cdot A(x,t)} \tag{4}$$

where in this context β is an auxiliary variable of no physical significance.

exercise:
How can we recover the desired space average $\langle A \rangle$ from $\left\langle e^{\beta \cdot A} \right\rangle$?

exercise

$$\langle A \rangle = \frac{\partial}{\partial \beta} \left\langle e^{\beta \cdot A} \right\rangle \Big|_{\beta=0}$$

characteristic function with time

as $t \to \infty$ we expect:

$$\left\langle e^{\beta A} \right\rangle \to (\text{const}) e^{ts(\beta)}$$

the rate of growth characteristic function is given by

$$s(\beta) = \lim_{t \to \infty} \frac{1}{t} \ln \left\langle e^{\beta A} \right\rangle \tag{5}$$

exercise: How can we calculate $\langle a \rangle$?

calculating moments

We can use derivatives of $s(\beta)$ to calculate the expectation value of the observable, its variance, and higher moments of the integrated observable

for example,

$$\left.\frac{\partial s}{\partial \beta}\right|_{\beta=0} = \lim_{t\to\infty} \frac{1}{t} \langle A \rangle = \langle a \rangle \tag{6}$$

Pseudo-cycles and shadowing

pseudo-cycles

dynamical zeta function expanded:

$$1/\zeta = \prod_p (1 - t_p) = 1 - \sum_{\{p_1 p_2 \dots p_k\}}{}' (-1)^{k+1} t_{p_1} t_{p_2} \dots t_{p_k} \tag{7}$$

$t_\pi = (-1)^{k+1} t_{p_1} t_{p_2} \dots t_{p_k}$ is a product of the prime cycle weights t_p

pseudo-cycle label

$$\pi = p_1 + p_2 + \cdots + p_k \tag{8}$$

series (7) compactly written

$$1/\zeta = 1 - \sum_\pi{}' t_\pi \,. \tag{9}$$

products t_π are weights of pseudo-cycles,

sequences of shorter cycles that shadow a cycle with the symbol sequence $p_1 p_2 \dots p_k$ along the segments $p_1,\ p_2, \dots,\ p_k$

pseudo-cycle weight

pseudo-cycle weight $= \prod$(weights of prime cycles) comprising it,

$$t_\pi = (-1)^{k+1} \frac{1}{|\Lambda_\pi|} e^{\beta A_\pi - s T_\pi} z^{n_\pi} , \qquad (10)$$

pseudo-cycle integrated observable A_π, period T_π, stability Λ_π:

$$\begin{aligned} \Lambda_\pi &= \Lambda_{p_1}\Lambda_{p_2}\cdots\Lambda_{p_k} , \qquad T_\pi = T_{p_1} + \ldots + T_{p_k} \\ A_\pi &= A_{p_1} + \ldots + A_{p_k} , \qquad n_\pi = n_{p_1} + \ldots + n_{p_k} , \end{aligned} \qquad (11)$$

cycle expansion

complete binary symbolic dynamics Euler product (7)

$$\begin{aligned}1/\zeta &= (1-t_0)(1-t_1)(1-t_{01})(1-t_{001})(1-t_{011}) \\ &\quad \times (1-t_{0001})(1-t_{0011})(1-t_{0111})(1-t_{00001})(1-t_{00011}) \\ &\quad \times (1-t_{00101})(1-t_{00111})(1-t_{01011})(1-t_{01111})\ldots\end{aligned} \tag{12}$$

the first few terms of the expansion (9) ordered by increasing total pseudo-cycle length:

$$\begin{aligned}1/\zeta &= 1-t_0-t_1-t_{01}-t_{001}-t_{011}-t_{0001}-t_{0011}-t_{0111}-\ldots \\ &\quad +t_{0+1}+t_{0+01}+t_{01+1}+t_{0+001}+t_{0+011}+t_{001+1}+t_{011+1} \\ &\quad -t_{0+01+1}-\ldots\end{aligned} \tag{13}$$

cycle expansion

regroup the terms into the

- **fundamental** contributions t_f
- **curvature** corrections

split into prime cycles p of period $n_p{=}n$ grouped together with pseudo-cycle **shadows**

$$\begin{aligned}
1/\zeta &= 1 - t_0 - t_1 - [(t_{01} - t_{0+1})] - [(t_{001} - t_{0+01}) + (t_{011} - t_{01+1})] \\
&\quad -[(t_{0001} - t_{0+001}) + (t_{0111} - t_{011+1}) \\
&\qquad +(t_{0011} - t_{001+1} - t_{0+011} + t_{0+01+1})] - \ldots \\
&= 1 - \sum_f t_f - \sum_n \hat{c}_n \ . \qquad (14)
\end{aligned}$$

curvature corrections

$-\ t_{0}$				
$-\ t_{1}$				
$-\ t_{10}$	$+\ t_{1}\,t_{0}$			
$-\ t_{100}$	$+\ t_{10+0}$			
$-\ t_{101}$	$+\ t_{10+1}$			
$-\ t_{1000}$	$+\ t_{100+0}$			
$-\ t_{1001}$	$+\ t_{100+1}$	$+\ t_{110+0}$	$-\ t_{1+10+0}$	
$-\ t_{1011}$	$+\ t_{101+1}$			
$-\ t_{10000}$	$+\ t_{1000+0}$			
$-\ t_{10001}$	$+\ t_{1001+0}$	$+\ t_{1000+1}$	$-\ t_{0+100+1}$	
$-\ t_{10010}$	$+\ t_{100+10}$			
$-\ t_{10101}$	$+\ t_{101+10}$			
$-\ t_{10011}$	$+\ t_{1011+0}$	$+\ t_{1001+1}$	$-\ t_{0+101+1}$	
$-\ t_{10111}$	$+\ t_{1011+1}$			
$-\ t_{100000}$	$+\ t_{10000+0}$			
$-\ t_{100001}$	$+\ t_{10001+0}$	$+\ t_{10000+1}$	$-\ t_{0+1000+1}$	
$-\ t_{100010}$	$+\ t_{10010+0}$	$+\ t_{1000+10}$	$-\ t_{0+100+10}$	
$-\ t_{100011}$	$+\ t_{10011+0}$	$+\ t_{10001+1}$	$-\ t_{0+1001+1}$	
$-\ t_{100101}$	$-\ t_{100110}$	$+\ t_{10010+1}$	$+\ t_{10110+0}$	
	$+\ t_{10+1001}$	$+\ t_{100+101}$	$-\ t_{0+10+101}$	$-\ t_{1+10+100}$
$-\ t_{101110}$	$+\ t_{10110+1}$	$+\ t_{1011+10}$	$-\ t_{1+101+10}$	
$-\ t_{100111}$	$+\ t_{10011+1}$	$+\ t_{10111+0}$	$-\ t_{0+1011+1}$	
$-\ t_{101111}$	$+\ t_{10111+1}$			

Evaluation of traces and spectral determinants

exact cycle weight

weight of prime cycle p repeated r times is

$$t_p(z,\beta,r) = \frac{e^{r\beta A_p}\, z^{r\,n_p}}{\left|\det\left(\mathbf{1}-M_p^r\right)\right|} \qquad \text{(discrete time)} \qquad (15)$$

$$t_p(s,\beta,r) = \frac{e^{r(\beta A_p - sT_p)}}{\left|\det\left(\mathbf{1}-M_p^r\right)\right|} \qquad \text{(continuous time)} \qquad (16)$$

trace formula, determinant, expanded

trace formula

$$\mathrm{tr}\,\frac{z\mathcal{L}}{1-z\mathcal{L}}\bigg|_N = \sum_{n=1}^{N} C_n z^n\,, \qquad C_n = \mathrm{tr}\,\mathcal{L}^n \tag{17}$$

spectral determinant

$$\det(1-z\mathcal{L})|_N = 1 - \sum_{n=1}^{N} Q_n z^n\,, \quad Q_n = n\text{th cumulant}\,, \tag{18}$$

truncated to prime cycles p and their repeats r such that $n_p r \le N$

convergence of cycle expansions

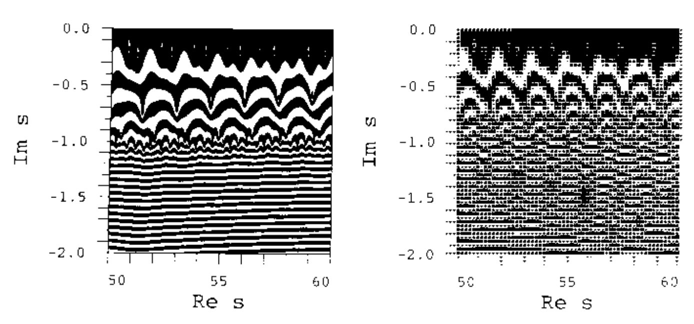
3-disk repeller escape rates computed from N-truncated cycle expansions

- spectral determinant
- dynamical zeta functions

spectral determinant $\det(s - \mathcal{A})$ convergence is **super-exponential**

N	. $\det(s-\mathcal{A})$	$1/\zeta(s)$	$1/\zeta(s)_{\text{3-disk}}$
1	0.39	0.407	
2	0.4105	0.41028	0.435
3	0.410338	0.410336	0.4049
4	0.4103384074	0.4103383	0.40945
5	0.4103384077696	0.4103384	0.410367
6	0.410338407769346482	0.4103383	0.410338
7	0.4103384077693464892		0.4103396
8	0.410338407769346489338468		
9	0.4103384077693464893384613074		
10	0.4103384077693464893384613078192		

3-disk spectral determinant vs $1/\zeta(s)$



complex s plane contour plots of the logarithm of
(left) $|1/\zeta(s)|$
(right) $|\det(s - \mathcal{A})|$
eigenvalues of the evolution operator $\mathcal{L}$ are the centers of elliptic neighborhoods

spectral determinant is entire and reveals further families of zeros

Cycle formulas for dynamical averages

eigenvalue conditions

eigenvalue conditions for

dynamical zeta function (9)
spectral determinant (18)

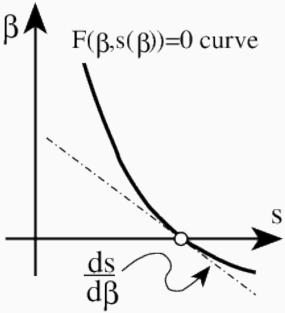
$$0 = 1 - {\sum_{\pi}}' t_\pi \,, \qquad t_\pi = t_\pi(\beta, s(\beta)) \tag{19}$$

$$0 = 1 - \sum_{n=1}^{\infty} Q_n \,, \qquad Q_n = Q_n(\beta, s(\beta)) \,, \tag{20}$$

are implicit equations for an eigenvalue $s = s(\beta)$ of form

$$0 = F(\beta, s(\beta))$$

eigenvalue condition → expectation value



eigenvalue condition is satisfied on the curve $F = 0$ on the (β, s) plane

expectation value of the observable is given by the slope of the curve

eigenvalue condition → expectation value

the cycle averaging formulas for the slope and curvature of $s(\beta)$ are obtained as in (6), by taking derivatives of the eigenvalue condition

the chain rule for the first derivative yields

$$\begin{aligned} 0 &= \frac{d}{d\beta} F(\beta, s(\beta)) \\ &= \frac{\partial F}{\partial \beta} + \frac{ds}{d\beta} \left. \frac{\partial F}{\partial s} \right|_{s=s(\beta)} \quad \Longrightarrow \quad \frac{ds}{d\beta} = -\frac{\partial F}{\partial \beta} \Big/ \frac{\partial F}{\partial s} \,, \end{aligned} \tag{21}$$

and for the second derivative of $F(\beta, s(\beta)) = 0$

$$\frac{d^2 s}{d\beta^2} = -\left[\frac{\partial^2 F}{\partial \beta^2} + 2 \frac{ds}{d\beta} \frac{\partial^2 F}{\partial \beta \partial s} + \left(\frac{ds}{d\beta} \right)^2 \frac{\partial^2 F}{\partial s^2} \right] \Big/ \frac{\partial F}{\partial s} \,. \tag{22}$$

cycle averaging formulas

denote expectations for eigenvalue condition $F = 0$ by

$$\begin{aligned} \langle A \rangle_F &= -\left.\frac{\partial F}{\partial \beta}\right|_{\beta, s=s(\beta)}, \qquad \langle T \rangle_F = \left.\frac{\partial F}{\partial s}\right|_{\beta, s=s(\beta)}, \\ \langle A^2 \rangle_F &= -\left.\frac{\partial^2 F}{\partial \beta^2}\right|_{\beta, s=s(\beta)}, \qquad \langle TA \rangle_F = \left.\frac{\partial^2 F}{\partial s \partial \beta}\right|_{\beta, s=s(\beta)} \end{aligned} \tag{23}$$

cycle averaging formulas for
expectation of the observable, its variance:

$$\langle a \rangle = \frac{\langle A \rangle_F}{\langle T \rangle_F} \tag{24}$$

$$\Delta = \frac{1}{\langle T \rangle_F} \langle (A - T \langle a \rangle)^2 \rangle_F \,, \tag{25}$$

example : dynamical zeta function cycle averaging formulas

for the dynamical zeta function we obtain

$$\begin{aligned}
\langle A\rangle_\zeta &:= -\frac{\partial}{\partial\beta}\frac{1}{\zeta} = \sum{}' A_\pi t_\pi \\
\langle T\rangle_\zeta &:= \frac{\partial}{\partial s}\frac{1}{\zeta} = \sum{}' T_\pi t_\pi\,, \quad \langle n\rangle_\zeta := -z\frac{\partial}{\partial z}\frac{1}{\zeta} = \sum{}' n_\pi t_\pi\,,
\end{aligned} \tag{26}$$

$\langle A\rangle_F$ evaluated on pseudo-cycles (11), with pseudo-cycle weights $t_\pi = t_\pi(z,\beta,s(\beta))$ evaluated at the eigenvalue $s(\beta)$

$$\langle A\rangle_\zeta = \sum_\pi{}'(-1)^{k+1}\frac{A_{p_1}+A_{p_2}\cdots+A_{p_k}}{|\Lambda_{p_1}\cdots\Lambda_{p_k}|} \tag{27}$$

$\langle T\rangle_\zeta$ is of the same form

example: cycle expansion for the mean cycle period

for complete binary symbolic dynamics
the mean cycle period is given by

$$\langle T \rangle_\zeta = \frac{T_0}{|\Lambda_0|} + \frac{T_1}{|\Lambda_1|} + \left(\frac{T_{01}}{|\Lambda_{01}|} - \frac{T_0 + T_1}{|\Lambda_0 \Lambda_1|} \right) + \left(\frac{T_{001}}{|\Lambda_{001}|} - \frac{T_{01} + T_0}{|\Lambda_{01} \Lambda_0|} \right) + \left(\frac{T_{011}}{|\Lambda_{011}|} - \frac{T_{01} + T_1}{|\Lambda_{01} \Lambda_1|} \right) + \dots \tag{28}$$

note: the cycle expansions for averages are grouped into the same shadowing combinations as the dynamical zeta function cycle expansion (14), with nearby pseudo-cycles nearly canceling each other

Lyapunov exponents

formula for Lyapunov exponent

Construction of the evolution operator for the evaluation of the Lyapunov spectra for a *d*-dimensional flow: we need an extension of the evolution equations to a flow in the tangent space

All that remains is to determine the value of the Lyapunov exponent

$$\lambda = \langle \ln |f'(x)| \rangle = \left. \frac{\partial s(\beta)}{\partial \beta} \right|_{\beta=0} = s'(0) \tag{29}$$

How?

example : cycle expansion formula for Lyapunov exponents

we have related the Lyapunov exponent for a 1-dimensional map to the leading eigenvalue of an evolution operator

now the cycle averaging formula (27) yields an exact explict expression for the Lyapunov exponent in terms of prime cycles:

$$\lambda = \frac{1}{\langle n \rangle_\zeta} \sum{}' (-1)^{k+1} \frac{\log|\Lambda_{p_1}| + \cdots + \log|\Lambda_{p_k}|}{|\Lambda_{p_1} \cdots \Lambda_{p_k}|} \tag{30}$$

big picture recap

Since detailed prediction is impossible in chaotic dynamics, averages are useful to describe the system.

The key idea is to express expectation values of observables as derivatives of evolution operators leading eigenvalue

Dynamical averages can thus be extracted from the eigenvalues of appropriately constructed evolution operators

Questions?