Das Problem: dancers and drifters

pipe flow

(a) instantaneous global state of a fluid (marked by a ‘swirl’)
Das Problem: dancers and drifters

(a) $\theta$

(b) $\theta$

(c) $\theta$

(d) $\theta$

symmetry: a pipe flow solution translated or rotated or reflected is also a solution

(b) the state translated by downstream shift $d$ (fluid states are $SO(2)_z$ equivariant in a stream-wise periodic pipe),
Das Problem: dancers and drifters

Symmetry: a pipe flow solution translated or rotated or reflected is also a solution.

(c) The state translated by $d$ and rotated azimuthal by $\phi$ (the two states are $SO(2)_\theta \times SO(2)_z$ equivariant).
Das Problem: dancers and drifters

symmetry: a pipe flow solution translated or rotated or reflected is also a solution

(d) the state reflected and rotated azimuthally by $\phi$ (the two states are $O(2)_\theta$ equivariant).
Das Problem: dancers and drifters

(a) \( \theta \)

(b) \( \theta \)

(c) \( \theta \)

(d) \( \theta \)

states may also be symmetry-related by time evolution

relative equilibrium: solution that retains its shape while rotating and traveling downstream with constant \( c \).
Das Problem: dancers and drifters

states may also be symmetry-related by time evolution

relative periodic orbit: $\mathcal{M}_p$ a *time dependent*, shape-changing state of the fluid that after a period $T_p$ reemerges as (b), (c), or (d), the initial state translated by $d_p$, rotated by $\phi_p$ and possibly also azimuthally reflected
Das Problem: don’t be stupid

with a continuous symmetry,
   there are families of $\infty$-many equivalent states

you do not want to compute the same solution over and over, do you?

so, you must reduce any continuous symmetry
Happy families are all alike; every unhappy family is unhappy in its own way
everybody, her mother, and Robert MacKay knows how to do this except the author of
masters of group theory
Das Problem: a 5-dimensional drifting attractor

**complex Lorenz equations**

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{y}_1 \\
\dot{y}_2 \\
\dot{z}
\end{bmatrix}
= \begin{bmatrix}
-\sigma x_1 + \sigma y_1 \\
-\sigma x_2 + \sigma y_2 \\
(\rho_1 - z)x_1 - \rho_2 x_2 - y_1 - ey_2 \\
\rho_2 x_1 + (\rho_1 - z)x_2 + ey_1 - y_2 \\
-bz + x_1 y_1 + x_2 y_2
\end{bmatrix}
\]

\[\rho_1 = 28, \quad \rho_2 = 0, \quad b = 8/3, \quad \sigma = 10, \quad e = 1/10\]

- A typical \(\{x_1, x_2, z\}\) trajectory
- superimposed: a trajectory whose initial point is close to the relative equilibrium \(Q_1\)
continuous symmetry induces drifts

- generic chaotic trajectory (blue)
- $E_0$ equilibrium
- $E_0$ unstable manifold - a cone of such (green)
- $Q_1$ relative equilibrium (red)
- $Q_1$ unstable manifold, one for each point on $Q_1$ (brown)
- relative periodic orbit $01$ (purple)
what to do?

it's a mess

the goal

reduce this messy strange attractor to something simple
the goal attained
started in five dimensions : reduced it to one (!)

but it will cost you
must learn how to reduce (quotient) the SO(2) symmetry

1D return map!
the goal attained

started in five dimensions: reduced it to one (!)

but it will cost you

must learn how to reduce (quotient)
the SO(2) symmetry

how? hang on, that’s what we’ll explain here
symmetries of dynamics

**a flow** \( \dot{x} = \nu(x) \) is **G-equivariant** if

\[
\nu(x) = g^{-1} \nu(gx), \quad \text{for all } g \in G.
\]

**definition:** Lie group

a topological group \( G \) such that

1. \( G \) has the structure of a smooth differential manifold
2. composition map \( G \times G \to G : (g, h) \to gh^{-1} \) is smooth

mystified?

just think “aha, like the rotation group \( \text{SO}(3) \)…”
example: $SO(2)$ invariance

complex Lorenz equations

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{y}_1 \\
\dot{y}_2 \\
\dot{z}
\end{bmatrix}
= \begin{bmatrix}
-\sigma x_1 + \sigma y_1 \\
-\sigma x_2 + \sigma y_2 \\
(\rho_1 - z)x_1 - \rho_2 x_2 - y_1 - ey_2 \\
\rho_2 x_1 + (\rho_1 - z)x_2 + ey_1 - y_2 \\
-bz + x_1 y_1 + x_2 y_2
\end{bmatrix}
\]

invariant under a $SO(2)$ rotation by finite angle $\phi$:

\[
g(\phi) = \begin{pmatrix}
\cos \phi & -\sin \phi & 0 & 0 & 0 \\
\sin \phi & \cos \phi & 0 & 0 & 0 \\
0 & 0 & \cos \phi & -\sin \phi & 0 \\
0 & 0 & \sin \phi & \cos \phi & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]
example: abelian group $SO(2)$

$SO(2)$: rotations in a plane

reflection $(x, y) \rightarrow (-x, y)$ excluded ($\det g = -1$)

if the group $G$ actions consists of two such rotations which commute, the group $G$ is an Abelian group that sweeps out a $T^2$ torus
example: continuous symmetries of pipe flow

pipe flow
- periodic streamwise, spanwise
- eqs. under azimuthal flip invariant

a) $\text{SO}(2)_z \times \text{O}(2)_\theta$ symmetry
b) laminar sol. is invariant
for any $x \in \mathcal{M}$, the **group orbit** $\mathcal{M}_x$ of $x$ is the set of all group actions

$$\mathcal{M}_x = \{g \cdot x \mid g \in G\} \subset \mathcal{M}$$

states in $\mathcal{M}_x$ are physically equivalent
example: group orbit of a pipe flow relative equilibrium
\[ \hat{x}' = \text{Kerswell et al. } N2_M1 \text{ solution, } (Re = 2400, \text{ stubby } L = 2.5D \text{ pipe}) \]

a very smooth, almost laminar solution

**SO(2) \times SO(2) symmetry**
\[ \Rightarrow \text{ group orbit is 2-torus} \]
projected on
- 2 \( \hat{x}' \) group tangents
- 3. axis along the curvature direction

2d group orbit (in 100,000 dimension state space) traced out by
- equal increment translations in \( \theta \) (dashed blue)
- equal increments in \( z \) (solid red)
example: group orbit of a pipe flow turbulent state

\( \hat{x}' \) is Kerswell et al. N2_M1 relative equilibrium
( \( Re = 2400 \), stubby \( L = 2.5D \) pipe)

a turbulent state

\( SO(2) \times SO(2) \) symmetry
\( \Rightarrow \) group orbit is 2-torus

group orbits of nonlinear states are highly contorted
group orbit $\mathcal{M}_x$ of $x$ is the set of all group actions

$$\mathcal{M}_x = \{g x \mid g \in G\}$$
foliation by group orbits

any point on the manifold $M_{x(t)}$ is equivalent to any other
foliation by group orbits

action of a symmetry group foliates the state space into a union of group orbits

each group orbit an equivalence class
the goal

replace each group orbit by a unique point in a lower-dimensional

\[ \mathcal{M}/G \]  symmetry reduced state space
symmetry reduction

full state space

\[ \mathcal{M} \times (\tau) \times x(0) \times (\tau) \]

reduced state space

\[ \hat{\mathcal{M}} \times \hat{x}(0) \]
Axial shifts of TW state

$g(\tau) x(0) = x(\tau) \in \mathcal{M}_{TW}$
Reduction of TW orbit to point by shifts

relative equilibrium is made stationary by a counter-rotating ‘frame’

* ‘pedestrian’ = polite word for ‘applied mathematician’
(a) $N$-continuous parameters symmetry: each state space point $x$ owns $(N+1)$ tangent vectors: $v(x)$ along the time flow $x(t)$ and the $N$ group tangents $t_1(x), t_2(x), \cdots, t_N(x)$ along space, tangent to the $N$-dimensional group orbit $\mathcal{M}_x$. 
symmetries of dynamics

(b) each point has a trajectory (blue) under time evolution
symmetries of dynamics

(c) each point has a group orbit (green) of symmetry-related states. For $\text{SO}(2)$, this is topologically a circle. Any two points on a group orbit are physically equivalent, but may lie far from each other in state space.
(d) together, time-evolution and group actions trace out a wurst of physically equivalent solutions
A relative periodic orbit \( p \) is an orbit in state space \( \mathcal{M} \) which exactly recurs

\[
x_p(t) = g_p x_p(t + T_p), \quad x_p(t) \in \mathcal{M}_p
\]

for a fixed relative period \( T_p \) and a fixed group action \( g_p \in G \) that “rotates” the endpoint \( x_p(T_p) \) back into the initial point \( x_p(0) \).
relative periodic orbit : state space visualization

each cycle point
\[ x_p(0) = g_p x_p(T_p) \]

exactly recurs at a fixed

relative period
\[ T_p \]

but shifted by a fixed

group action
\[ g_p \]

(green dashes) group orbit
(blue) relative periodic orbit orbit
(arrows) velocity, group tangents
relative periodic orbit: state space visualization

Group action parameters

\[ \phi = (\phi_1, \phi_2, \cdots \phi_N) \]

are irrational:

trajectory sweeps out ergodically the group orbit without ever closing into a periodic orbit
example: pipe flow relative periodic orbit \( \text{rpo}_{36.72} \)
symmetry reduction: full state space trajectory \( x(t) \)

\[ \Rightarrow \]
redduced state space trajectory \( \hat{x}(t) \), continuous group induced drifts quotiented out

full state space

traced for two periods:
fills quasi-periodically a highly contorted 2-torus

closes a periodic orbit in one period
a relative periodic orbit of the Kuramoto-Sivashinsky flow, $128d$
state space traced for four periods $T_p$, projected on
a stationary state space coordinate frame $\{v_1, v_2, v_3\}$; a mess
relativity for pedestrians

try a co-moving coordinate frame?

(a relative periodic orbit of the Kuramoto-Sivashinsky flow projected on

a co-moving \{\tilde{v}_1, \tilde{v}_2, \tilde{v}_3\} frame
relativity for pedestrians

no good global co-moving frame!

beautiful, but this is no symmetry reduction at all;

all other relative periodic orbits require their own frames, moving at different velocities!
method of moving frames / slices

cut group orbits by a hypersurface (spatial analogue of time Poincaré section), so

each group orbit of symmetry-equivalent points represented by the single point

cut how?
you are observing turbulence in a pipe flow, or your defibrillator has a mesh of sensors measuring electrical currents that cross your heart, and

you have a precomputed pattern, and are sifting through the data set of observed patterns for something like it

here you see a pattern, and there you see a pattern that seems much like the first one

how ‘much like the first one?’
take the first pattern

‘template’ or ‘reference state’

a point $\hat{x}'$ in the state space $\mathcal{M}$

and use the symmetries of the flow to

slide and rotate the ‘template’

act with elements of the symmetry group $G$ on $\hat{x}' \rightarrow g(\phi) \hat{x}'$

until it overlies the second pattern (a point $x$ in the state space)

distance between the two patterns

$$|x - g(\phi) \hat{x}'| = |\hat{x} - \hat{x}'|$$

is minimized
idea: the closest match

template: Sophus Lie

(1) rotate bearded guy $x$ traces out the group orbit $M_x$

(2) replace the group orbit by the closest match $\hat{x}$ to the template pattern $\hat{x}'$

the closest matches $\hat{x}$ lie in the $(d-N)$ symmetry reduced state space $\hat{M}$
assume that $G$ is a subgroup of the group of orthogonal transformations $O(d)$, and measure distance $|x|^2 = \langle x|x \rangle$ in terms of the Euclidean inner product.

numerical fluids: PDE discretization independent L2 distance is the energy norm

$$\|u - v\|^2 = \langle u - v|u - v \rangle = \frac{1}{V} \int_{\Omega} dx \ (u - v) \cdot (u - v)$$

experimental fluid:

image discretization independent distance is Hamming distance, or ???
minimal distance
is a solution to the extremum conditions

\[ \frac{\partial}{\partial \phi_a} |x - g(\phi) \hat{x}'|^2 \]

but what is

\[ \frac{\partial}{\partial \phi_a} g(\phi) ? \]
Lie algebras for pedestrians

an element of a compact Lie group:

\[ g(\phi) \propto e^{\phi \cdot T}, \quad \phi \cdot T = \sum \phi_a T_a, \ a = 1, 2, \cdots, N \]

\( \phi \cdot T \): *Lie algebra* element
\( \phi_a \): parameters of the transformation.

**infinitesimal transformations**

\[ g = e^{\delta \phi \cdot T} \simeq 1 + \phi \cdot T, \quad |\delta \phi| \ll 1 \]

**Lie algebra**

- \( T_a \) are *generators* of infinitesimal transformations
- here \( T_a \) are \([d \times d]\) antisymmetric matrices
- \( T_a \) are elements of the Lie algebra of \( G \)
symmetries of dynamics

each state space point $x$ has

$\text{time tangent vector } v(x) \text{ along the time flow } x(t)$
symmetries of dynamics

each state space point $x$ has

group tangent vectors $t_1(x), t_2(x), \cdots, t_N(x)$ along the $N$-dimensional space group orbit $\mathcal{M}_x$
example: \( \text{SO}(2) \) invariance of complex Lorenz equations

complex Lorenz equations are invariant under \( \text{SO}(2) \) rotation by finite angle \( \phi \):

\[
g(\phi) = \begin{pmatrix}
\cos \phi & -\sin \phi & 0 & 0 & 0 \\
\sin \phi & \cos \phi & 0 & 0 & 0 \\
0 & 0 & \cos \phi & -\sin \phi & 0 \\
0 & 0 & \sin \phi & \cos \phi & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

\( \text{SO}(2) \) Lie algebra has one generator of infinitesimal rotations

\[
T = \begin{pmatrix}
0 & -1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]
now have the ‘slice condition’

**group tangent fields**

flow field at the state space point $x$ induced by the action of the group is given by the set of $N$ tangent fields

$$t_a(x)_i = (T_a)_{ij}x_j$$

**slice condition**

$$\frac{\partial}{\partial \phi_a} |x - g(\phi) \hat{x}'|^2 = 2 \langle \hat{x} - \hat{x}' | t'_a \rangle = 0, \quad t'_a = T_a \hat{x}'$$
traveling wave

\[ t_{\text{Ref}} = \partial_z \mathbf{x}_{\text{Ref}} \]

\[ \mathbf{x}_{\text{Ref}} \cdot t_{\text{Ref}} = 0 \]
Reduce all TWs into a single slice

How? - several speeds $c$, possibly unknown

$$\mathbf{x}_i(0) = S_z(-c_i t) \mathbf{x}_i(t)$$

How? - several speeds $c$, possibly unknown
flow within the slice
slice fixed by \( \hat{x}' \)

**reduced state space** \( \hat{\mathcal{M}} \) flow \( \hat{v}(\hat{x}) \)

\[
\begin{align*}
\hat{v}(\hat{x}) &= v(\hat{x}) - \dot{\phi}(\hat{x}) \cdot t(\hat{x}), \quad \hat{x} \in \hat{\mathcal{M}} \\
\dot{\phi}_a(\hat{x}) &= (v(\hat{x})^T t'_a) / (t(\hat{x})^T \cdot t').
\end{align*}
\]

- \( v \): velocity, full space
- \( \hat{v} \): velocity component in slice
- \( \dot{\phi} \cdot t \): velocity component normal to slice
- \( \dot{\phi} \): reconstruction equation for the group phases

**Cartan derivative**

\[
g^{-1} \dot{g}x = e^{-\dot{\phi} \cdot T} \frac{d}{dT} e^{\phi \cdot T} x = \dot{\phi} \cdot t(x)
\]
flow within the slice

slice hyperplane \( \hat{\mathcal{M}} \) through the template point \( \hat{x}' \), normal to its group tangent \( t' \), intersects all group orbits (dotted lines) in a neighborhood of \( \hat{x}' \)

state space trajectory point \( x(t) \) (solid black line) and the reduced state space trajectory \( \hat{x}(t) \) (solid green line) belong to the same group orbit \( \mathcal{M}_{x(t)} \), and are equivalent up to a moving frame group rotation \( g(t) \)
flow within the slice

full-space trajectory $x(\tau)$
rotated into the reduced state space $\hat{x}(\tau) = g(\phi)^{-1}x(\tau)$
by appropriate moving frame angles $\{\phi(\tau)\}$
relative periodic orbit $\rightarrow$ periodic orbit

full state space relative periodic orbit $x(\tau)$ is rotated into the reduced state space periodic orbit
relative equilibria and relative periodic orbits together

Method Summary
- **TW**
  - $N \rightarrow (N-1) \text{ dim } (N \rightarrow \infty)$
  - Automatic removal of strong shift (gives $c$ for TW)
  - $TW \rightarrow \text{ point}$
- **RPO**
  - $RPO \rightarrow \text{ PO}$
symmetry reduction by the method of slices

blue point: the template $\hat{x}'$
symmetry reduction by the method of slices

pink points: equivalent to $\hat{x}$ up to a shift, so a relative periodic orbit (green) in the $d$-dimensional full state space $\mathcal{M}$ closes into a periodic orbit (blue) in the slice $\hat{\mathcal{M}}$
symmetry reduction by the method of slices

slice $\hat{\mathcal{M}} = \mathcal{M}/G$: a $(d-1)$-dimensional slab transverse to the template group tangent $t'$
symmetry reduction by the method of slices

typical group orbit (dotted) crosses the slice hyperplane \textit{transversally}, with group tangent \( t = t(\hat{x}) \)
symmetry reduction achieved!

- all points equivalent by symmetries are represented by a single point
- families of solutions are mapped to a single solution
  - relative equilibria become equilibria
  - relative periodic orbits become periodic orbits
die Lösung : complex Lorenz flow reduced

full state space

reduced state space

ergodic trajectory was a mess, now the topology is revealed relative periodic orbit 01 now a periodic orbit
triumph: all pipe flow solution in one happy family

a typical turbulent state $\hat{x}'$ breaks all symmetries
plot relative equilibria and unstable manifolds

all in the same projection
inset: an expanded view
blue loop: $T = 4.93$
relative periodic orbit

$T = 10.96$ and $T = 36.92$
relative periodic orbits embedded in turbulence

first ‘turbulent’ relative periodic orbits for pipe flows!
portrait of complex Lorenz flow in reduced state space

any choices of the slice $\hat{x}'$ exhibit flow discontinuities
take-home message

rotation into a slice is not an average over 3D pipe azimuthal angle

it is the full snapshot of the flow embedded in the \( \infty \)-dimensional state space

NO information is lost by symmetry reduction

- not modeling by a few degrees of freedom
- no dimensional reduction
group tangent of a generic trajectory orthogonal to the slice tangent at a sequence of instants $\tau_k$

$$t(\tau_k)^T \cdot t' = 0$$
slice hyperplane is almost never a global slice; it is valid up to slice border, a \((d-2)\)-dimensional hypersurface (red) of points \(\hat{x}^*\) whose group orbits graze the slice, i.e. points whose tangents \(t^* = t(\hat{x}^*)\) lie in \(\hat{\mathcal{M}}\)
slice trouble 1

group orbits beyond the slice border miss the slice hyperplane: the “missing chunk” is here indicated by the dashed lines.
example: group orbit of a pipe flow turbulent state

$\hat{x}'$ is Kerswell et al N2_M1 relative equilibrium

( $Re = 2400$, stubby $L = 2.5D$ pipe)

**SO(2) × SO(2) symmetry**

$\Rightarrow$ group orbit is 2-torus

a turbulent state

**distance extremum condition**

$$\frac{\partial}{\partial \phi_a} |x - g(\phi) \, \hat{x}'|^2 = 0$$

group orbits of highly nonlinear states are highly contorted:
many extrema, multiple sections by a slice
How good is your slice?

hyperplane of points $x^*$ defined by being normal to the quadratic Casimir-weighted vector $T^2 \hat{x}'$, such that from the template vantage point their group orbits are not transverse, but locally ‘horizontal,’

$$\langle t(x^*)|t'\rangle = -\langle x^*|T^2 \hat{x}'\rangle = 0$$

(for simplicity, specialize to the SO(2) case)
inflection hyperplane

\( S \) : set of all points \( \hat{x}^* \) which are both

(a) in the slice

(b) whose group tangent \( t(\hat{x}^*) \) is also in the slice

\[
\langle \hat{x}^* | t' \rangle = 0
\]

\[
\langle t(\hat{x}^*) | t' \rangle = -\langle \hat{x}^* | T^2 \hat{x} \rangle = 0
\]

\( S \) is the locus of inflection points, a hyperplane through which

- curvature of the distance function changes sign
- local minimum turns into a local maximum
slice is good up to inflection hyperplane
slice trouble 2

slice may cut a relative periodic orbit multiple times

here a single relative periodic orbit is intersected by a slice in 3 separate sections of the relative periodic orbit torus, and 3 sections that appear to connect to a closed loop
construct a global atlas by deploying a set of linear Poincaré sections and slices,

each a local chart in the neighborhood of an important equilibrium and/or periodic orbit
summary

conclusion

- symmetry reduction by method of slices: efficient, allows exploration of high-dimensional flows hitherto unthinkable
- stretching and folding of unstable manifolds in reduced state space organizes the flow

to be done

- construct Poincaré sections and return maps
- find all (relative) periodic orbits up to a given period
- use the information quantitatively (periodic orbit theory)
take-home message

if you have a symmetry

use it!

without symmetry reduction,
no understanding of pipe, Couette, ..., flows is possible
amazing theory! amazing numerics! and still... frustration...

“Ask your doctor if taking a pill to solve all your problems is right for you.”