Structures of Shear Turbulence

Fabian Waleffe

Depts of Mathematics and Engineering Physics
University of Wisconsin
Madison, WI, USA
Historical Overview

Adapted from


- Mean Flow Era 1883-1936
  (Reynolds, Prandtl, von Karman,...)
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  (hot wire anemometry, 2-point correlations,...)
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- CFD Era 1986-
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- CFD Era 1986-
- Dynamical Era: Self-Sustaining Process
  Exact Coherent Structures
  Periodic Solutions,...
"... at some point in the tube, always at a considerable distance from the intake, the colour band would all at once mix up with the surrounding water..."

Linear Stability - Nonlinear instability:
"... the critical velocity was very sensitive to disturbance in the water before entering the tubes.... This at once suggested the idea that the condition might be one of instability for disturbances of a certain magnitude and stability for smaller disturbances".
Reynolds similarity:  \[ R = \frac{UL}{\nu} \]

Mean flow + fluctuations (Reynolds 1894)
\[ \mathbf{v}(x, y, z, t) = \overline{U}(y) \hat{x} + \mathbf{u}(x, y, z, t) \]

Reynolds Stress:
\[ \tau = \nu \frac{d \overline{U}}{dy} - \overline{uv} \]

Law of the wall (Prandtl 1925) (wall unit scaling)

log law, velocity defect law (von Karman 1930)
Mean Flow Era Concepts (2)

- ‘eddy-viscosity’: \(-\overline{uv} \approx \tilde{\nu} \frac{dU}{dy}\)

- ‘Mixing length’ \(\ell\): \(\tilde{\nu} \approx u_* \ell\), \(u_* = \left|\overline{uv}\right|^{1/2}\)

\[\Rightarrow \tilde{\nu} = \ell^2 \left|\frac{dU}{dy}\right|\]

- Turbulence = random interactions of "eddies"

- Richardson Cascade (1922)
  "Big whorls have little whorls, Which feed on their velocity,
Little whorls have lesser whorls, And so on to viscosity"

  momentum transport \(\rightarrow\) energy cascade
Turbulent motion ‘not as random’ as molecular motion
\[ \langle u(x)v(x) \rangle \rightarrow 2\text{-point correlation} \langle u_i(x+r)u_j(x) \rangle \]
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isotropic turbulence "fundamental"
GI Taylor 1935, grid turbulence; Karman-Howarth 1938
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Kolmogorov: energy dissipation rate per unit mass \( \mathcal{E} \)

\[ \langle (u(x + r) - u(x))^n \rangle \propto (\mathcal{E}r)^{n/3} \]

if \[ \eta = (\nu^3/\mathcal{E})^{1/4} \ll r \ll L \] (inertial range)

(OK for \( n = 2 \), departures for \( n > 2 \), intermittency...)
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No momentum transport, only cascade of energy!
‘Horseshoe’ structure as self-consistent ‘turbulence molecule’ “optimized” for vortex stretching by mean shear

*Theodorsen, 1952*
2-point correlations $\rightarrow$ Structures

Townsend 1956 ‘The Structure of Turbulent Shear Flows’
Upper Bounds Era (GFD era!)

- 'Outline of a theory of turbulent convection' Malkus 1954
- 'Outline of a theory of turbulent shear flows' Malkus 1956
- Marginal Stability, Optimum transport
- Upper bound theory (Howard, Busse, Malkus & Ierley & L.M. Smith,...)
- Background field approach (Doering & Constantin, Kerswell,...)
‘Optimum’ transport structure

Busse, JFM 1970
Visualization Era: Streaks

Streaks with $100^+$ spacing

Kline, Reynolds, Schraub & Runstadler, JFM 1967
(diagram from Smith & Walker, 1997)
Visualization Era: Streaks everywhere!

Streaks in Turbulent Boundary Layer

Cantwell, Coles & Dimotakis, JFM 1978
Boundary Layer Transition: Streaks!

DNS of ‘Natural’ transition

*Rai & Moin, AIAA 1991*
Typical vortex structure: Theodorsen’s horseshoe!

Head & Bandyopadhyay, JFM 1981
Kim, Moin & Moser, JFM 1987

$R_T \approx 180$, $R_m \approx 5600$
DNS near-wall Structures

Asymmetric CS educed from KMM channel data at $R_T = 180$

Derek Stretch, 1990
Visualization/DNS Era

baguettes and croissants! not spaghetti!

3D ‘Inverse’ cascade: buffer layer → outer flow

Steve Robinson, ARFM 1991
...buffer layer to outer space!

STS 114: Return to Flight, launched July 26, 2005...
Mechanistic Era

Acarlar & Smith
JFM 1987
(synthetic streak regeneration)
+ Benney 1984
Cartoon of Self-Sustaining Process

- **Flow**
- **Streaks**
- **Rolls**
- **Streak Instability**
Self-Sustaining Process

advection of mean shear
Streamwise Rolls

Streaks

exp (i \alpha x) mode

instability of \( U(y,z) \)

nonlinear self-interaction

\( O(1/R) \)

\( O(1) \)

\( O(1/R) \)


Add artificial roll forcing $O(1/R^2)$ to Navier-Stokes Equations
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1D-1C Laminar flow $\rightarrow$ 2D-3C "streaky flow"

$$v = U(y, z)\hat{x} + \frac{1}{R} [V(y, z)\hat{y} + W(y, z)\hat{z}]$$
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2D-3C streaky flow inflectionally unstable (flapping flag)
SSP as Computational Method

- Add artificial roll forcing $O(1/R^2)$ to Navier-Stokes Equations
- 1D-1C Laminar flow $\rightarrow$ 2D-3C "streaky flow"
  \[ \mathbf{v} = U(y, z)\hat{x} + \frac{1}{R} \left[ V(y, z)\hat{y} + W(y, z)\hat{z} \right] \]
- 2D-3C streaky flow inflectionally unstable (flapping flag)
- Track 3D-3C solution that bifurcates from marginally stable streaky flow
  (Newton’s method, continuation, huge nonlinear eigenvalue problem)
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2D-3C streaky flow inflectionally unstable (flapping flag)

Track 3D-3C solution that bifurcates from marginally stable streaky flow

(Newton's method, continuation, huge nonlinear eigenvalue problem)

Continue till no need for artificial roll forcing
Bifurcation from streaky flow (PCF)

\[ V_{\text{max}} = \frac{F_r}{R} \]
Traveling Wave Solutions (1/2 PPF)

Waleffe, JFM 2001, PoF 2003
= ‘Exact Coherent Structures’ (ECS)
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... a feature not a bug! (more tomorrow!)
Traveling Wave Solutions

= ‘Exact Coherent Structures’ (ECS)

‘optimum’: \( R^+ = h^+ = 44, \quad L_x^+ = 273, \quad L_z^+ = 105 \)

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generic:

- Plane Couette flow (wall driven)
- Plane Poiseuille flow (pressure driven)
- free-slip, no-slip, any-slip
- Pipe flow
Traveling Waves in Pipe Flow

Faisst & Eckhardt, PRL 2003, Wedin & Kerswell JFM 2004

Hof et al. Science, Sept. 2004
ECS come in pairs

Bifurcation Diagram (saddle-node $F_r \equiv 0$):
upper and lower branches

Normalized wall shear rate (drag) in Plane Couette Flow vs $R$
ECS pairs

- **upper branch ECS** \( \approx \) backbone of turbulence
  
capture ‘statistics’ pretty well (mean and rms profiles)
  
  Waleffe PoF 2003, Jimenez *et al.* PoF 2005

- **lower branch ECS** \( \approx \) backbone of separatrix
  
  (laminar and turbulent separated by stable manifold(s) of lower branch(es))

- scaling of **lower branch ECS** \( \leftrightarrow \) transition threshold
Outstanding evidence for $1/R$ (Pipe flow)

Turbulence threshold scaling. The amplitude required to establish turbulence scales inversely with the Reynolds number of flow when the injected flux is normalized by the flux flowing through the pipe. (Adapted from ref. 2)

Hof, Juel & Mullin, PRL, Dec 2003; Physics Today Feb 2004
Fourier in $\theta = x - ct$ (**traveling wave**)

\begin{align*}
u(\theta, y, z) &= u_0(y, z) + u_1(y, z)e^{i\theta} + u_2(y, z)e^{i2\theta} + \cdots \\
v(\theta, y, z) &= v_0(y, z) + v_1(y, z)e^{i\theta} + \cdots \\
w(\theta, y, z) &= w_0(y, z) + w_1(y, z)e^{i\theta} + \cdots
\end{align*}

In SSP theory:

\begin{align*}
u_0(y, z) &= O(1), \quad v_0, w_0 = O\left(\frac{1}{R}\right), \quad u_1, v_1, w_1 = O\left(\frac{1}{R}\right)
\end{align*}
$R$-Scaling of harmonics (Rigid Rigid Couette)

\[ \max_y RMS_z \left( u_0(y, z) - \overline{U}(y) \right), \quad (v_0, w_0), (u_1, v_1, w_1), \ldots \]

*Waleffe & Wang, 2004*
$R$-Scaling dropping higher harmonics

$$\max_y RMS_z \left( u_0(y,z) - \overline{U}(y) \right), \quad (v_0, w_0), (u_1, v_1, w_1), \ldots$$

Waleffe & Wang, 2005

Trouble in the 1st harmonic?...
Structure of Lower branch @ $R = 6200$ (RRC)

$u_0$ large, $Q = \nabla^2 p/2$ small

Waleffe & Wang, 2004, 2005
Critical layer! $u_0(y, z) - c = 0$
\[(\alpha, \gamma) = (1.14, 2.505), \ R = [6196, 31599], \ x = 0, \ w_1\]
LBS Rolls @ $R = 6200$ (RRC)
(\alpha, \gamma) = (1.14, 2.505), R = [6196, 31599], v_0
SSP exact as $R \to \infty$ (but...)

$O(1/R)$ rolls $\longrightarrow O(1)$ streaks $\longrightarrow$ streak instability
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- $O(1/R)$ rolls $\to$ $O(1)$ streaks $\to$ streak instability

- $\approx$ Benney’s ‘Mean Flow-First Harmonic Theory’
  (inviscid wavepackets, $\epsilon \to$ viscous traveling wave, $1/R$)
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- $O(1/R)$ rolls $\to O(1)$ streaks $\to$ streak instability

- $\approx$ Benney’s ‘Mean Flow-First Harmonic Theory’
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- but... 2D Critical Layer $u_0(y, z) - c = 0$
  complicates scaling and asymptotics
SSP asymptotics: coupled 2D modes

\[
(\partial_t + \mathbf{v} \cdot \nabla)\mathbf{v} + \nabla p = \frac{1}{R} \nabla^2 \mathbf{v}, \quad \nabla \cdot \mathbf{v} = 0
\]

as \( R \to \infty \), there are SF TWS

\[
\mathbf{v}(x, y, z, t) \sim u_0(y, z) \hat{x} + \left( v_0(y, z) \hat{y} + w_0(y, z) \hat{z} \right) + e^{i \alpha(x-ct)} \hat{\mathbf{v}}_1(y, z) + c.c.
\]
SSP asymptotics: leading order eqns

- advection-diffusion of streaky flow $u_0(y, z)$:

$$v_0 \frac{\partial u_0}{\partial y} + w_0 \frac{\partial u_0}{\partial z} = \frac{1}{R} \nabla^2 u_0$$
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- Streamwise rolls: $v_0, w_0 \sim 1/R \Rightarrow u_0(y, z)$
advection-diffusion of streaky flow $u_0(y, z)$:

$$v_0 \frac{\partial u_0}{\partial y} + w_0 \frac{\partial u_0}{\partial z} = \frac{1}{R} \nabla^2 u_0$$

Streamwise rolls: $v_0(y, z) = \partial_z \Psi_0$, $w_0(y, z) = -\partial_y \Psi_0$

$$\frac{1}{R} \nabla^4 \Psi_0 = J(\nabla^2 \Psi_0, \Psi_0)$$

Rolls decoupled from streaky flow

$$J(A, B) = \partial_y A \partial_z B - \partial_z A \partial_y B$$
SSP asymptotics: leading order eqns

- advection-diffusion of streaky flow $u_0(y, z)$:

$$v_0 \frac{\partial u_0}{\partial y} + w_0 \frac{\partial u_0}{\partial z} = \frac{1}{R} \nabla^2 u_0 - \frac{\partial \overline{u_1 v_1^x}}{\partial y} - \frac{\partial \overline{u_1 w_1^x}}{\partial z}$$

- Streamwise rolls: $v_0(y, z) = \partial_z \Psi_0$, $w_0(y, z) = -\partial_y \Psi_0$

$$\frac{1}{R} \nabla^4 \Psi_0 = J(\nabla^2 \Psi_0, \Psi_0) + \frac{\partial^2}{\partial y \partial z} (\overline{v_1^2 x} - \overline{w_1^2 x}) + \left( \frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial y^2} \right) \overline{v_1 w_1^x}$$

1st harmonic feeds on streaks to sustain rolls

$v_1 \sim 1/R \quad v_0, w_0 \sim 1/R$
SSP asymptotics: leading order eqns

- **advection-diffusion of streaky flow** \( u_0(y, z) \):

\[
v_0 \frac{\partial u_0}{\partial y} + w_0 \frac{\partial u_0}{\partial z} = \frac{1}{R} \nabla^2 u_0 - \frac{\partial u_1 v_1^x}{\partial y} - \frac{\partial u_1 w_1^x}{\partial z}
\]

- **Streamwise rolls**:

\[
v_0(y, z) = \partial_z \Psi_0, \quad w_0(y, z) = -\partial_y \Psi_0
\]

\[
\frac{1}{R} \nabla^4 \Psi_0 = J(\nabla^2 \Psi_0, \Psi_0) + \frac{\partial^2}{\partial y \partial z} (\overline{v_1^{2x}} - \overline{w_1^{2x}}) + \left( \frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial y^2} \right) \overline{v_1 w_1^x}
\]

- **First Harmonic neutrally stable** \( v_1 = e^{i\alpha x} \hat{v}_1(y, z) + c.c. \)

\[
(v_0 - c\hat{x}) \cdot \nabla v_1 + v_1 \cdot \nabla v_0 + \nabla p_1 = \frac{1}{R} \nabla^2 v_1, \quad \nabla \cdot v_1 = 0
\]
Conclusions

Theory catching up at last!
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- What about ‘turbulence’?
  unstable solutions...
- The neglected lower branch states: ‘large’ scale, 2D self-sustained critical layer
  drag only 10-20% higher than laminar! control?