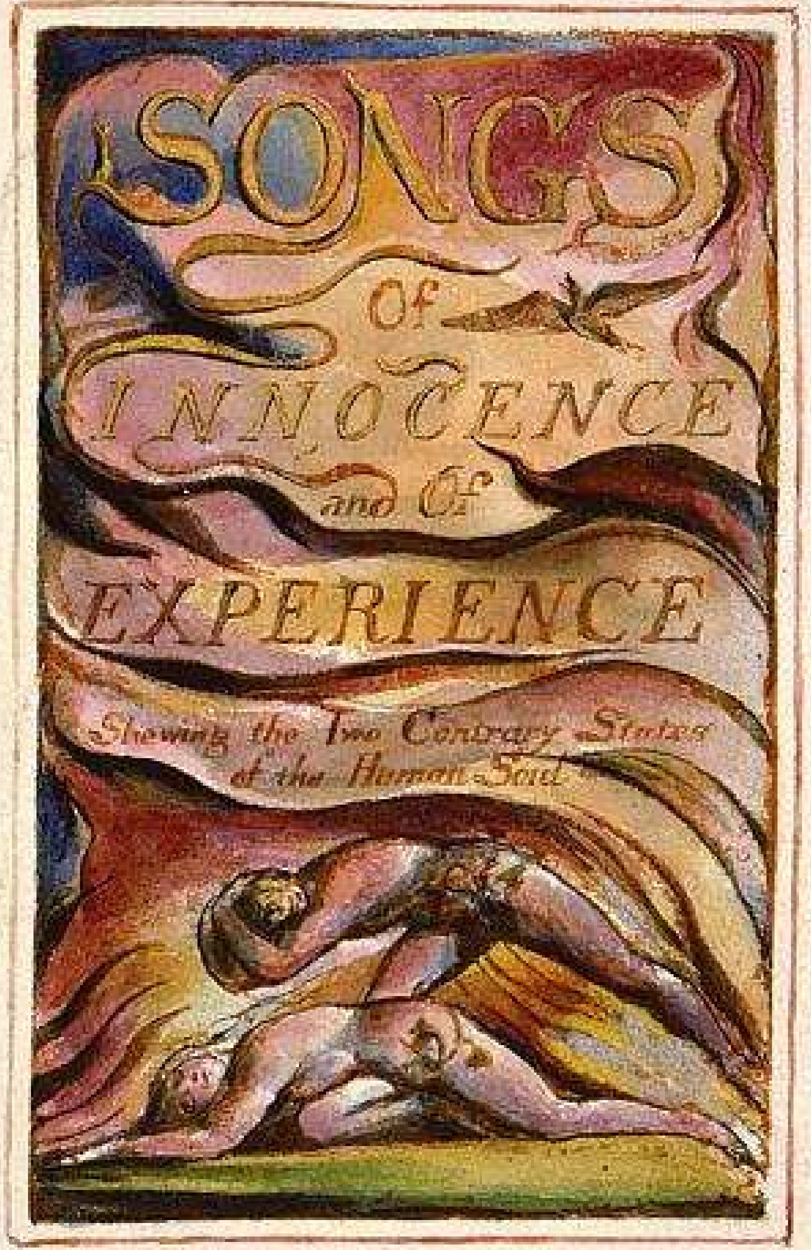


Turbulence?

a stroll through 61,506 dimensions

Predrag Cvitanović

YouTube

School of Physics

Georgia Tech, Atlanta GA, USA

22 September 2022

Dreams of grand schemes

Navier-Stokes

$$\rho \frac{\partial u_i}{\partial t} + \rho u_j \frac{\partial u_i}{\partial x_j} = \rho X_i - \frac{\partial p}{\partial x_i} + \mu \nabla^2 u_i$$

Einstein

$$R_{ik} - \frac{1}{2} g_{ik} R = \frac{8\pi k}{c^4} T_{ik}$$

$$R^{i}_{\ klm} = \frac{\partial \Gamma^i_{km}}{\partial x^l} - \frac{\partial \Gamma^i_{kl}}{\partial x^m} + \Gamma^i_{nl}\Gamma^n_{km} - \Gamma^i_{nm}\Gamma^n_{kl}$$

QUANTUM

Yang-Mills

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F_a^{\mu\nu}$$

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\, C_{abc} A^b_\mu A^c_\nu$$

Now, go solve the problem of turbulence

17th century mathematics

Newton 1686

Newton 1686

$$m \frac{d\vec{v}}{dt} = F$$

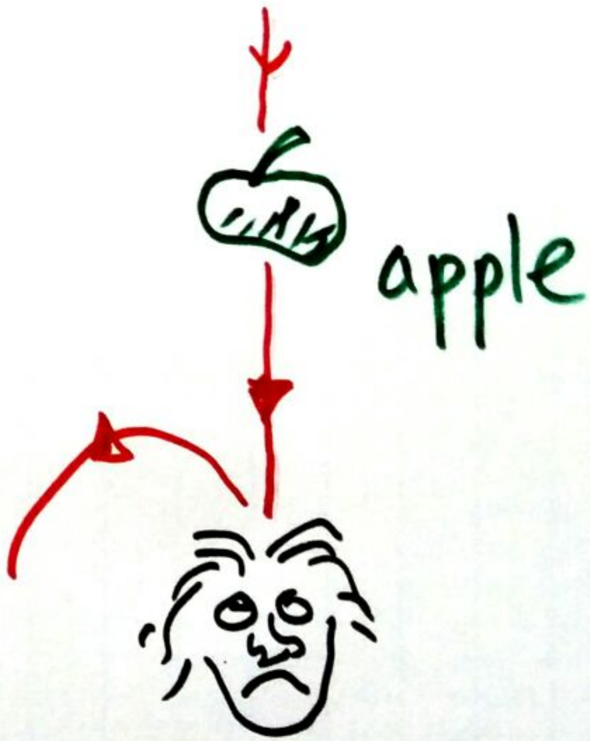
Newton 1686

$$m \frac{d\vec{v}}{dt} = F$$

force

mass

acceleration



18th century mathematics

Euler 1757

$$\Delta m \frac{d\vec{v}}{dt} = \Delta \vec{F}$$

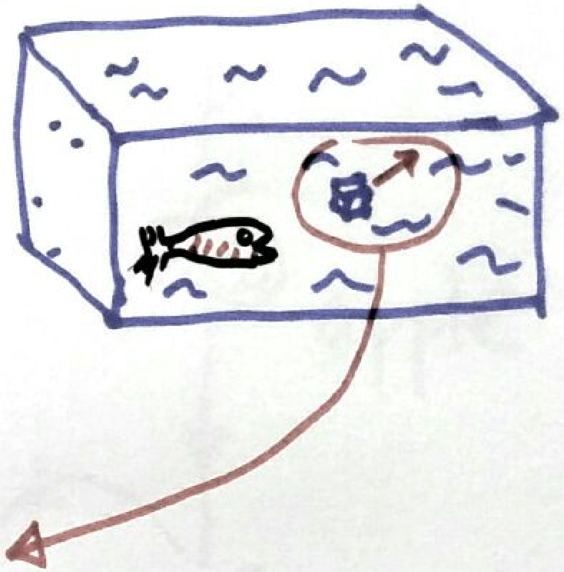
Euler 1757

$$\Delta m \frac{d\vec{v}}{dt} = \Delta F$$

force

mass 'package'

acceleration



Euler 1757

$$\Delta m \frac{d\vec{v}}{dt} = \Delta \vec{F}$$

force
(pressure + external)

mass
'package'

acceleration

$$\Delta m \quad \frac{d\vec{v}}{dt} = \frac{\partial v}{\partial t} + (\vec{v} \cdot \vec{\nabla}) v$$

moving
"packet"

19th century mathematics

Navier 1822

$$\frac{\partial v}{\partial t} + (\vec{v} \cdot \vec{\nabla}) v = f + \frac{1}{Re} \nabla^2 v$$

Navier 1822

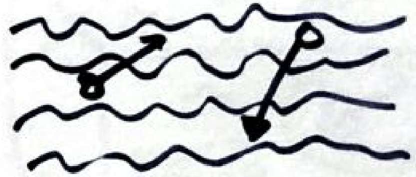
$$\frac{\partial v}{\partial t} + (\vec{v}\cdot\vec{\nabla})v = f + \frac{1}{Re}\nabla^2 v$$

viscosity!

acceleration

force

molecules



diffuse,
slow fluid
down

Navier 1822

$$\frac{\partial v}{\partial t} + (\vec{v}\cdot\vec{\nabla})v = f + \frac{1}{Re}\nabla^2 v$$

viscosity!

Experimentalists:

1. this is WRONG!

Navier 1822

$$\frac{\partial v}{\partial t} + (\vec{v}\cdot\vec{\nabla})v = f + \frac{1}{Re}\nabla^2 v$$

viscosity!

Experimentalists:

1. this is WRONG!
2. his bridge fell down!

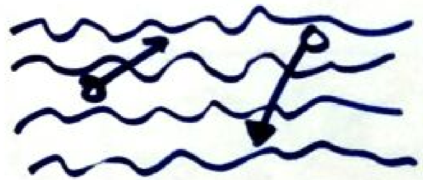
Darrigold *Worlds of Flow* (Oxford 2005)
(not sure that the German experimentalist mentioned in the talk ever existed :)

Navier-Stokes 1845

$$\frac{\partial v}{\partial t} + (\vec{v}\cdot\vec{\nabla})v = f + \frac{1}{Re}\nabla^2 v$$

acceleration — force — viscosity!

molecules



diffuse,
slow fluid
down

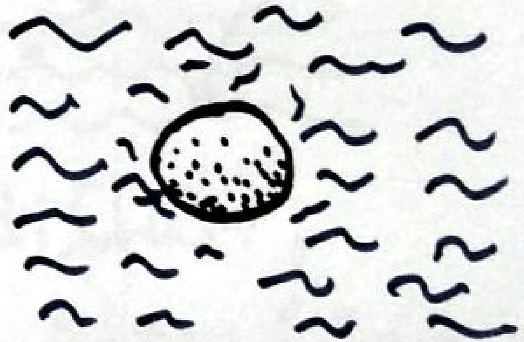
Darrigold <u>*Worlds of Flow*</u> (Oxford 2005)
"Many years elapsed before this equation acquired the fundamental status that we now ascribe to it."

Navier-Stokes 1845

$$\frac{\partial v}{\partial t} + (\vec{v} \cdot \vec{\nabla}) v = f + \frac{1}{Re} \nabla^2 v$$

Viscosity!

acceleration   force



In 1845 Stokes was the 4th to independently discover these equations.

Navier-Stokes 1845

$$\frac{\partial v}{\partial t} + (\vec{v}\cdot\vec{\nabla})v = f + \frac{1}{Re}\nabla^2 v$$

Viscosity!

$$\cdots + \sum_{j=1}^{3} v_j(x)\frac{\partial}{\partial x_j} v_i(x) = \cdots\cdots$$

Navier-Stokes 1845

$$\frac{\partial v}{\partial t} + (\vec{v}\cdot\vec{\nabla})v = f + \frac{1}{Re}\nabla^2 v$$

Viscosity!

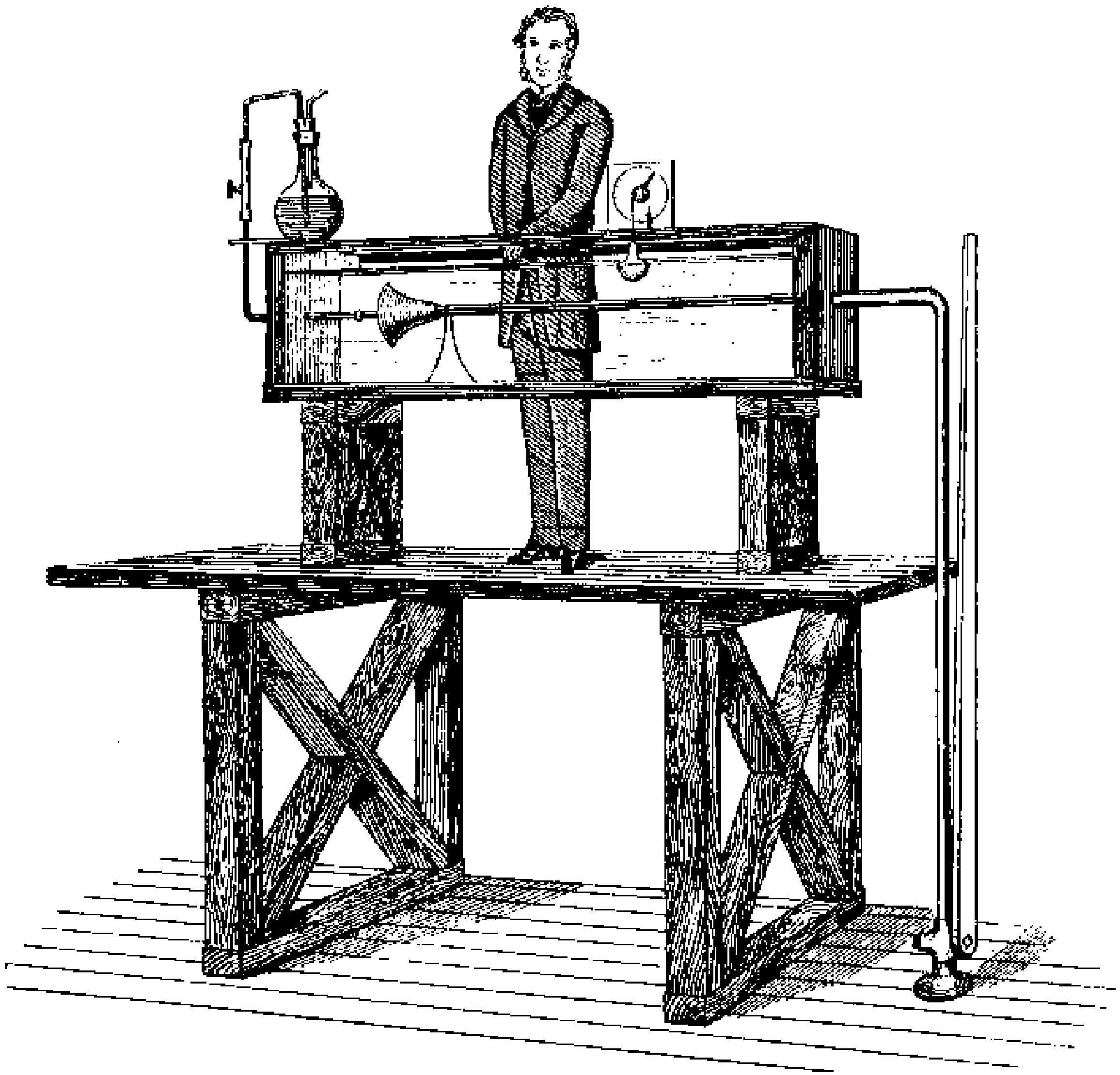
$$\cdots + \sum_{j=1}^{3} v_j(x)\frac{\partial}{\partial x_j} v_i(x) = \cdots\cdots$$

"$v_j v_i$"

NON-linear in velocity!!

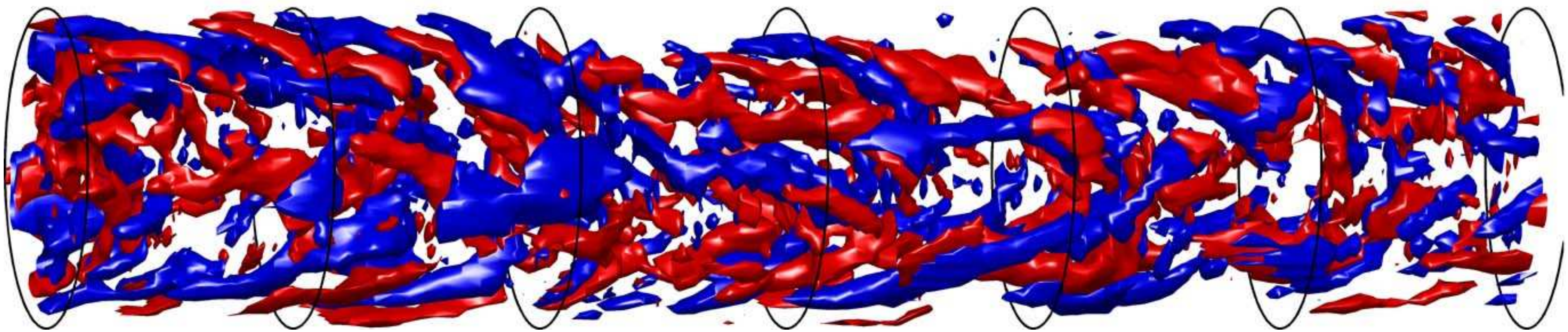
19th century experiments

1883: Osborne Reynolds demonstration



3rd millenium experiments: Unstable Coherent Structures

Stereoscopic Particle Image Velocimetry → 3-d velocity field over the entire pipe[1]



Observed structures resemble numerically computed traveling waves

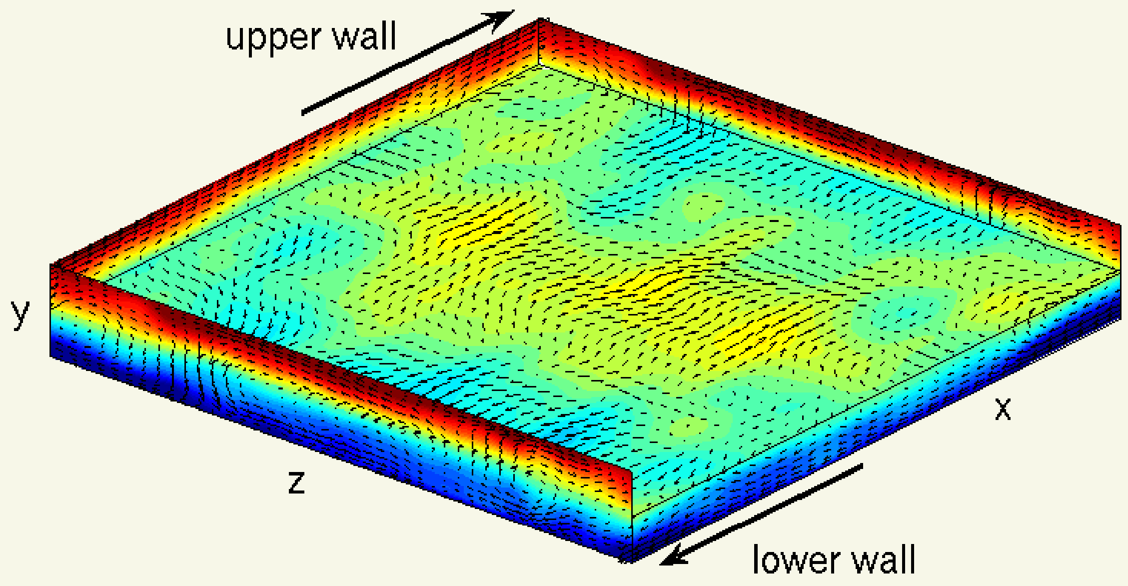
What lies beyond?

[1] Casimir W.H. van Doorne (PhD thesis, Delft 2004); Hof et al., Science (Sep 10, 2004)

3rd millenium numerical simmulations : Plane Couette flow

Navier-Stokes: $$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \frac{1}{Re} \nabla^2 \mathbf{u} \,, \quad \nabla \cdot \mathbf{u} = 0$$

BCs: no-slip at walls $y = \pm 1$



click here to see the online video

Turbulence: A walk through a repertoire of unstable recurrent patterns?

As a turbulent flow evolves, every so often we catch a glimpse of a familiar pattern:

⟹ other swirls ⟹

For any finite spatial resolution, the system follows approximately for a finite time a pattern belonging to a finite alphabet of admissible patterns. The long term dynamics = a walk through the space of such unstable patterns.

big deal!

20th century mathematics

The ultimate goal, however, must be a rational theory of statistical hydrodynamics where [$\cdots$] properties of turbulent flow can be mathematically deduced from the fundamental equations of hydromechanics.

—E. Hopf 1948

Q: How do you treat Navier-Stokes as a dynamical system?

The devil is in the details

Trouble with infinite-dimensional flows

Navier-Stokes equation **f**

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \nu \nabla^2 \mathbf{u}\,, \qquad \nabla \cdot \mathbf{u} = 0$$

requires at least 100,000-dimensional discretization,

ODE: Galerkin projection of Navier–Stokes

expand $\mathbf{u}$ (deviation of velocity from laminar)

$$\mathbf{u}(\mathbf{x},t) = a_n(t)\mathbf{\Phi}_n(\mathbf{x}), \quad n, = 1, \ldots, d$$

Galerkin projection of NS onto $\mathbf{\Phi}_m$ produces ODE in $\mathbb{R}^d$

$$\dot{a}_m = F(a)_m = L_{mn}\, a_n + N_{mnp}\, a_n a_p, \quad m, n, p = 1, \ldots, d\,.$$

where

- $L_{mn} = (\nu\nabla^2\mathbf{\Phi}_n - \partial\mathbf{\Phi}_n/\partial x, -\Phi_n^v \cdot \mathbf{e}_x, \mathbf{\Phi}_m)_\Omega$ and $N_{mnp} = -(\mathbf{\Phi}_n \cdot \nabla\mathbf{\Phi}_p, \mathbf{\Phi}_m)_\Omega$
- Indices range from 1 to $d \approx 10^5$ (2×32^3 to 2×48^3)
- ODE system too big to integrate

.

Turbulent flows cannot be modeled by a few modes

Attractor is "low dimensional," but has to be tracked in the full 10^3 to 10^5 dimensions

ODE vs. CFD reps. of Navier–Stokes

ODE formulation

- Closed-form, unconstrained, real-valued dynamical system
- Orthonormal $\boldsymbol{\Phi}_n$: $||a||^2 = ||\mathbf{u}||_\Omega^2$
- Impossible to integrate: F quadratic in $\mathbb{R}^d$, $d \approx 10^5$

CFD algorithm

- Efficient time integration of Navier–Stokes
- Constraints: pressure, BCs, complex symmetries
- No 1-order ODE formulation, no clear set of independent variables

THE POINT OF THIS TALK

!!! THE POINT OF THIS TALK !!!

UNLEARN: 3-d VISUALIZATION	THINK: ∞-d PHASE SPACE

instant in turbulent evolution:

a 3-d video frame,

each pixel a 3-d velocity field

instant in turbulent evolution:

a unique point

theory of turbulence =

geometry of the state space

[E. Hopf 1948]

3rd milleniumm theory

THINK IN STATE SPACE!

A: dual ODE/CDF representations

ODE in $\mathbb{R}^d$ CFD algorithm

$$
\begin{array}{ccc}
a(t) & \xrightarrow{\ \mathbf{u}(\mathbf{x},t)=a_n(t)\mathbf{\Phi}_n(x)\ } & \mathbf{u}(\mathbf{x},t) \\
\dot{a}=F(a)\ \big\downarrow & & \big\downarrow\ \mathrm{CFD} \\
a(t+T) & \xleftarrow[\ a_n=(\mathbf{u},\mathbf{\Phi}_n)_\Omega\ ]{} & \mathbf{u}(\mathbf{x},t+T)
\end{array}
$$

State space portraits = projections on well-chosen states $\hat{\mathbf{u}}_n$:

$\hat{a}_n(t) = (\mathbf{u}(t),\ \hat{\mathbf{u}}_n)_\Omega$ (integral over the box)

CFD/ODE: State space portraits

Visualize state space by projecting ODE $a(t)$ or CFD $\mathbf{u}(t)$ onto a few well-chosen $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ representative velocity fields

(e.g., a few equilibria and their unstable eigenvectors).

Construct $\{\hat{\mathbf{u}}_1, \hat{\mathbf{u}}_2, \hat{\mathbf{u}}_3\}$ by Gram-Schmidt orthogonalization and inner product

$$(\mathbf{u}_1, \mathbf{u}_2)_\Omega = \frac{1}{V}\int_0^{\mathcal{L}_x}\int_{-1}^{1}\int_0^{\mathcal{L}_z} \mathbf{u}_1 \cdot \mathbf{u}_2 \, dx\, dy\, dz$$

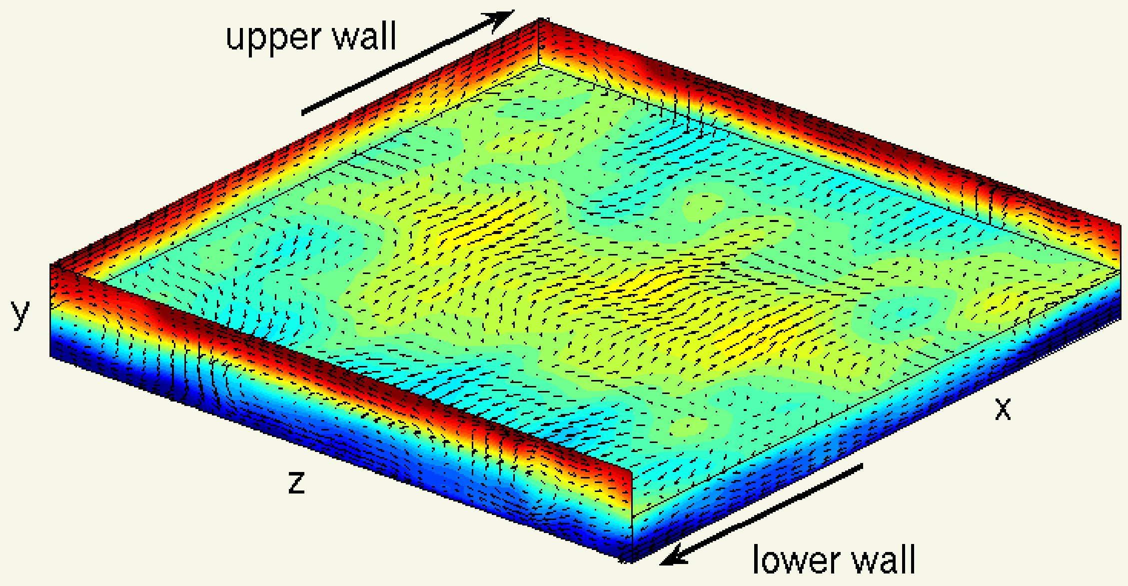
State space portraits = projections

$$\hat{a}_n(t) = (\mathbf{u}(t), \hat{\mathbf{u}}_n)_\Omega$$

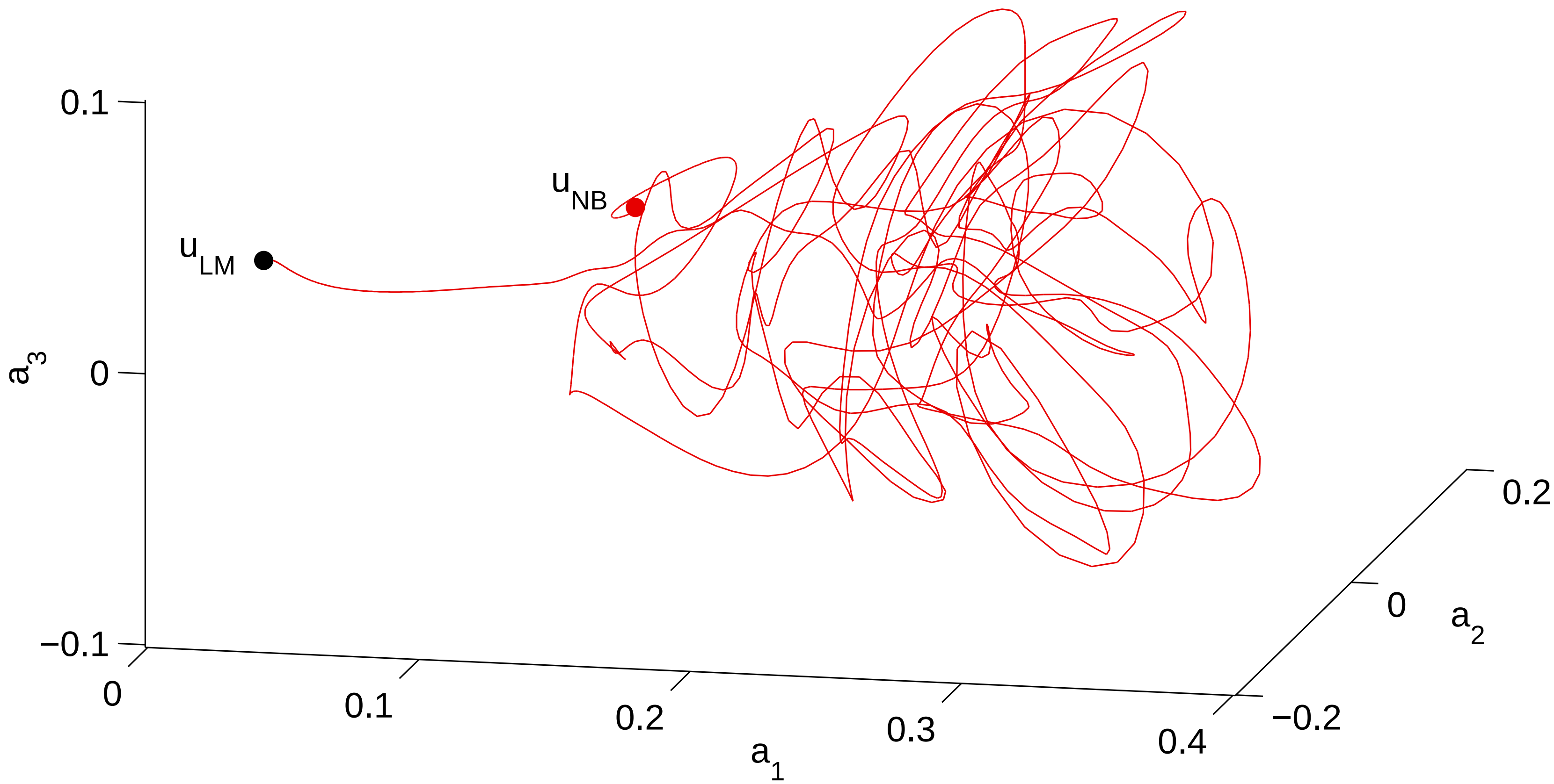
3rd millenium numerical simmulations : Plane Couette flow

Navier-Stokes: $$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} = -\nabla p + \frac{1}{Re}\nabla^2\mathbf{u}\,, \quad \nabla\cdot\mathbf{u} = 0$$

BCs: no-slip at walls $y = \pm 1$

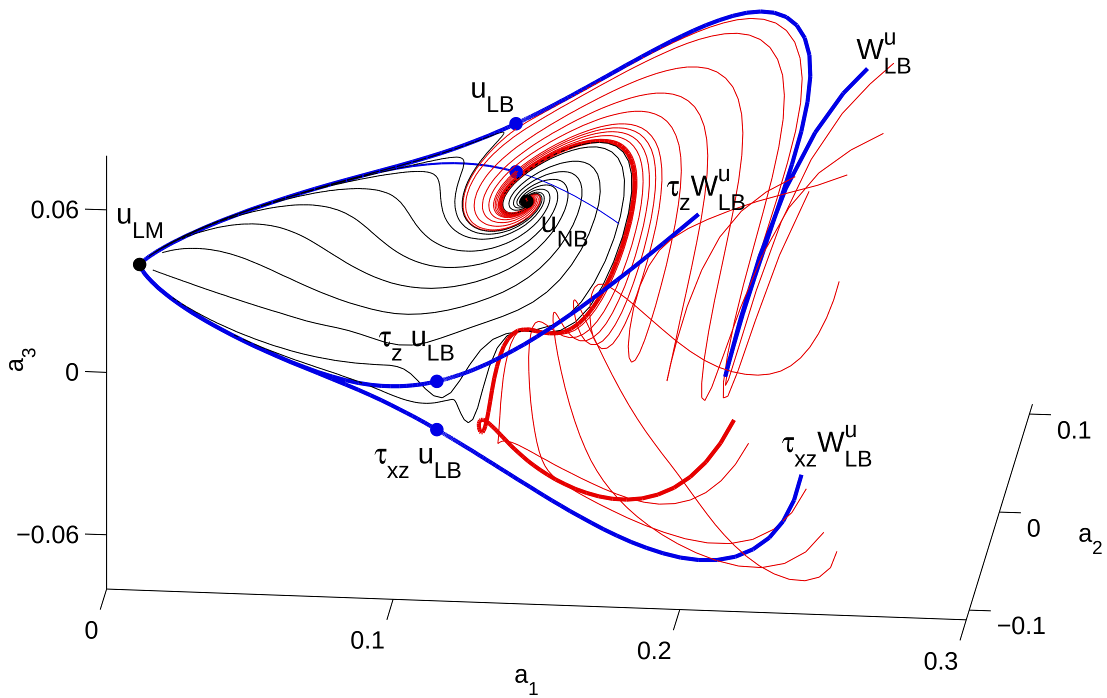


click here to see the online video

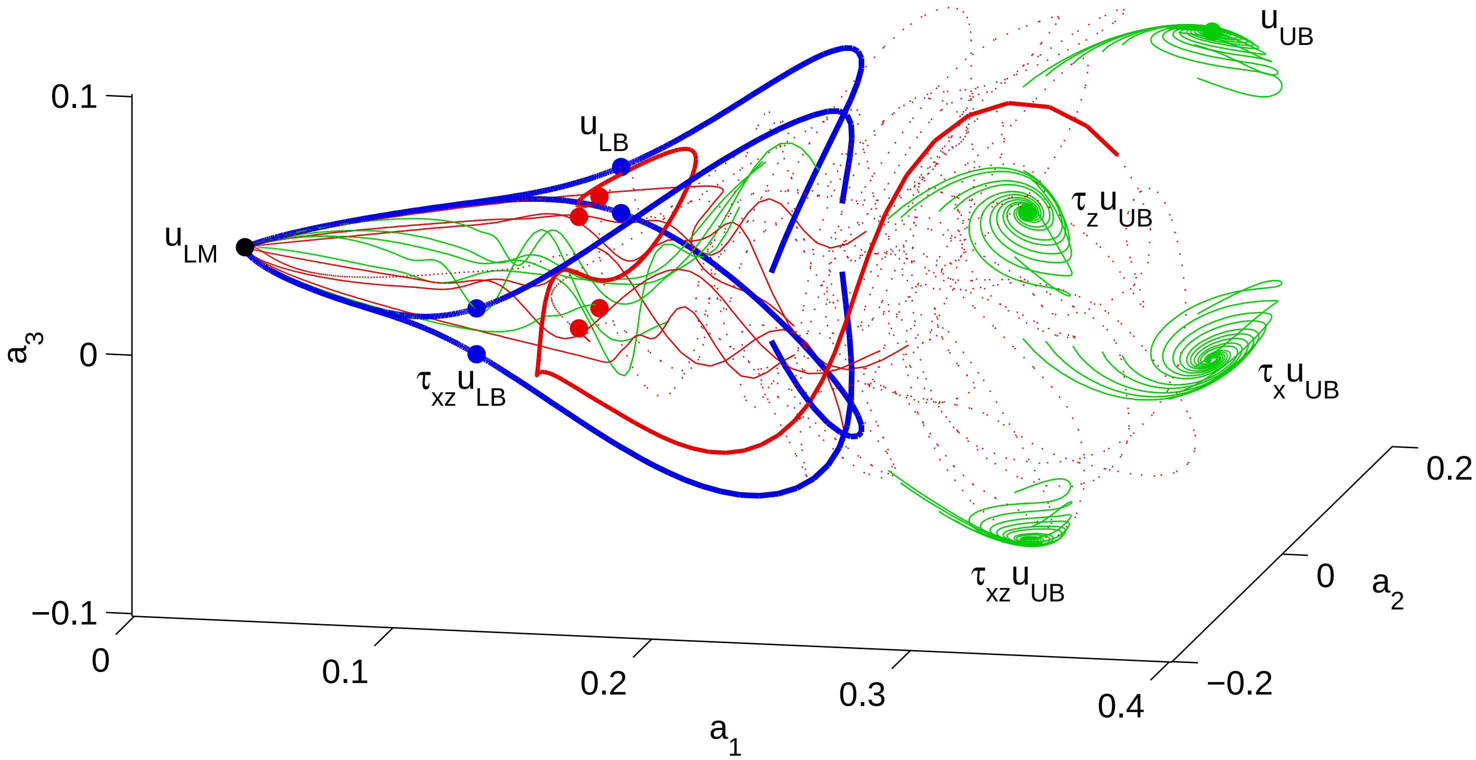


A transiently turbulent trajectory.

A stroll in 61,506 dimensions

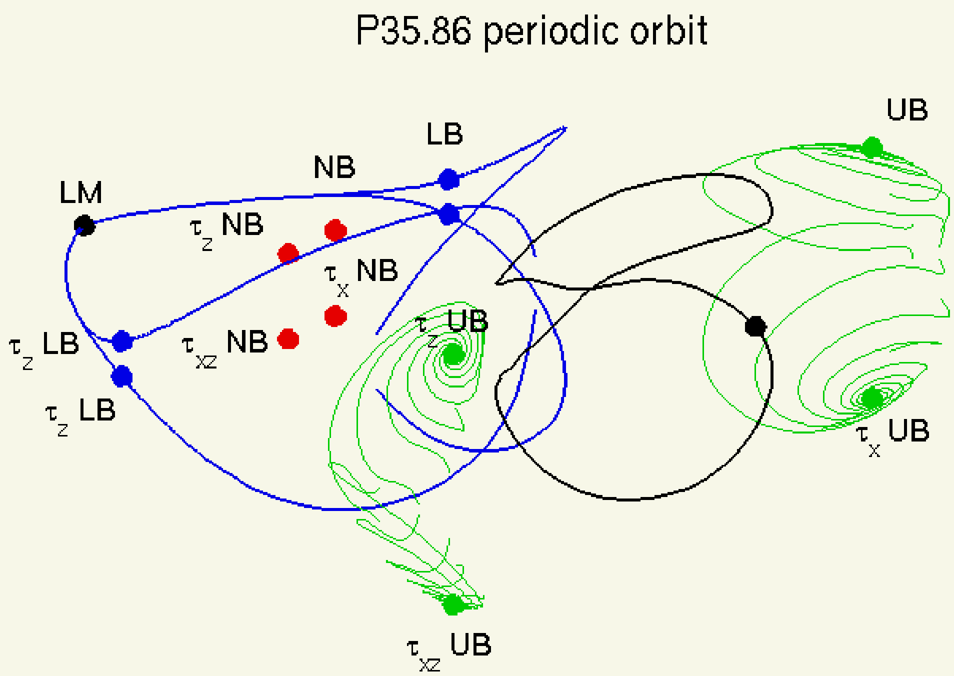


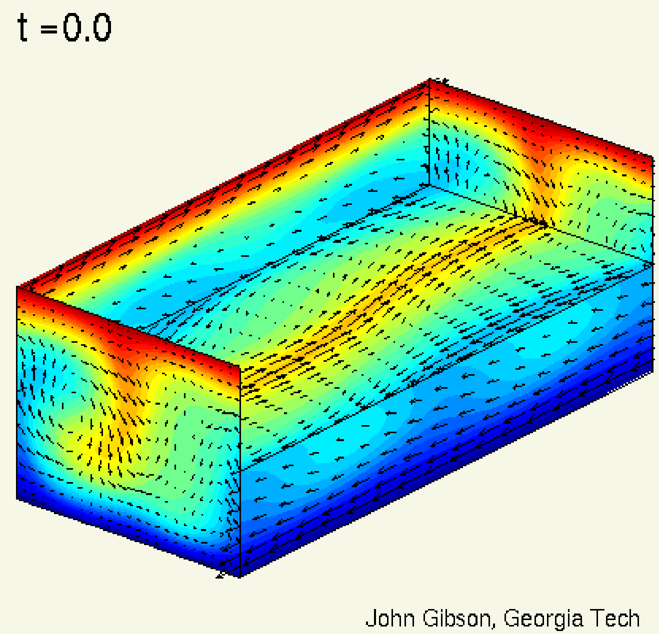
Unstable manifolds of projected from 61,506 dimensions to 3



A transiently turbulent trajectory in the $\mathbf{u}_{NB}$ unstable manifold, within the cage formed by $\mathbf{u}_{LB}$, $\mathbf{u}_{NB}$, $\mathbf{u}_{UB}$, their half-cell translations, and their unstable manifolds. The final decay to laminar of several other trajectories in the unstable manifolds of $\mathbf{u}_{NB}$ and $\mathbf{u}_{UB}$ are also shown.

Animation: T=35.86 periodic orbit

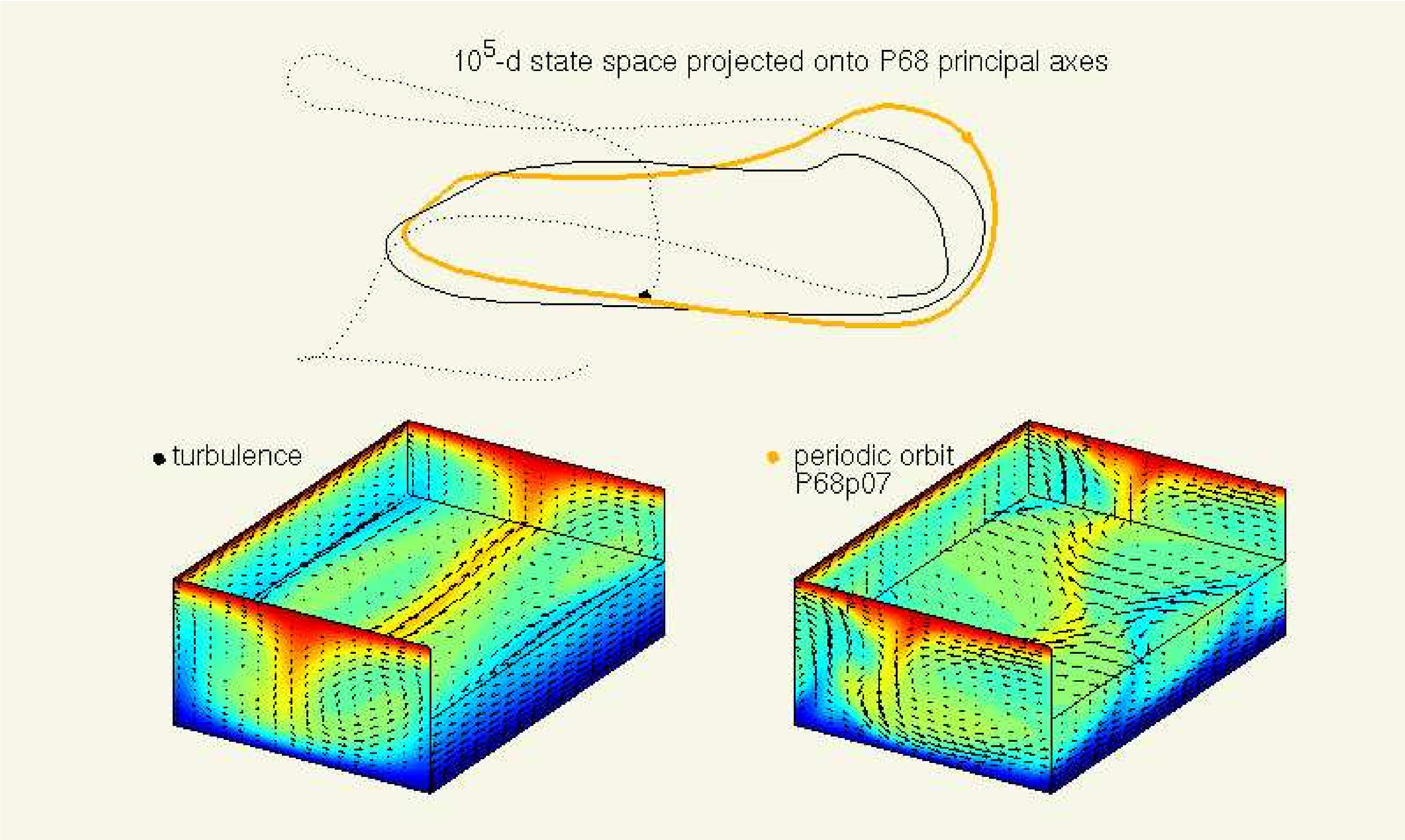




[click here to see the online video]

[ChaosBook.org/tutorials]

Periodic orbits shadow turbulence



A turbulent trajectory making a close pass to a periodic orbit.

click here to see the online video

ChaosBook.org/tutorials

In theory there is no difference between theory and practice. In practice there is. (Anonymous)

Pattern Formation and Control Lab

the team[1,2,3,4,5]

- Daniel Borrero
- Chris J. Crowley
- Roman O. Grigoriev
- Logan Kageorge
- Michael C. Krygier
- Ravi K. Pallantla
- Josh L. Pughe-Sanford
- Mike F. Schatz
- Balachandra Suri
- Jeffrey Tithof
- Wesley Toler

---

[1] B. Suri et al., Phys. Rev. E **98**, 023105 (2018).

[2] B. Suri et al., Phys. Rev. Lett. **125**, 064501 (2020).

[3] M. C. Krygier et al., J. Fluid Mech. **923**, A7 (2021).

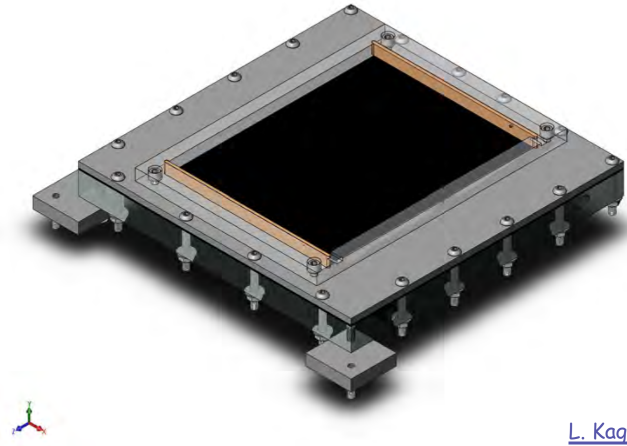
[4] C. J. Crowley et al., *Observing a dynamical skeleton of turbulence in Taylor-Couette flow experiments*, 2022.

[5] C. J. Crowley et al., Proc. Natl. Acad. Sci. **119**, 120665119 (2022).

3rd millennium experiment 1

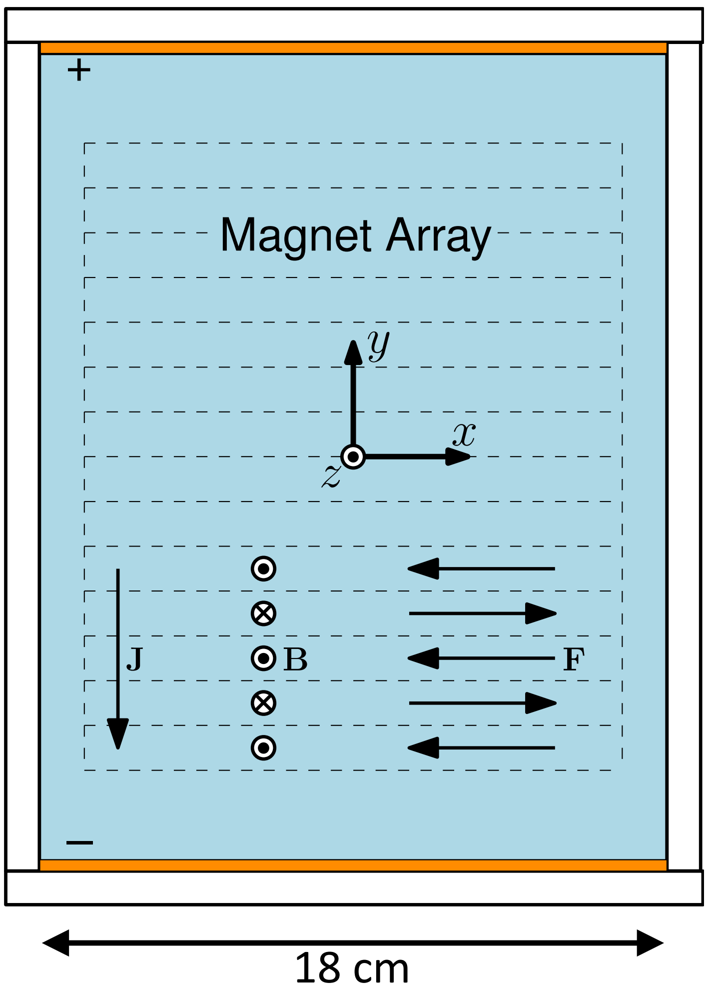
The Kolmogorov flow apparatus.

turbulence in 2 dimensions

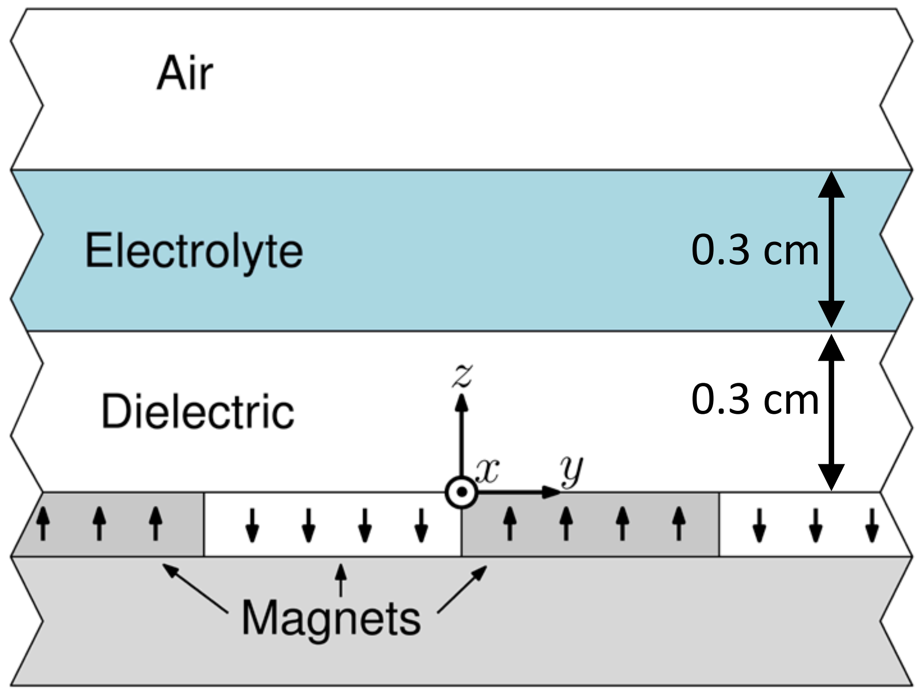


L. Kageorge PhD Thesis

Top View

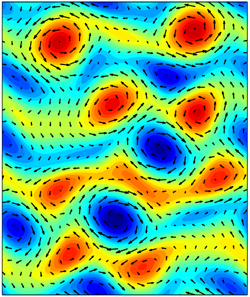


Cross Section View Along Y-Axis



L. Kageorge PhD Thesis

turbulence in 2D : 2-d video & state-space visualizations



(a)

$$\mathbf{u} = \begin{bmatrix} u_{(1,1)x} \\ u_{(2,1)x} \\ u_{(3,1)x} \\ \vdots \\ u_{(n,m)x} \\ u_{(1,1)y} \\ u_{(2,1)y} \\ u_{(3,1)y} \\ \vdots \\ u_{(n,m)y} \end{bmatrix}$$

(b)

Figure 1.1: (a) An illustrative example of a 2D velocity field. The heat map represents the vorticity of the velocity field, and helps guide the eye to the structure of the flow. (b) An example of how such a vector field can be converted into a state space vector by concatenating vector components.

click here to see the online video

turbulence in 2D : RPOs embedded in invariant measure

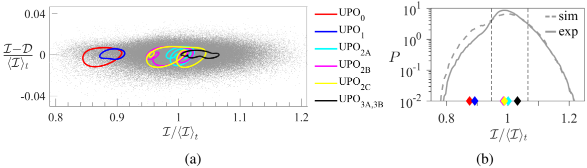
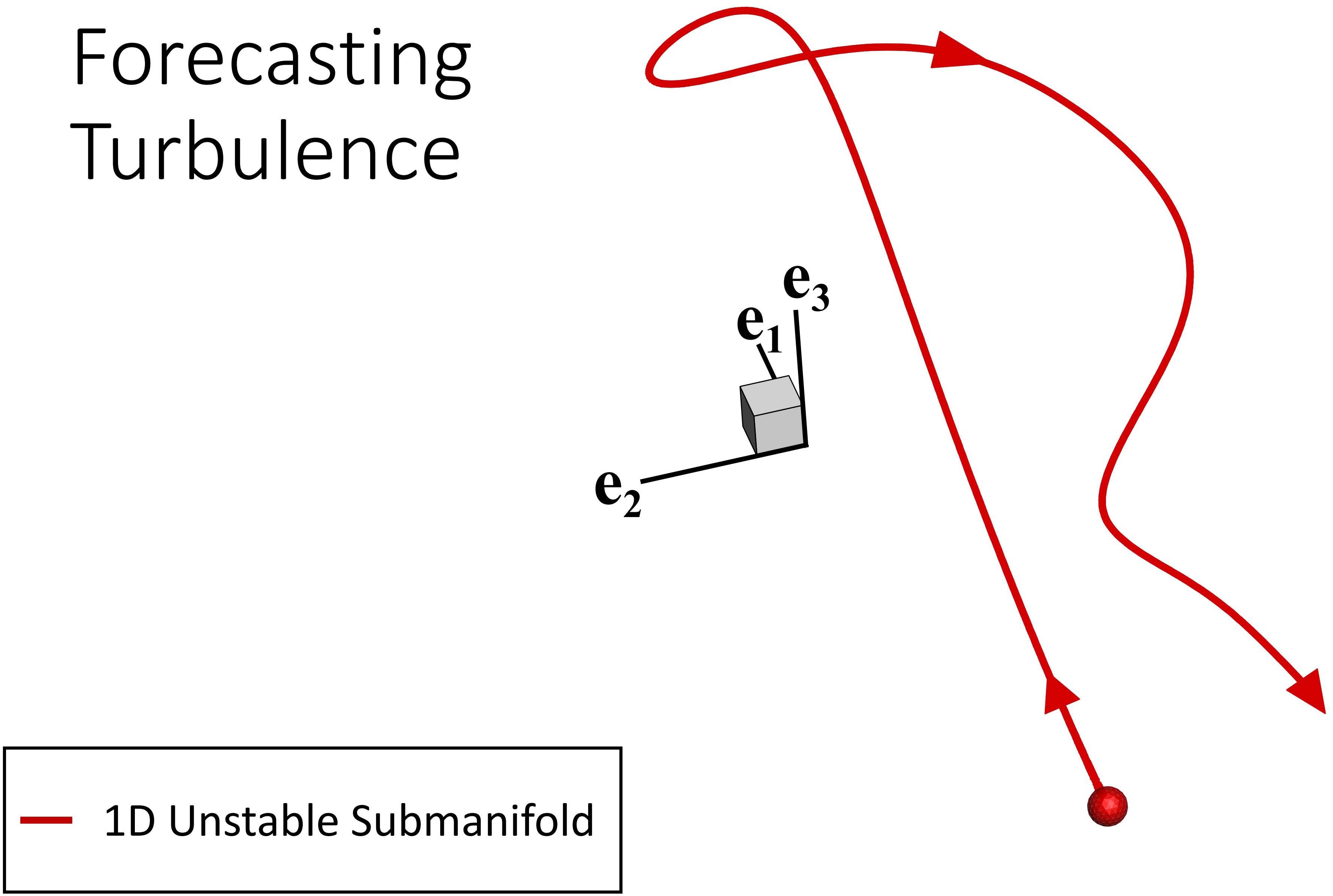


Figure 3.10: (a) Energy input rate $\mathcal{I}$ versus the difference between input and dissipation rates ($\mathcal{I} - \mathcal{D}$) for turbulent time series in experiment (scatter plot) and UPOs (closed loops). (b) Probability density function of $\mathcal{I}(t)$ for turbulent flow in experiment (solid gray) and DNS (dashed gray). Colored symbols show the mean values of $\mathcal{I}$ for each of the seven UPOs and the dashed black lines represent the range of $\mathcal{I}$ for UPO$_{2A\text{-}C}$ and UPO$_{3A,B}$.

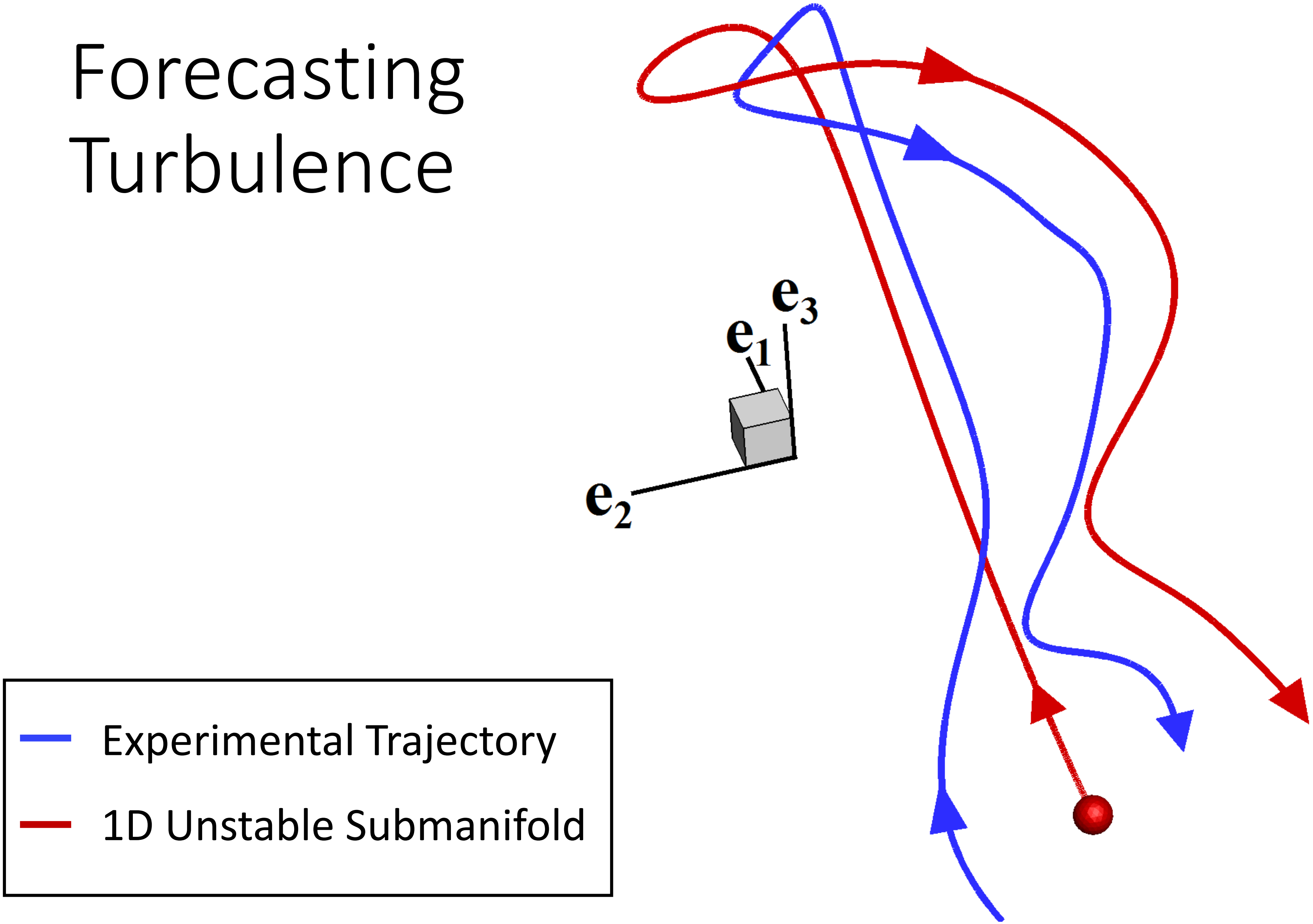
L. Kageorge PhD Thesis

Forecasting Turbulence

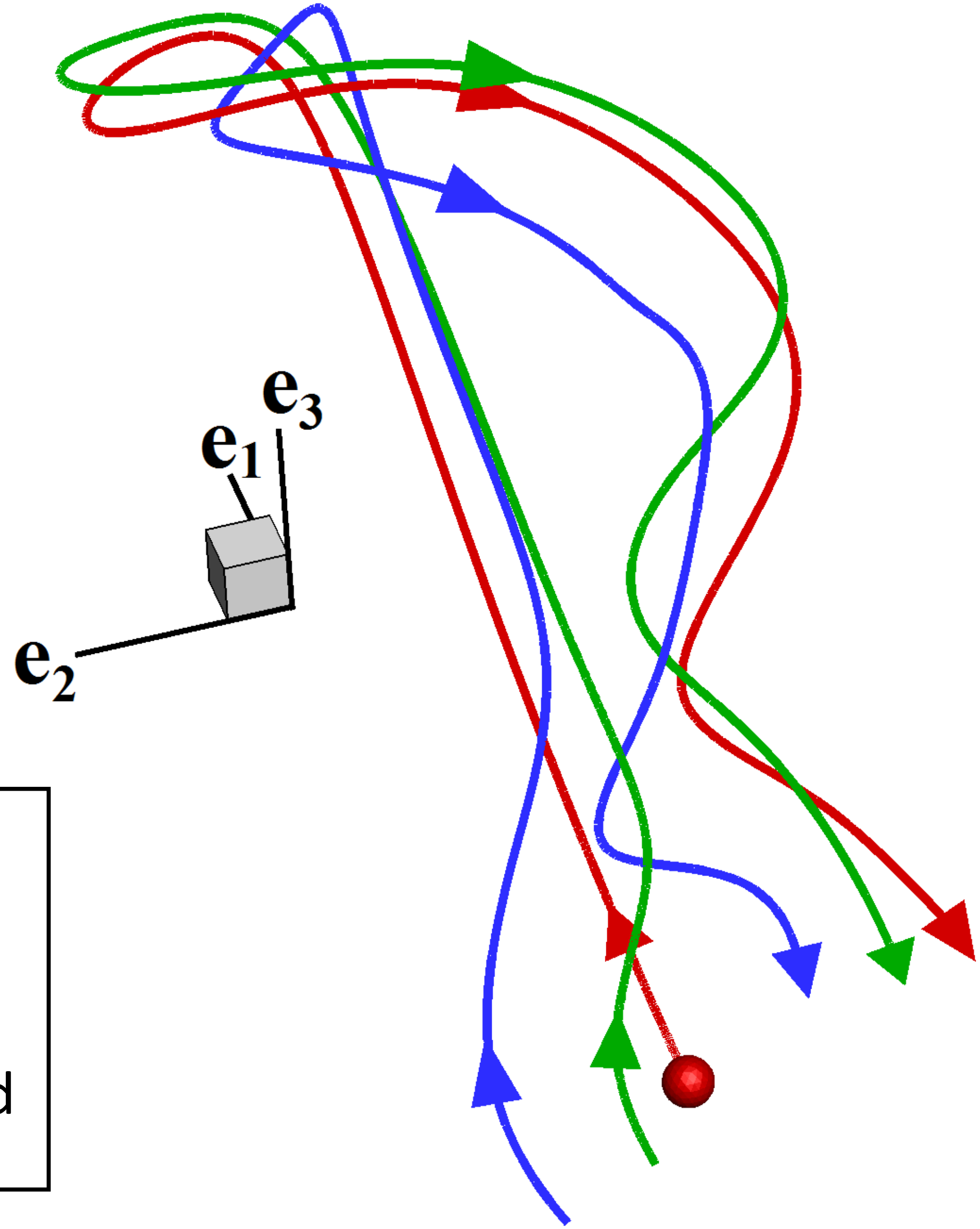


B Suri PhD Thesis

Forecasting Turbulence



Forecasting Turbulence



Numerical Trajectory

Experimental Trajectory

1D Unstable Submanifold

click here to see the online video

the big deal

the first experimental confirmation

1. of a Navier-Stokes predicted unstable manifold

3rd millennium experiment 2

turbulence in 3 dimensions : Taylor-Couette duct

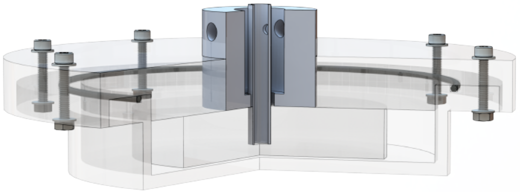


Figure 3.1: CAD model of the TCF cell. The cell is made of transparent PMMA allowing for unobstructed, optical access to the entire flow domain.

C.J. Crowley PhD Thesis

Taylor-Couette duct : full 3D flow visualization

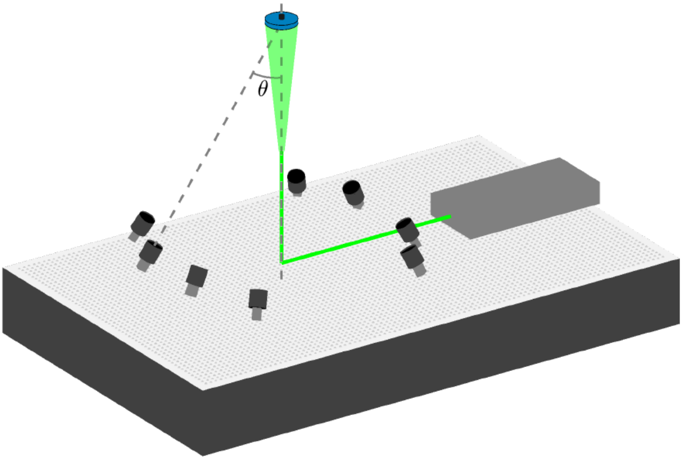


Figure 3.9: Camera configuration for 3D-3C measurements. The viewing angle, θ, is the angle between the camera and the z-axis of the TCF cell.

C.J. Crowley PhD Thesis

Taylor-Couette duct : a turbulent snapshot

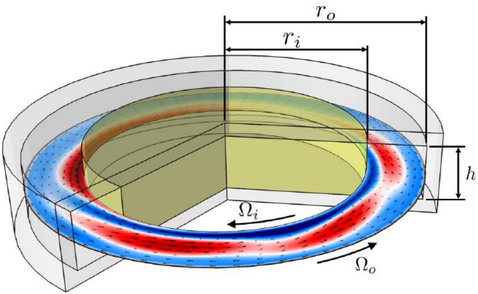
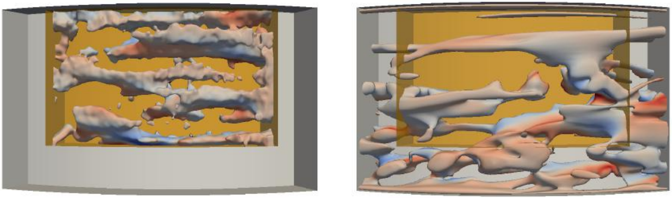


Figure 5.1: Turbulence is visualized in a laboratory flow between concentric, independently-rotating cylinders with radii r_i, r_o and corresponding angular velocities Ω_i, Ω_o. Fluid is confined between the cylinders and bounded axially by end caps co-rotating with the outer cylinder. The red-white-blue colors indicate the fluid's deviation from the mean azimuthal velocity component.

C.J. Crowley PhD Thesis

Taylor-Couette duct : experiment /DNS velocity isosurfaces



(a) (b)

Figure 4.5: A snapshot of a turbulent flow in experiment (a) and DNS (b). Each image shows a single isosurface of the perturbation field, $\tilde{\mathbf{v}}_\theta$, for $Re_\mathrm{i} = 650$ and $Re_\mathrm{o} = -1000$ inside a cylindrical subvolume. The color indicates the corresponding azimuthal velocity component. Red (blue) indicates flow in the same direction as the inner (outer) cylinder rotation.

C.J. Crowley PhD Thesis

Taylor-Couette duct : three state-space visualizations

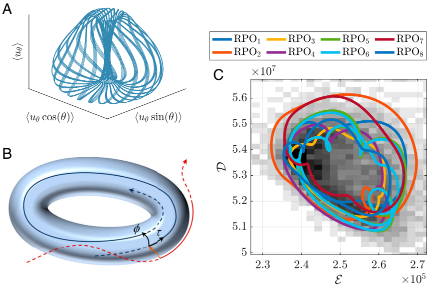
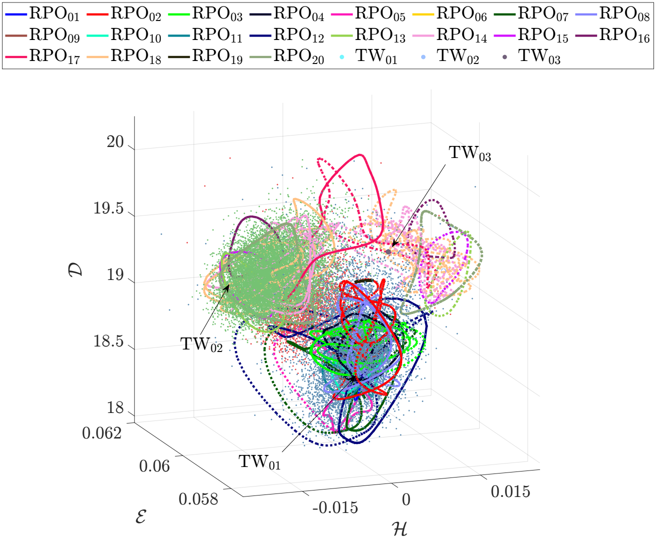


Fig. 2. Low-dimensional projections suggest that RPOs, i.e., solutions to the governing equations that recur indefinitely in time, are relevant to turbulence. (*A*) To demonstrate that RPOs are truly two-tori when rotational symmetry is not reduced, RPO_2 is plotted over 80 periods using the coordinates shown, where u_θ represents the azimuthal component of the flow velocity and $\langle \cdot \rangle$ indicates a spatial average. (*B*) Cartoon depicting how a portion of a turbulent trajectory (solid red curve) shadows, i.e., follows, an RPO (light blue surface) for a period of time. Shown in dark blue is the trajectory belonging to the RPO, which is most similar to the turbulent trajectory. The orange arrow relates a point on the turbulent trajectory to the point closest to it on the torus. (*C*) Using energy $\mathcal{E}$ and energy dissipation rate $\mathcal{D}$ of the flow as projection coordinates, eight RPOs are represented by closed trajectories (shown in color). The chaotic behavior of turbulence is indicated by the distribution (shown in gray) of visits to particular regions of the projection (darker regions have higher likelihood of visitation).

Taylor-Couette duct : RPOs embedded in invariant measure



Taylor-Couette duct : experiment trolls theory

examples of the experimental turbulent flow visiting (shadowing) relative periodic orbits (RPOs)

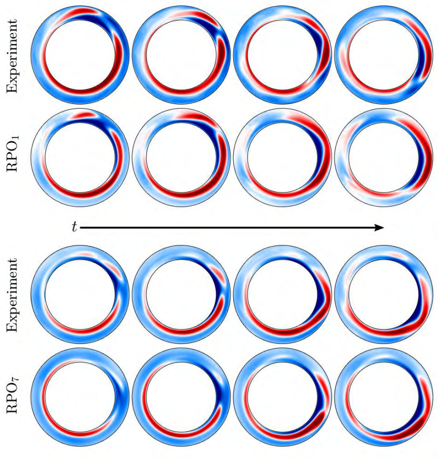


Figure 5.9: Experimental evidence that turbulence and RPOs, i.e., solutions to the governing equations that reccur indefinitely in time, co-evolve when the 'shadowing' criteria are met. Turbulence closely follows RPO_1 (top) and, during a different time interval, tracks RPO_7 (bottom).

experiment trolls theory :
a movie of the experimental turbulent flow visiting (shadowing) a relative periodic orbit

click here to see the online video

the big deal

walk!

the first experimental measurements

1. of a Navier-Stokes predicted unstable manifold
2. of shadowing by a Navier-Stokes predicted relative periodic orbit

Conclusions: geometry of Navier Stokes

dual ODE / CFD representations of Navier–Stokes

State space portraits

Computed eigenvalues, eigenfunctions of equilibrium states

Heteroclinic connections between equilibria

Turbulent dynamics

you can do this at home : channelflow.org

openpipeflow.org

orbithunter

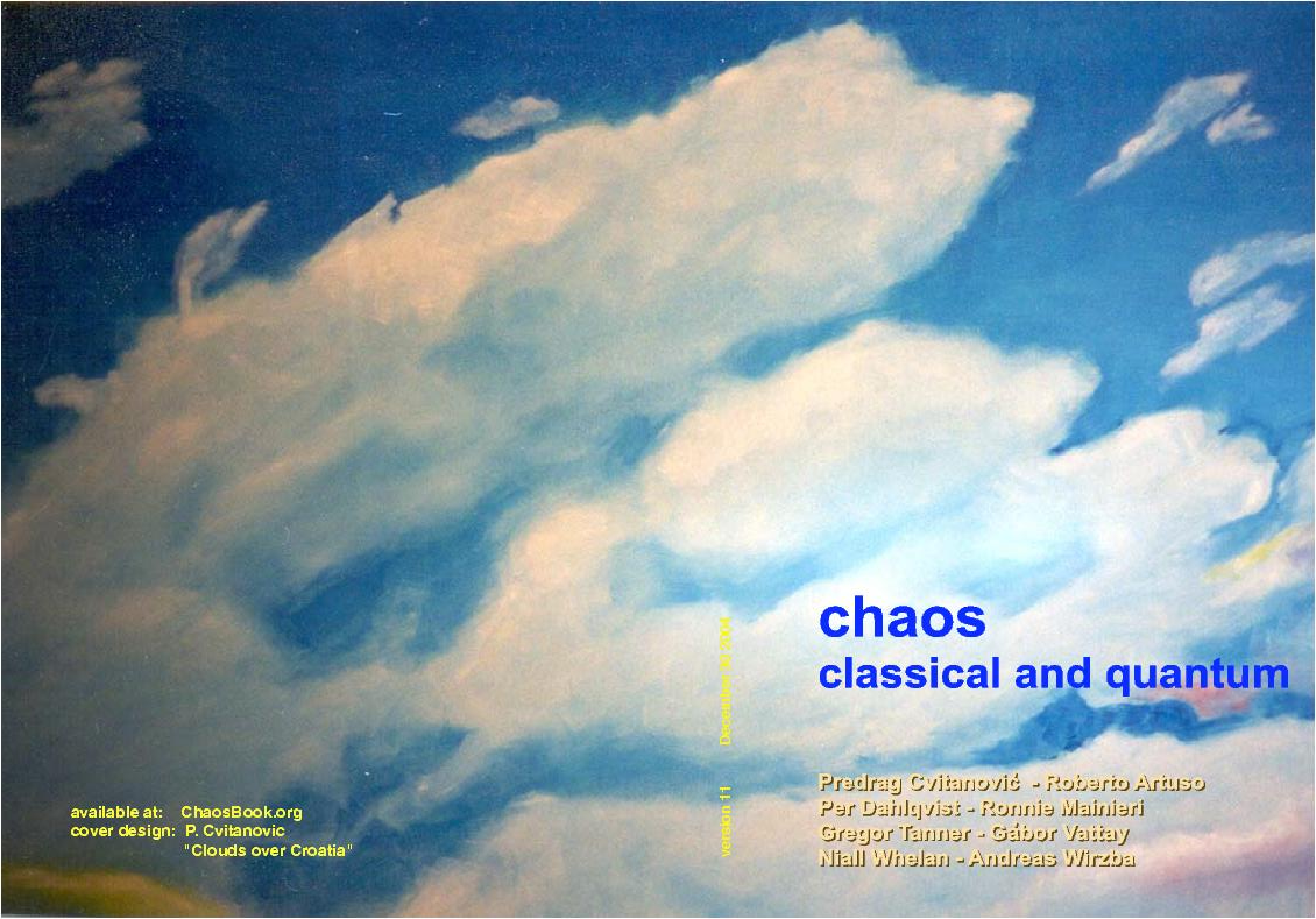
Future looks bright

next: 3rd millennium mathematics

In the seminal 1948 paper, E. Hopf presciently noted that
“The geometrical picture of the phase flow is, however, not the most important problem of the theory of turbulence. Of greater importance is the determination of the probability distributions associated with the phase flow”.

Hopf’s call for understanding probability distributions associated with the phase flow has indeed proven to be a key challenge, one in which dynamical systems theory has made the greatest progress.

see seminars on ChaosBook.org/overheads/spatiotemporal



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