Structures of Shear Turbulence

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Historical Overview

Adapted from

"A Brief History of Boundary Layer Structure Research," S.J. Kline 1997

- Mean Flow Era 1883-1936
  (Reynolds, Prandtl, von Karman,...)

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- Structural Era 1956-1986 (visualizations)

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- CFD Era 1986-

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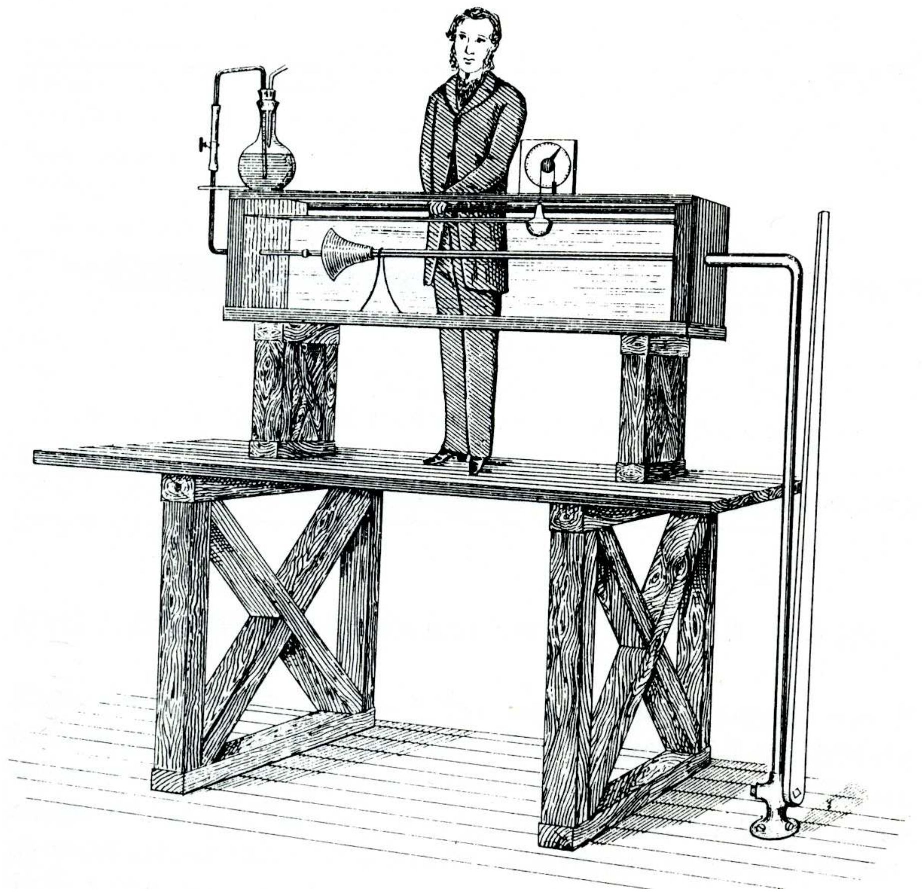
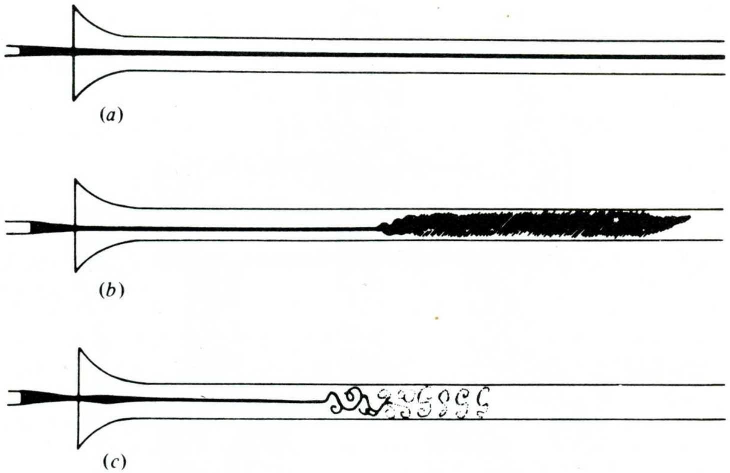
- CFD Era 1986-
- Dynamical Era: Self-Sustaining Process
  Exact Coherent Structures
  Periodic Solutions,...

Reynolds' Pipe Flow Experiments





Osborne Reynolds, Manchester 1883

Reynolds' Key Observations

- *"... at some point in the tube, always at a considerable distance from the intake, the colour band would all at once mix up with the surrounding water..."*
- Linear Stability - Nonlinear instability:
  "... the critical velocity was very sensitive to disturbance in the water before entering the tubes.... This at once suggested the idea that the condition might be one of instability for disturbances of a certain magnitude and stability for smaller disturbances".

Mean Flow Era Concepts

- Reynolds similarity: $R = \frac{UL}{\nu}$
- Mean flow + fluctuations (Reynolds 1894)

$$\boldsymbol{v}(x, y, z, t) = \overline{U}(y)\hat{\boldsymbol{x}} + \boldsymbol{u}(x, y, z, t)$$

- Reynolds Stress: $\tau = \nu \frac{d\overline{U}}{dy} - \overline{uv}$
- Law of the wall (Prandtl 1925) (wall unit scaling)
- log law, velocity defect law (von Karman 1930)

Mean Flow Era Concepts (2)

- 'eddy-viscosity': $-\overline{uv} \approx \tilde{\nu}\frac{d\overline{U}}{dy}$
- 'Mixing length' ℓ: $\tilde{\nu} \approx u_*\ell, \quad u_* = \left|\overline{uv}\right|^{1/2}$

$$\Rightarrow \tilde{\nu} = \ell^2 \left|\frac{d\overline{U}}{dy}\right|$$

- Turbulence= random interactions of "eddies"
- Richardson Cascade (1922)
  "Big whorls have little whorls, Which feed on their velocity,

  Little whorls have lesser whorls, And so on to viscosity"

momentum transport $\longrightarrow$ energy cascade

Statistical Era

- Turbulent motion ‘not as random’ as molecular motion $\langle u(x)v(x)\rangle \rightarrow$ 2-point correlation $\langle u_i(x+r)u_j(x)\rangle$

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- Kolmogorov: energy dissipation rate per unit mass $\mathcal{E}$

$$\langle (u(x+r)-u(x))^n\rangle \propto (\mathcal{E}r)^{n/3}$$

if $\quad \eta = (\nu^3/\mathcal{E})^{1/4} \ll r \ll L \quad$ (inertial range)

(OK for $n=2$, departures for $n>2$, intermittency...)

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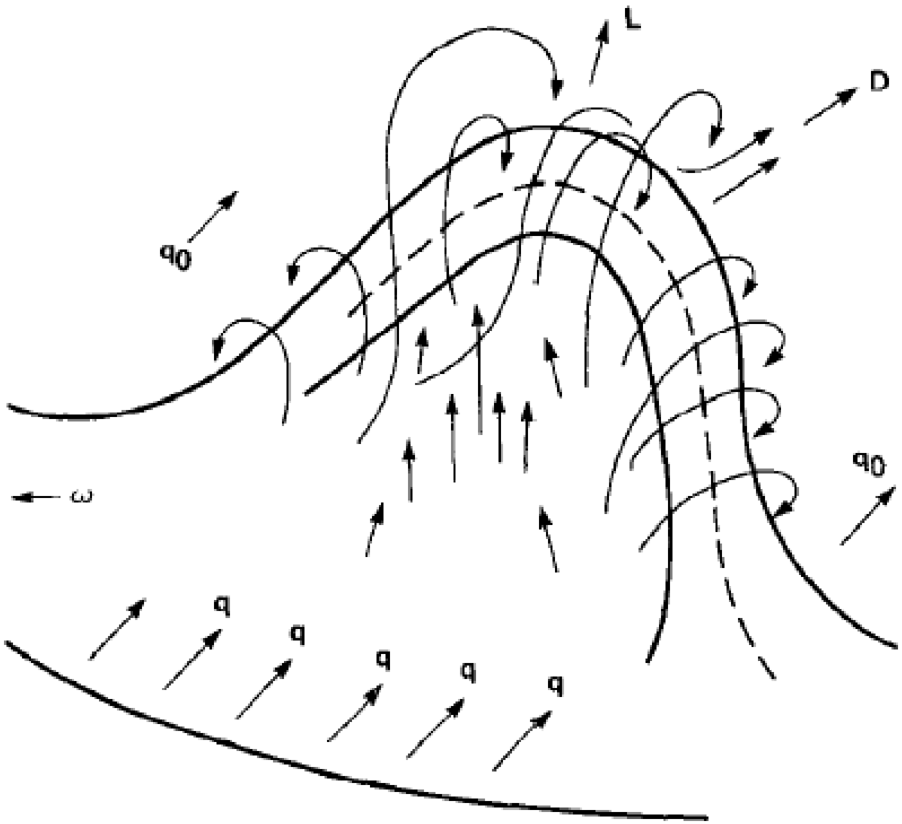
- No momentum transport, only cascade of energy!

‘Horseshoe’ structure as self-consistent ‘turbulence molecule’ “optimized” for vortex stretching by mean shear



Theodorsen, 1952

Structural Era

2-point correlations $\longrightarrow$ Structures

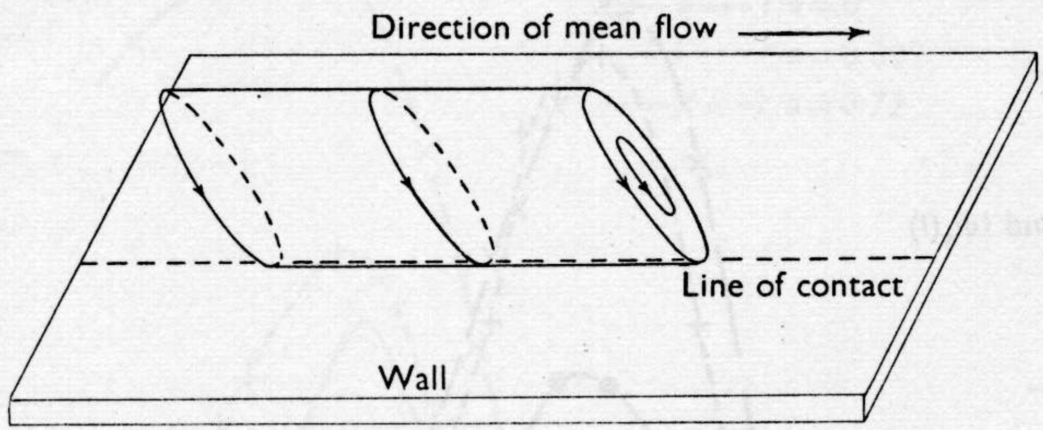


Fig. 9.8. An attached eddy.

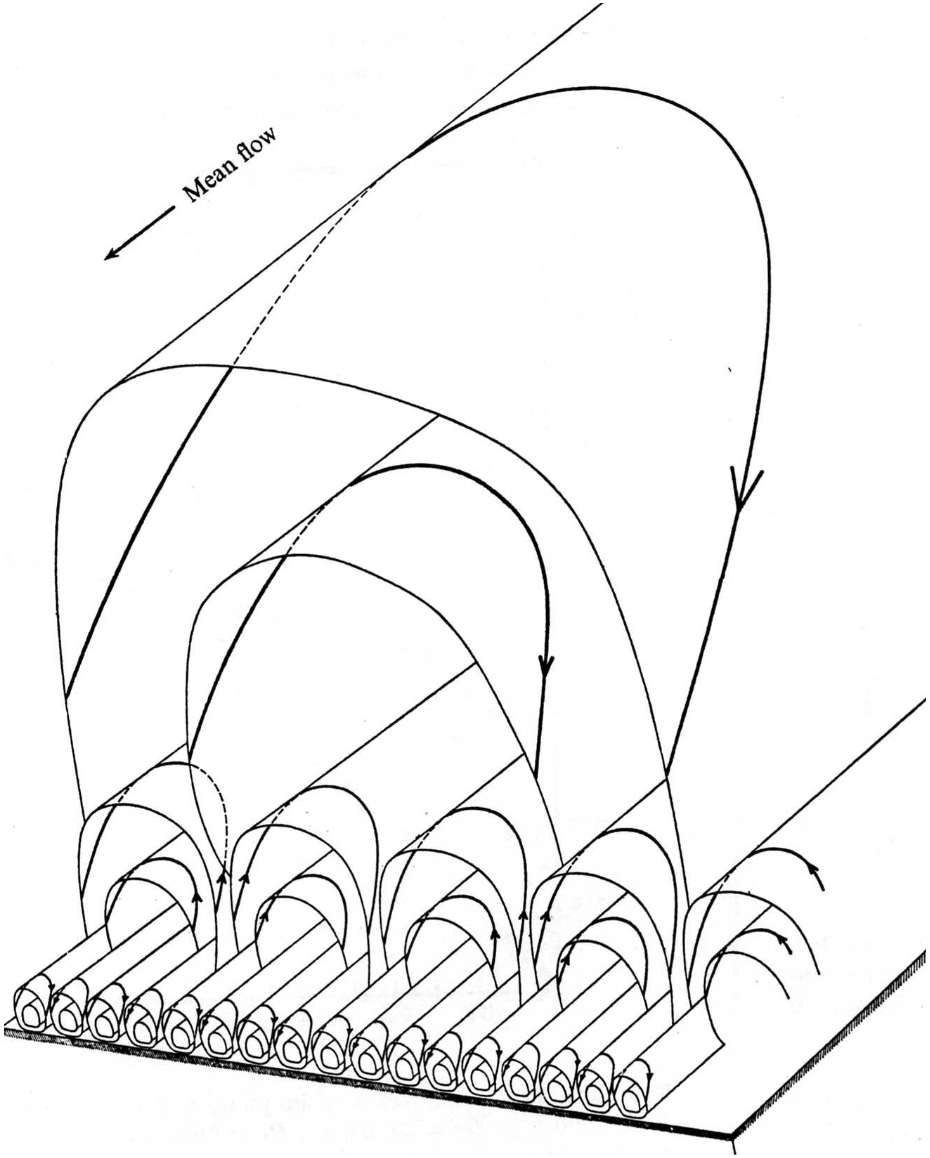
Townsend 1956 'The Structure of Turbulent Shear Flows'

Upper Bounds Era (GFD era!)

- *'Outline of a theory of turbulent convection'* Malkus 1954
  'Outline of a theory of turbulent shear flows' Malkus 1956
- Marginal Stability, Optimum transport
- Upper bound theory
  (Howard, Busse, Malkus & Ierley & L.M. Smith,...)
- Background field approach (Doering & Constantin, Kerswell...)

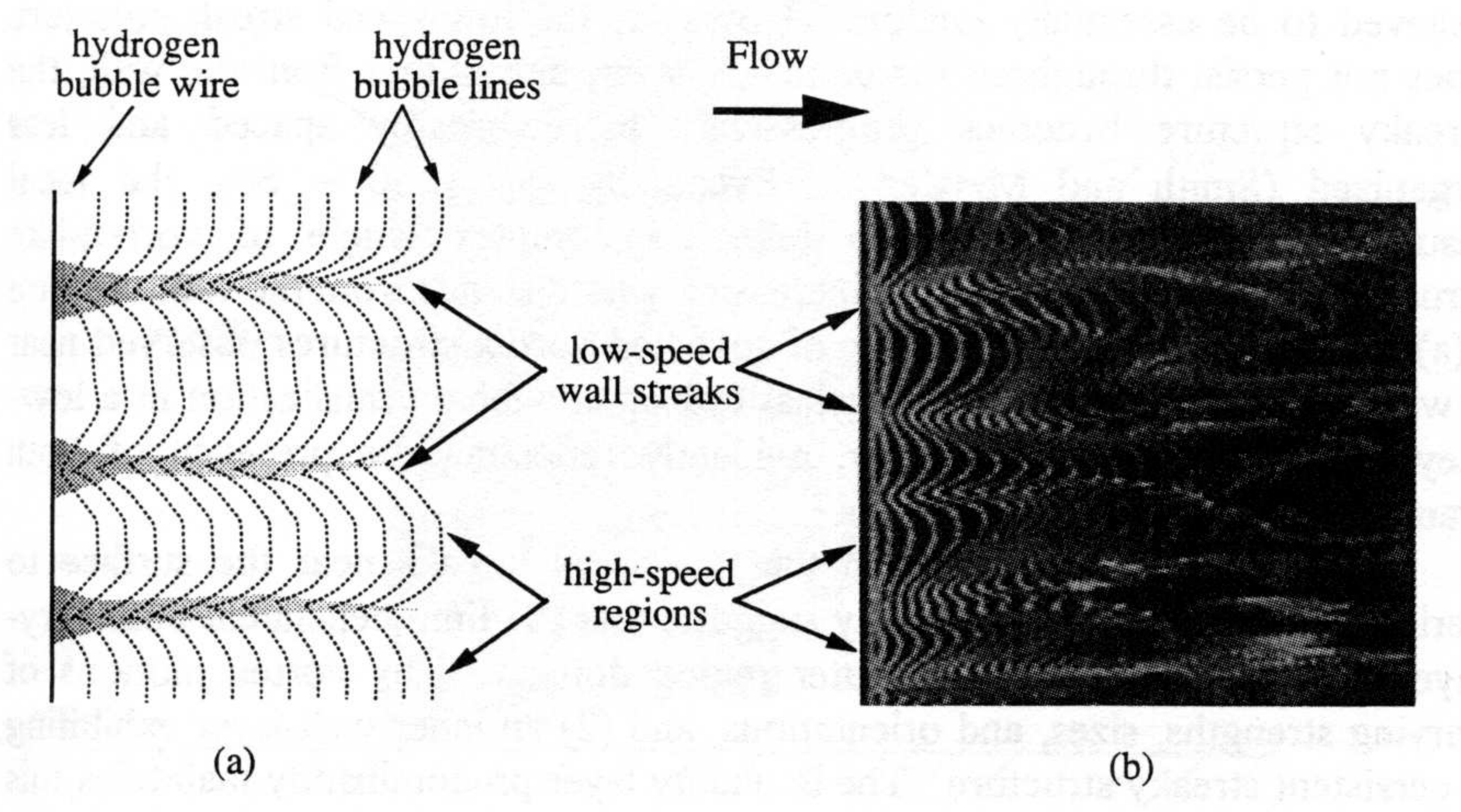
‘Optimum’ transport structure

Busse, JFM 1970



Visualization Era: Streaks

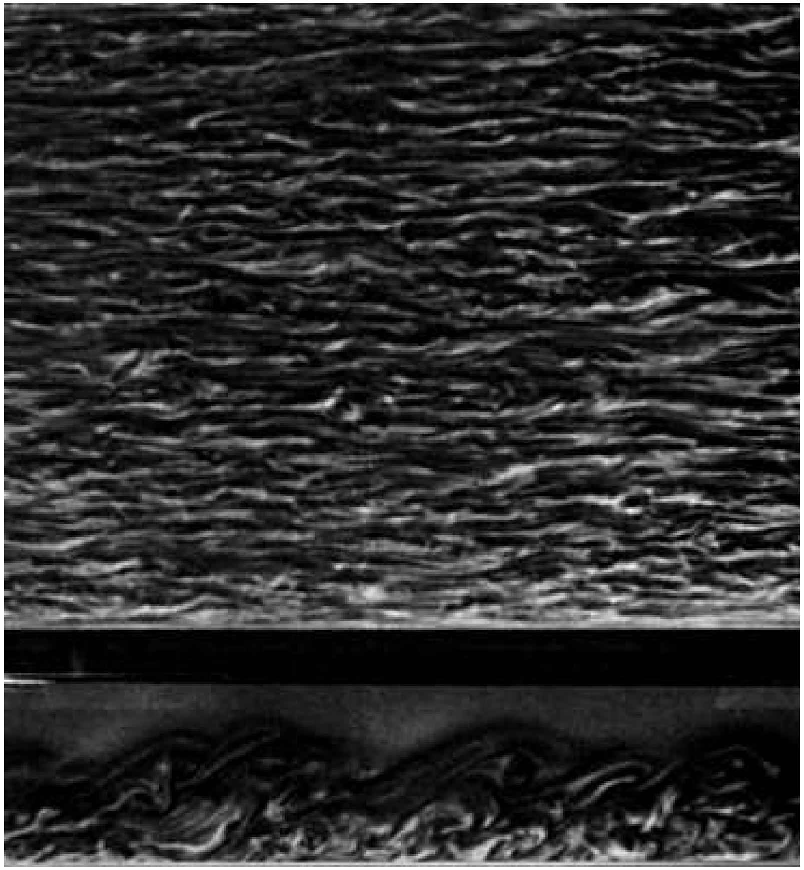
Streaks with 100^+ spacing



Kline, Reynolds, Schraub & Runstadler, JFM 1967
(diagram from Smith & Walker, 1997)

Visualization Era: Streaks everywhere!

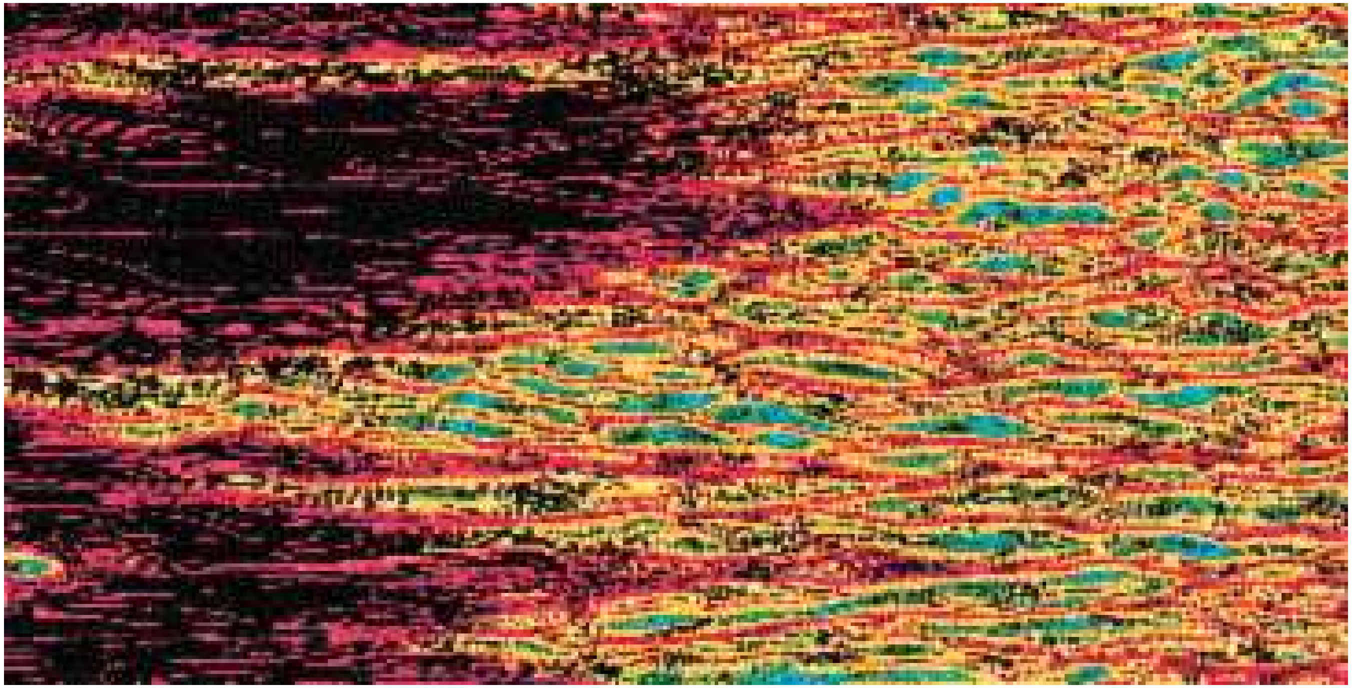
Streaks in Turbulent Boundary Layer



Cantwell, Coles & Dimotakis, JFM 1978

Boundary Layer Transition: Streaks!

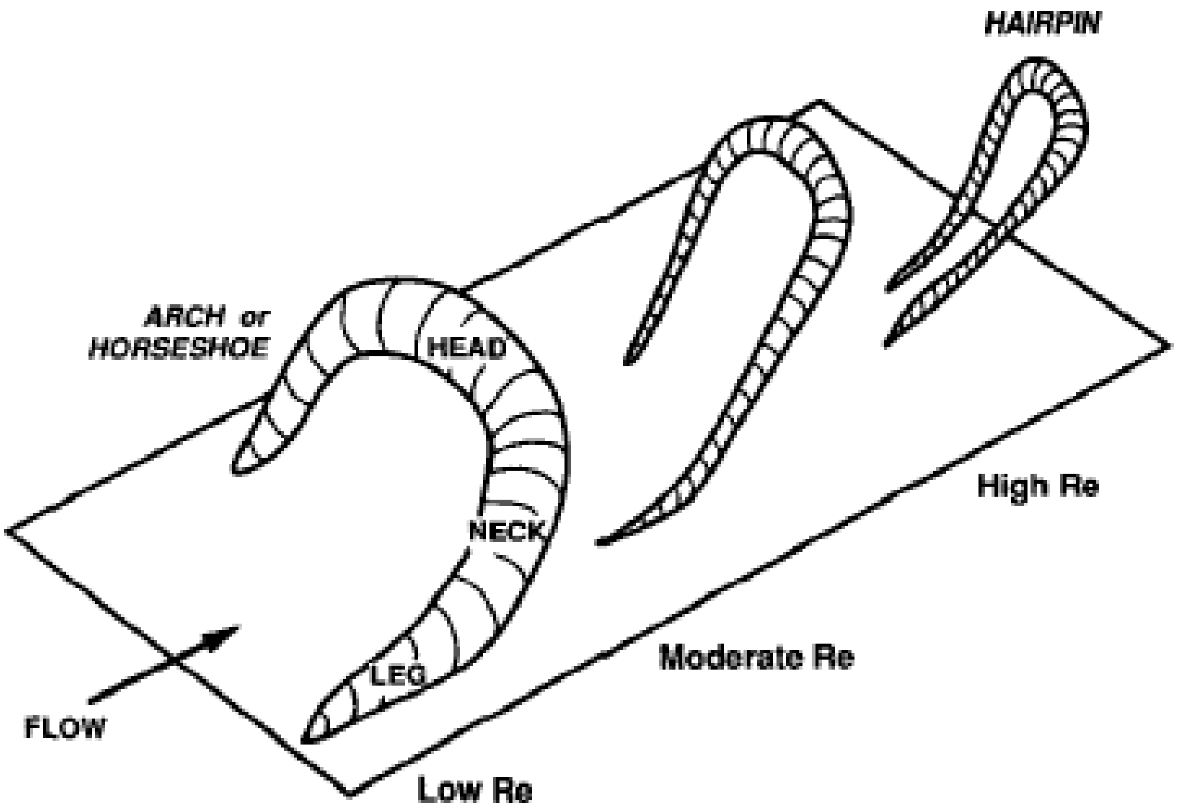
DNS of ‘Natural’ transition



Rai & Moin, AIAA 1991

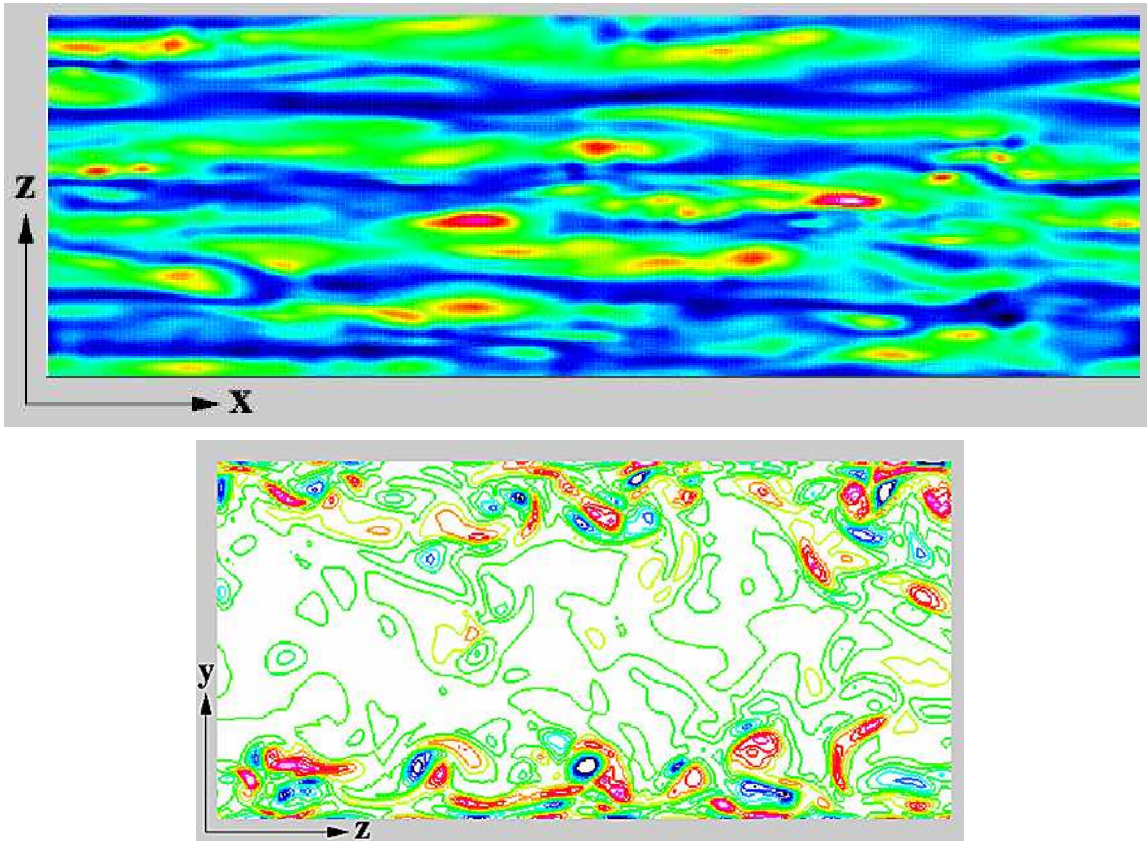
Visualization Era

Typical vortex structure: Theodorsen's horseshoe!



Head & Bandyopadhyay , JFM 1981

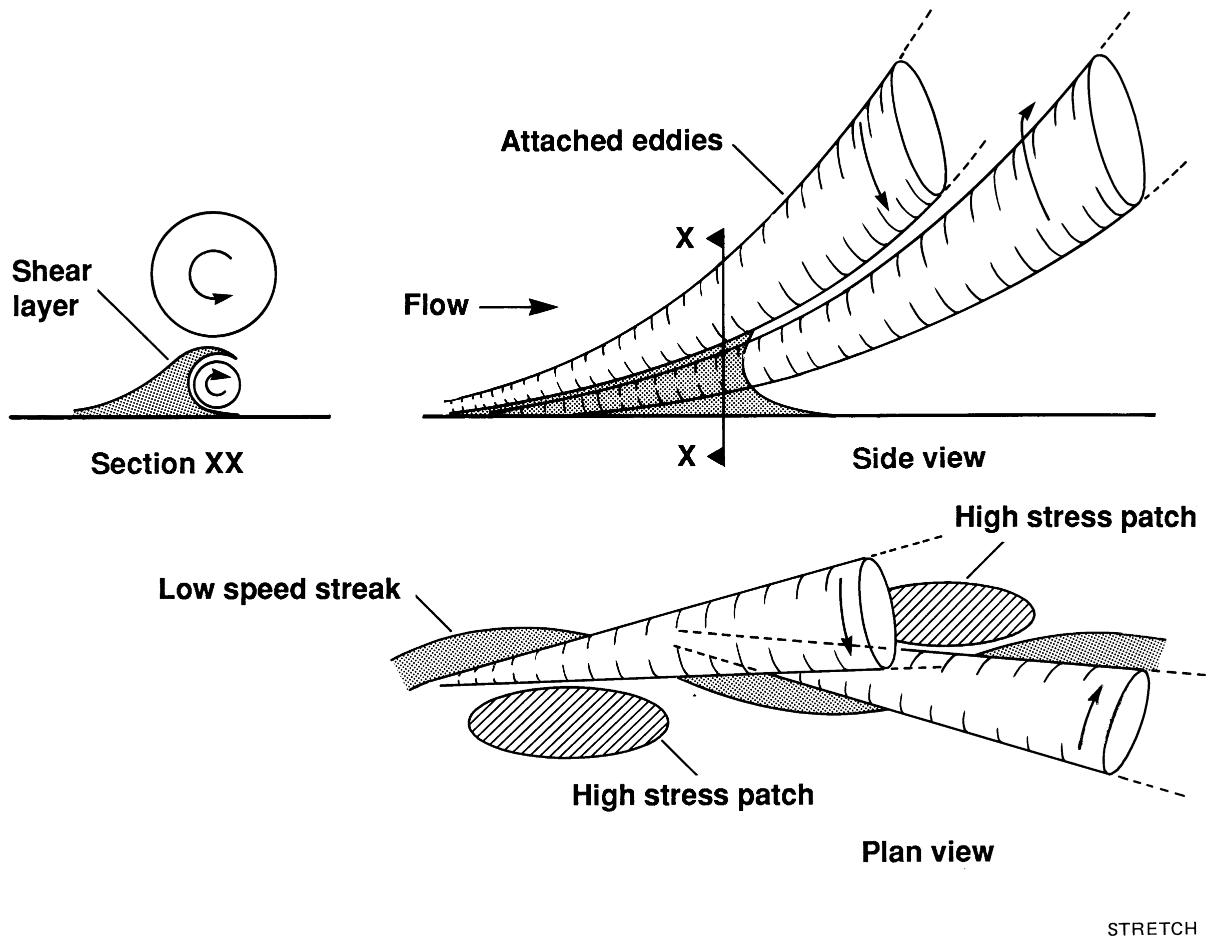
DNS of Turbulent Channel Flow



Kim, Moin & Moser, JFM 1987

$R_\tau \approx 180,\ R_m \approx 5600$

DNS near-wall Structures

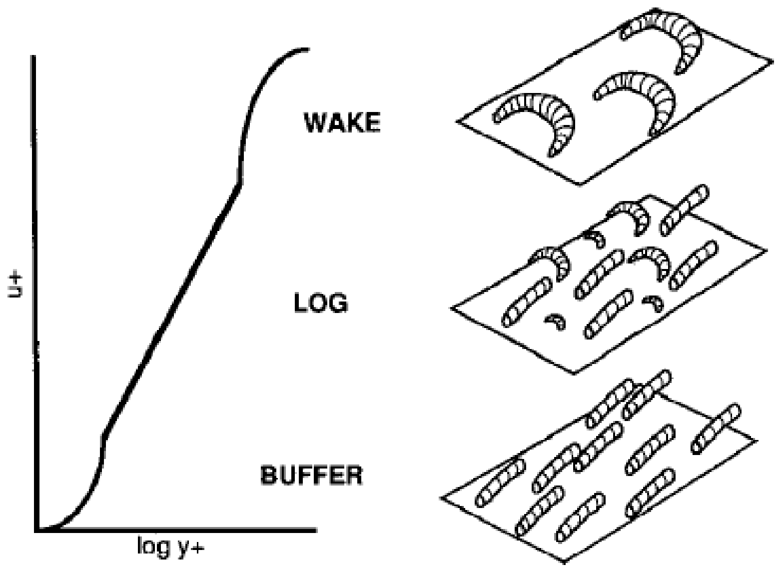


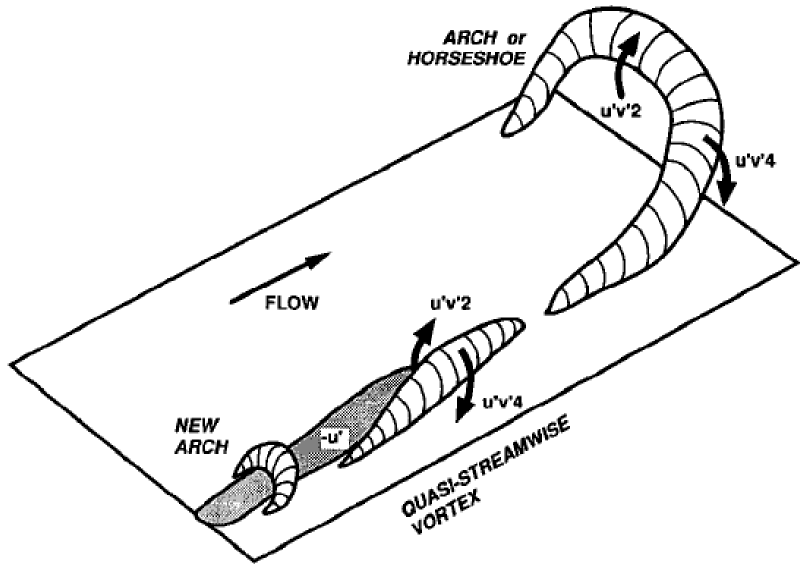
Asymmetric CS educed from KMM channel data at $R_\tau = 180$

Derek Stretch, 1990

Visualization/DNS Era

baguettes and croissants! not spaghetti!





3D 'Inverse' cascade: buffer layer → outer flow

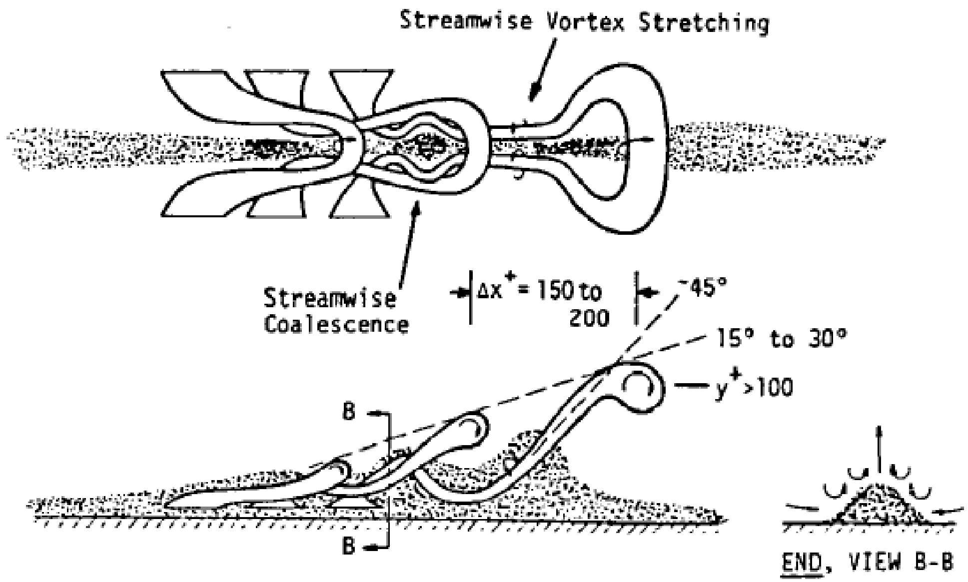
Steve Robinson, ARFM 1991

...buffer layer to outer space!

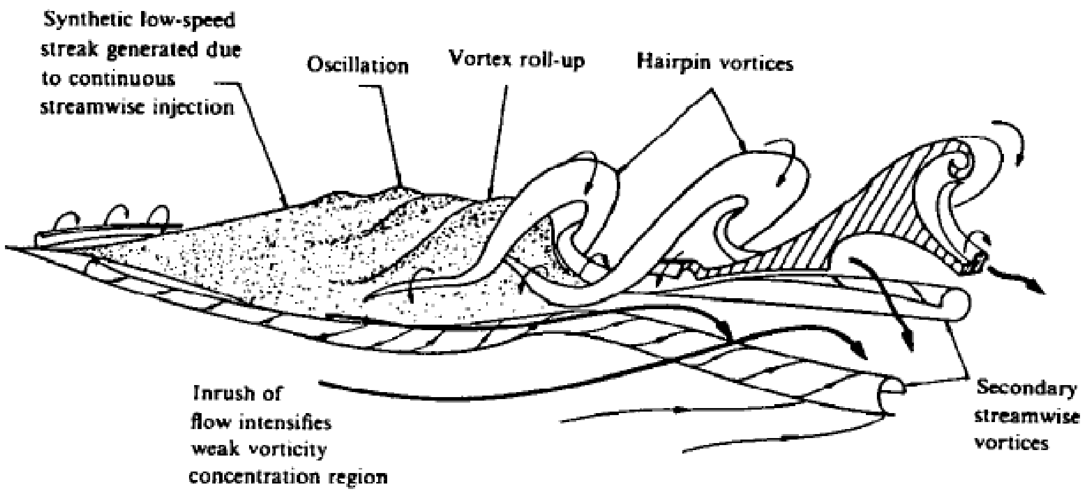
STS 114: Return to Flight, launched July 26, 2005...

Mechanistic Era

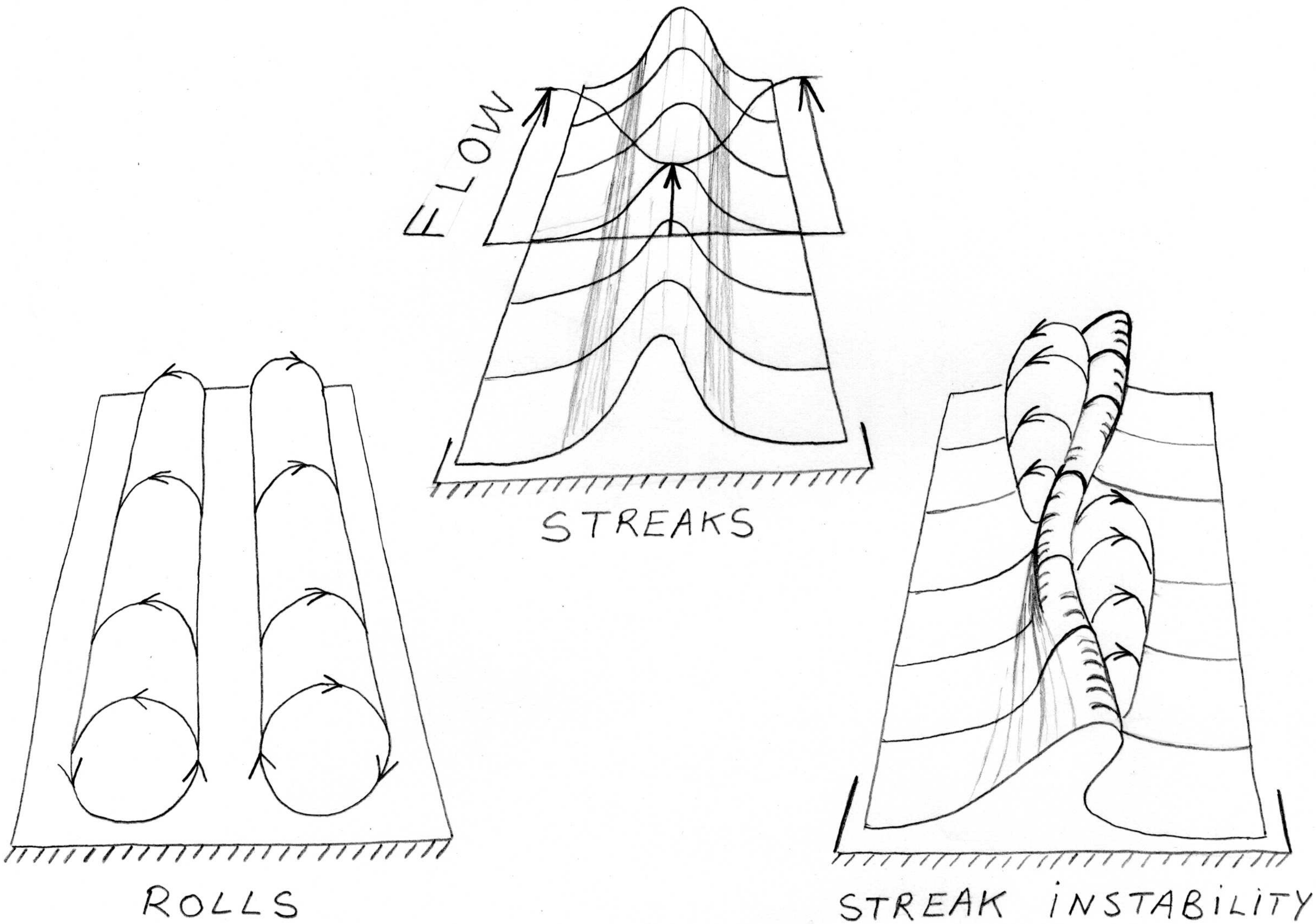
Acarlar & Smith JFM 1987 (synthetic streak regeneration) + Benney 1984



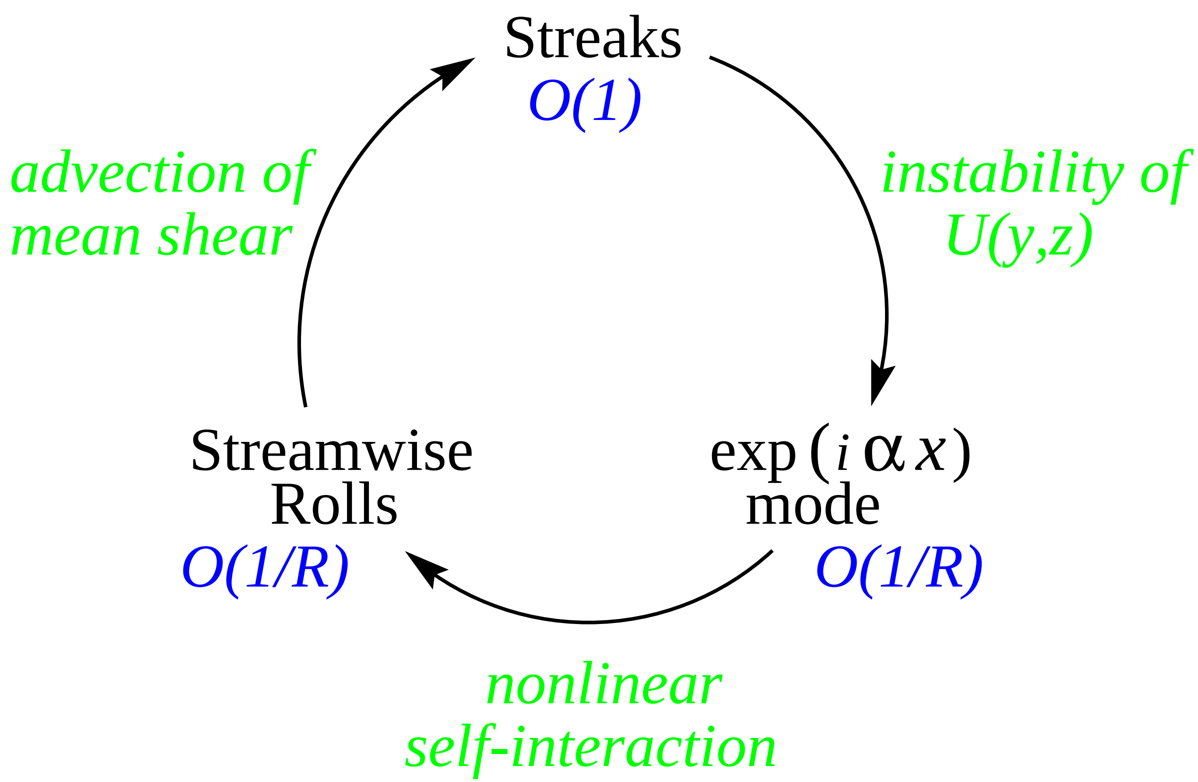
d) vortex ejection, stretching and interaction



Cartoon of Self-Sustaining Process



Self-Sustaining Process



Waleffe, Stud. Applied Math 1995, Phys. Fluids 1995, 1997

SSP as computational method, PRL 1998, JFM 2001, PoF 2003

SSP as Computational Method

- Add artificial roll forcing $O(1/R^2)$ to Navier-Stokes Equations

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- 1D-1C Laminar flow $\longrightarrow$ 2D-3C "streaky flow"

$$\boldsymbol{v} = U(y, z)\hat{\boldsymbol{x}} + \frac{1}{R}\left[V(y, z)\hat{\boldsymbol{y}} + W(y, z)\hat{\boldsymbol{z}}\right]$$

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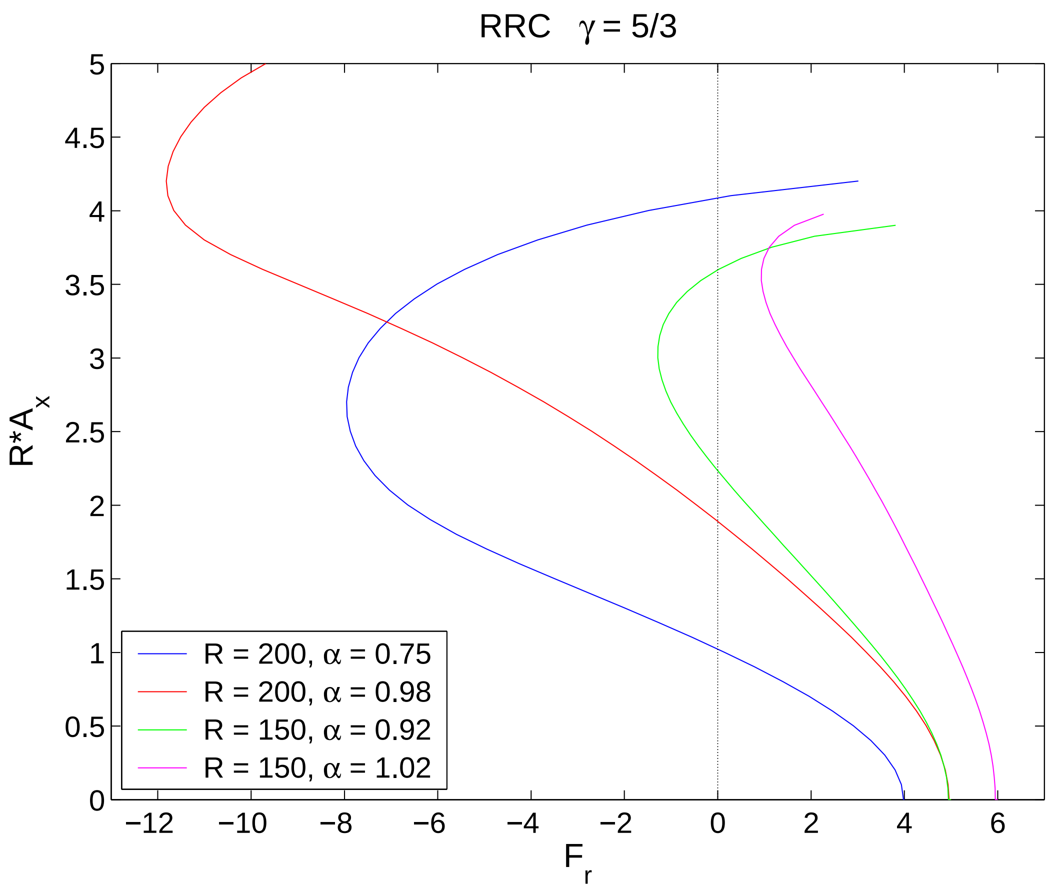
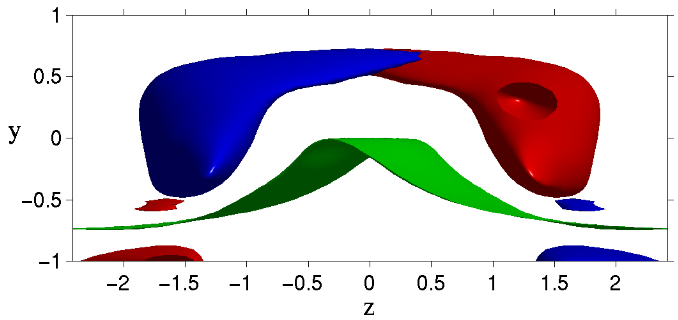
- Continue till no need for artificial roll forcing

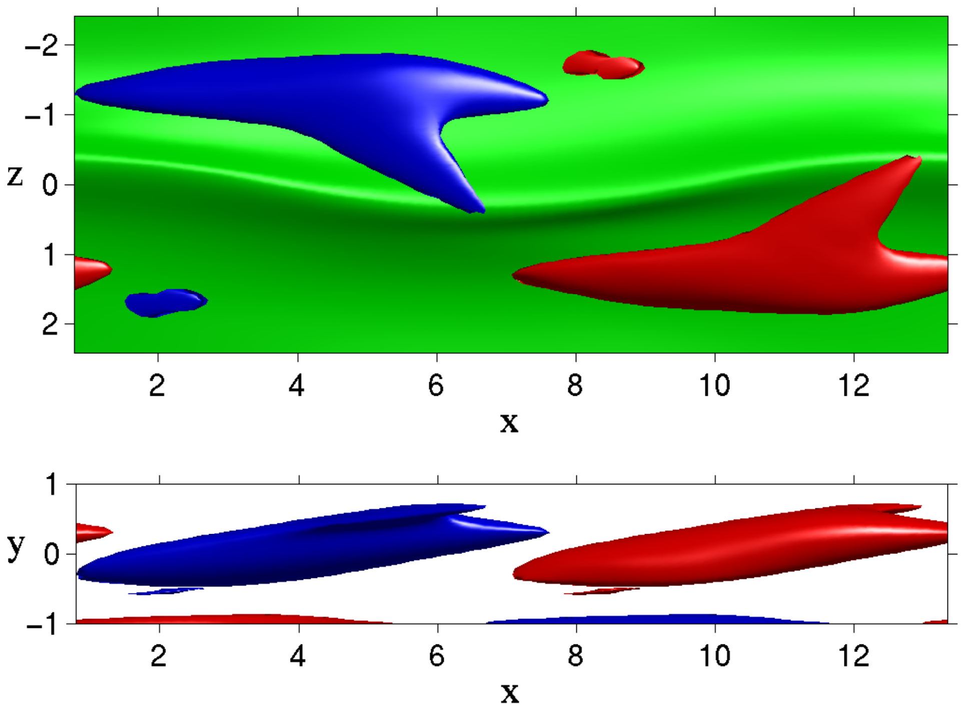
Bifurcation from streaky flow (PCF)



$$V_{\max} = \frac{F_r}{R}$$

Traveling Wave Solutions (1/2 PPF)





Waleffe, JFM 2001, PoF 2003

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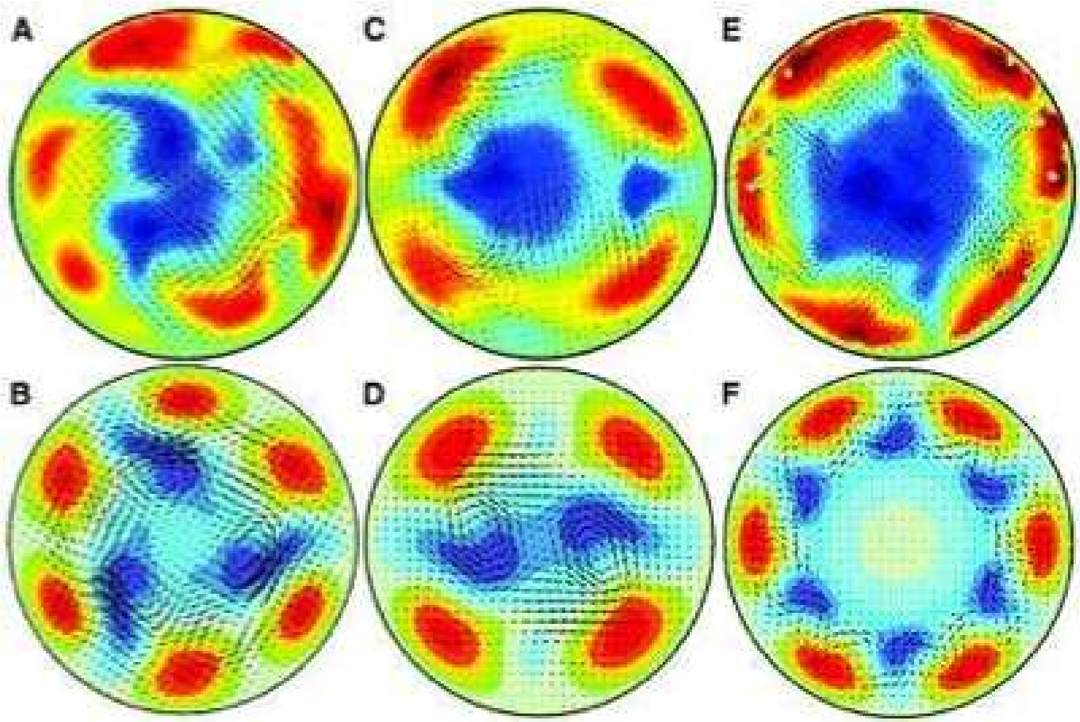
- unstable...
- … *a feature not a bug!* (more tomorrow!)
- generic:
  - Plane Couette flow (wall driven)
  - Plane Poiseuille flow (pressure driven)
  - free-slip, no-slip, any-slip
  - Pipe flow

Traveling Waves in Pipe Flow

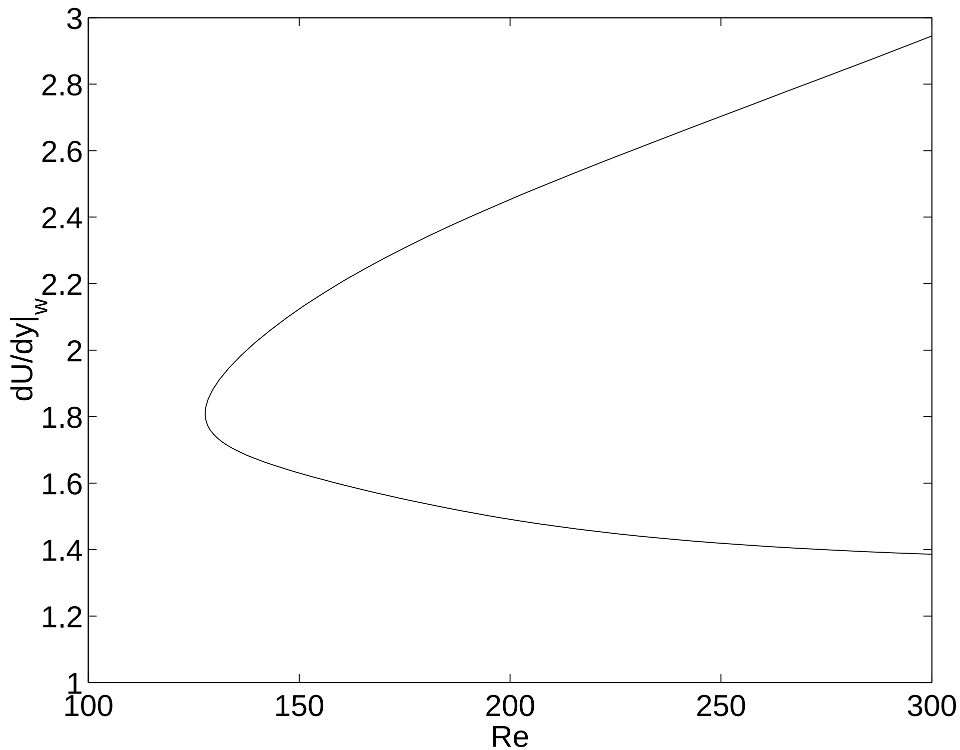


Faisst & Eckhardt, PRL 2003, Wedin & Kerswell JFM 2004

Hof *et al.* Science, Sept. 2004

ECS come in pairs

Bifurcation Diagram (saddle-node $F_r \equiv 0$): upper and lower branches



Normalized wall shear rate (drag) in Plane Couette Flow *vs* R

ECS pairs

- upper branch ECS $\approx$ backbone of turbulence

  capture 'statistics' pretty well (mean and rms profiles)

  Waleffe PoF 2003, Jimenez *et al.* PoF 2005

- lower branch ECS $\approx$ backbone of separatrix

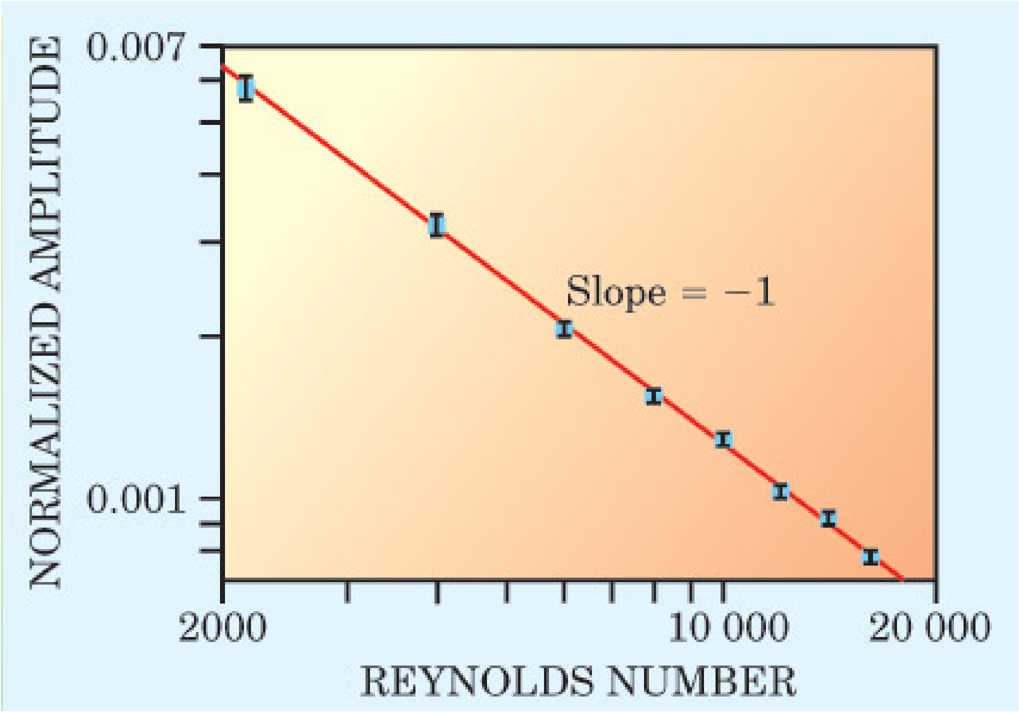
  (laminar and turbulent separated by stable manifold(s) of lower branch(es))

- scaling of lower branch ECS $\leftrightarrow$ transition threshold

Outstanding evidence for $1/R$ (Pipe flow)

Search and Discovery

Figure 2



Turbulence threshold scaling. The amplitude required to establish turbulence scales inversely with the Reynolds number of flow when the injected flux is normalized by the flux flowing through the pipe. (Adapted from ref. 2)

Hof, Juel & Mullin, PRL, Dec 2003; Physics Today Feb 2004

Fourier in $\theta = x - ct$ (*traveling wave*)

$$u(\theta, y, z) = u_0(y, z) + u_1(y, z)e^{i\theta} + u_2(y, z)e^{i2\theta} + \cdots$$

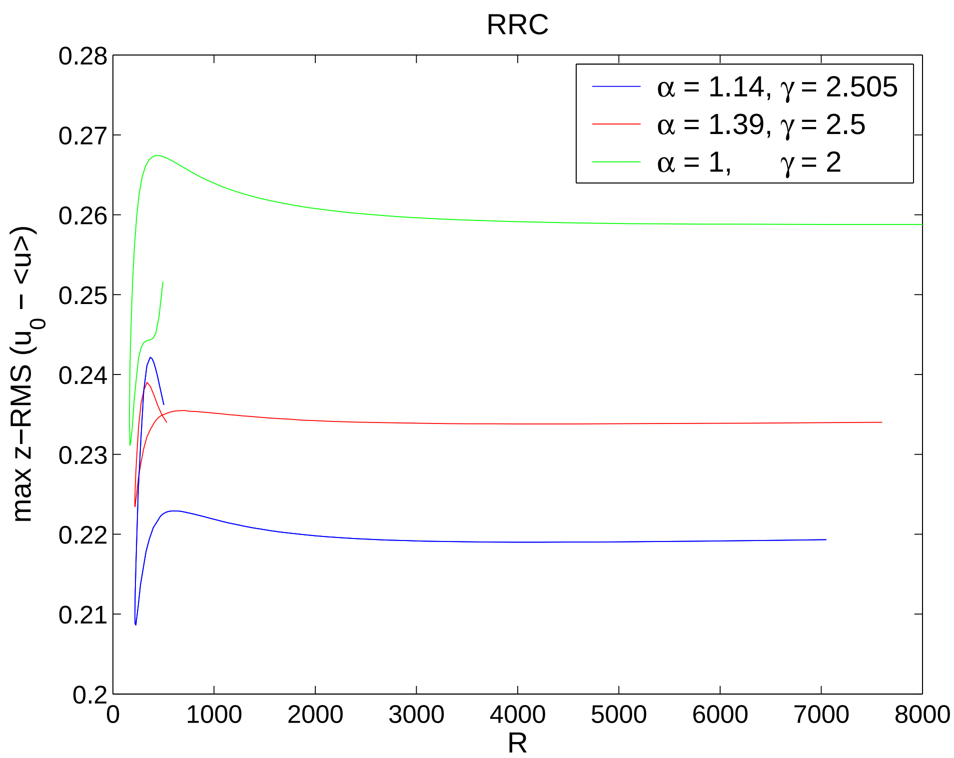
$$v(\theta, y, z) = v_0(y, z) + v_1(y, z)e^{i\theta} + \cdots$$

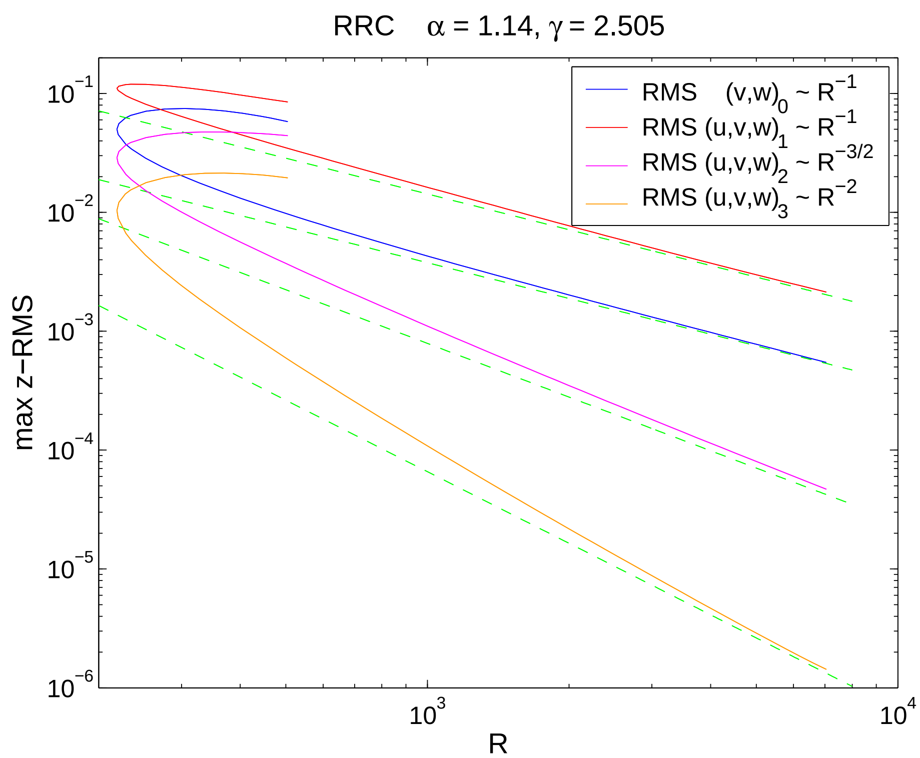
$$w(\theta, y, z) = w_0(y, z) + w_1(y, z)e^{i\theta} + \cdots$$

In SSP theory:

$$u_0(y, z) = O(1), \quad v_0, w_0 = O\left(\frac{1}{R}\right), \quad u_1, v_1, w_1 = O\left(\frac{1}{R}\right)$$

R-Scaling of harmonics (Rigid Rigid Couette)

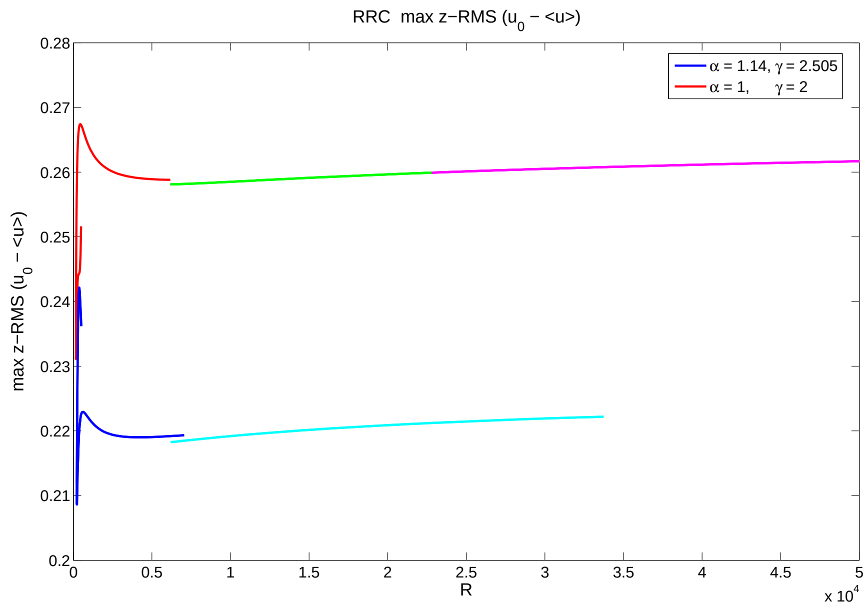


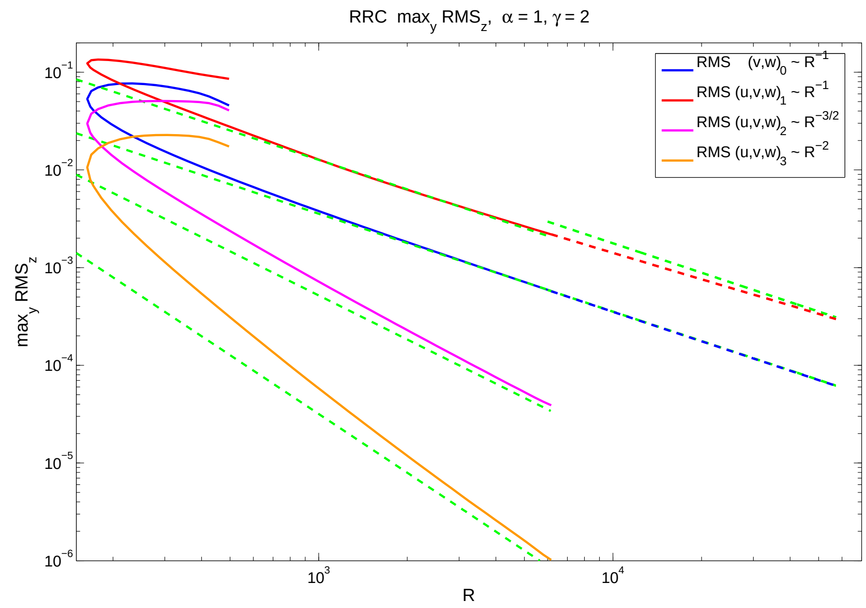


$$\max_y RMS_z \quad \left(u_0(y,z) - \overline{U}(y)\right), \qquad (v_0, w_0), (u_1, v_1, w_1), \ldots$$

Waleffe & Wang, 2004

R-Scaling dropping higher harmonics





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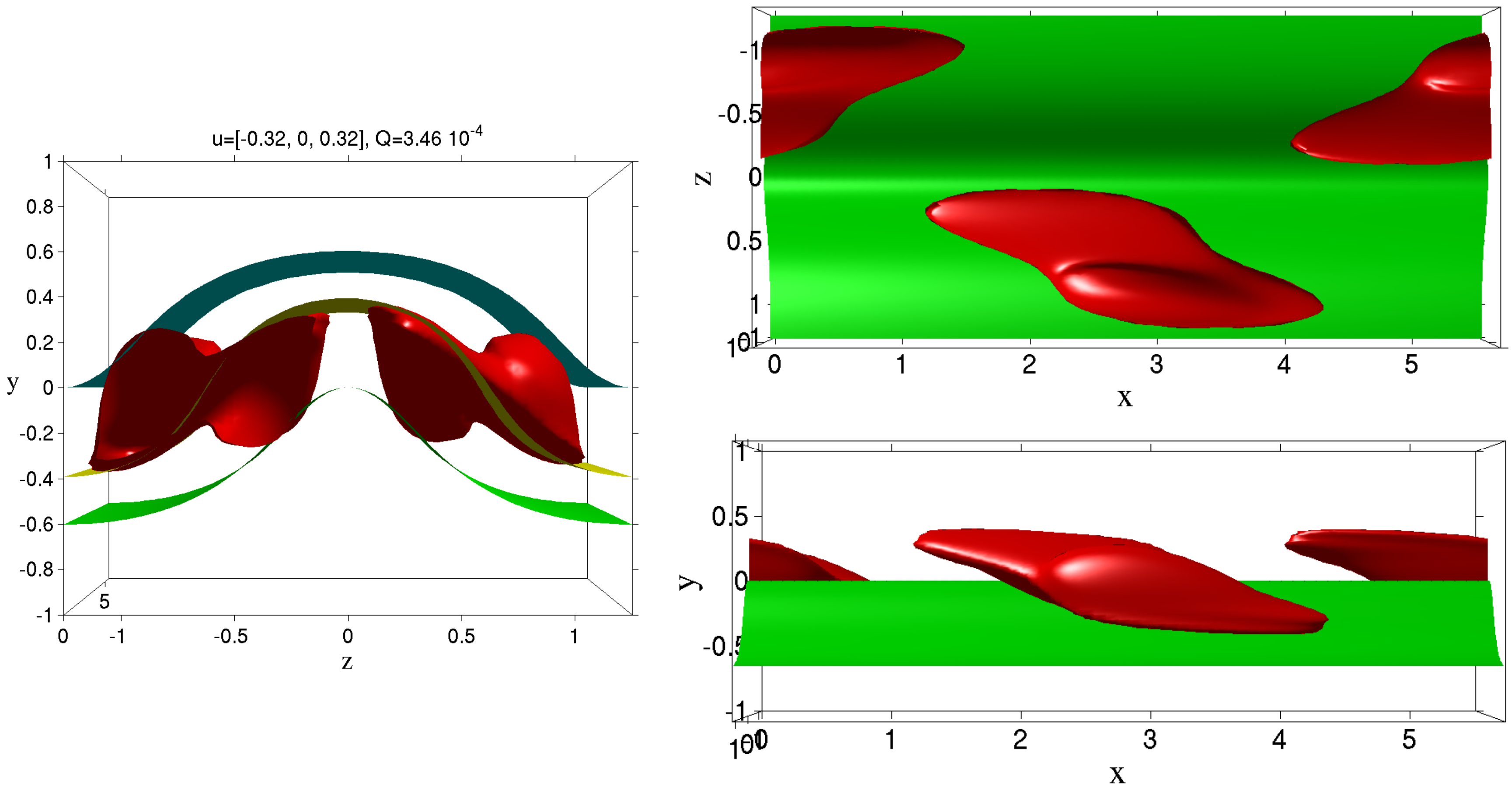
Waleffe & Wang, 2005

Trouble in the 1st harmonic?...

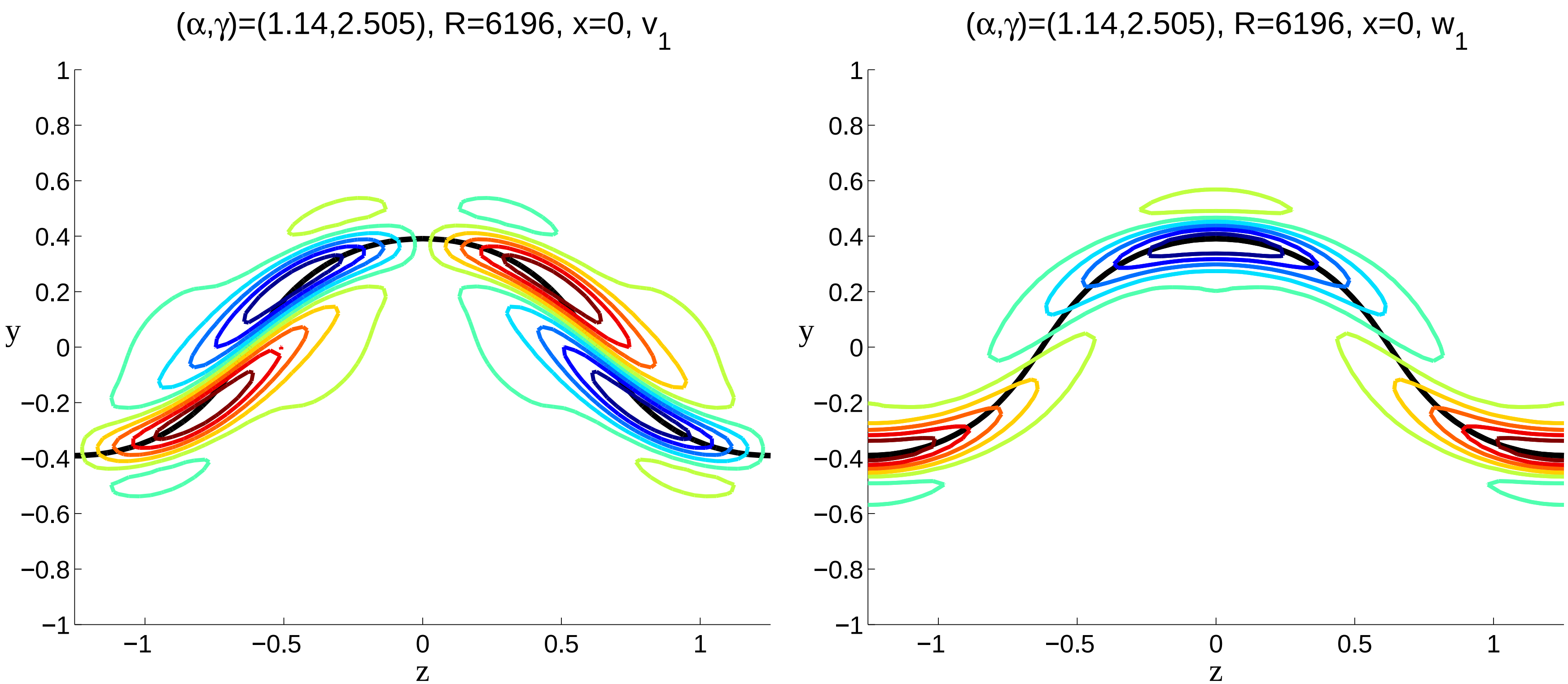
Structure of Lower branch @ $R = 6200$ (RRC)



u_0 large, $Q = \nabla^2 p/2$ small

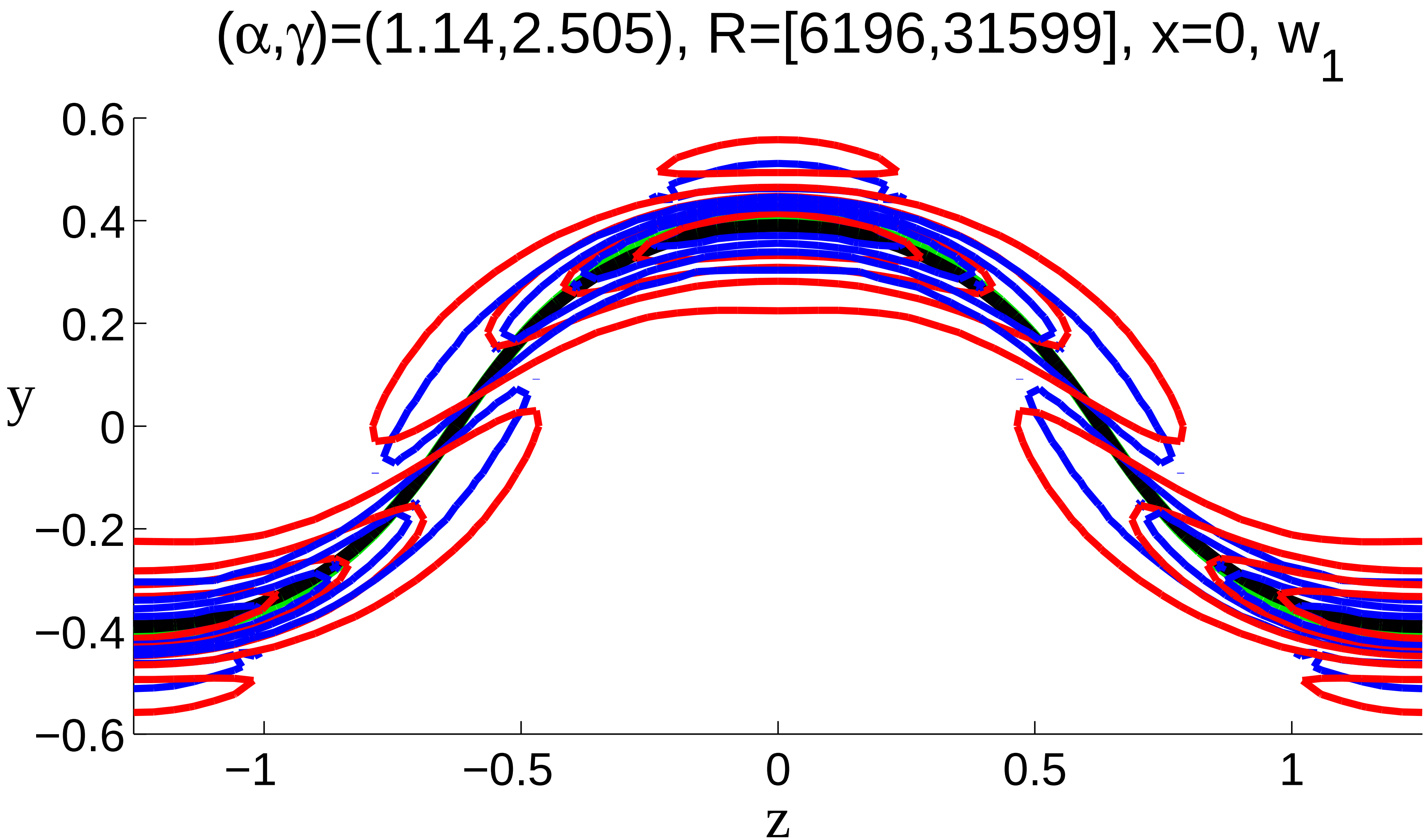
Waleffe & Wang, 2004, 2005

LBS 1st harmonic @ $R = 6200$ (RRC)

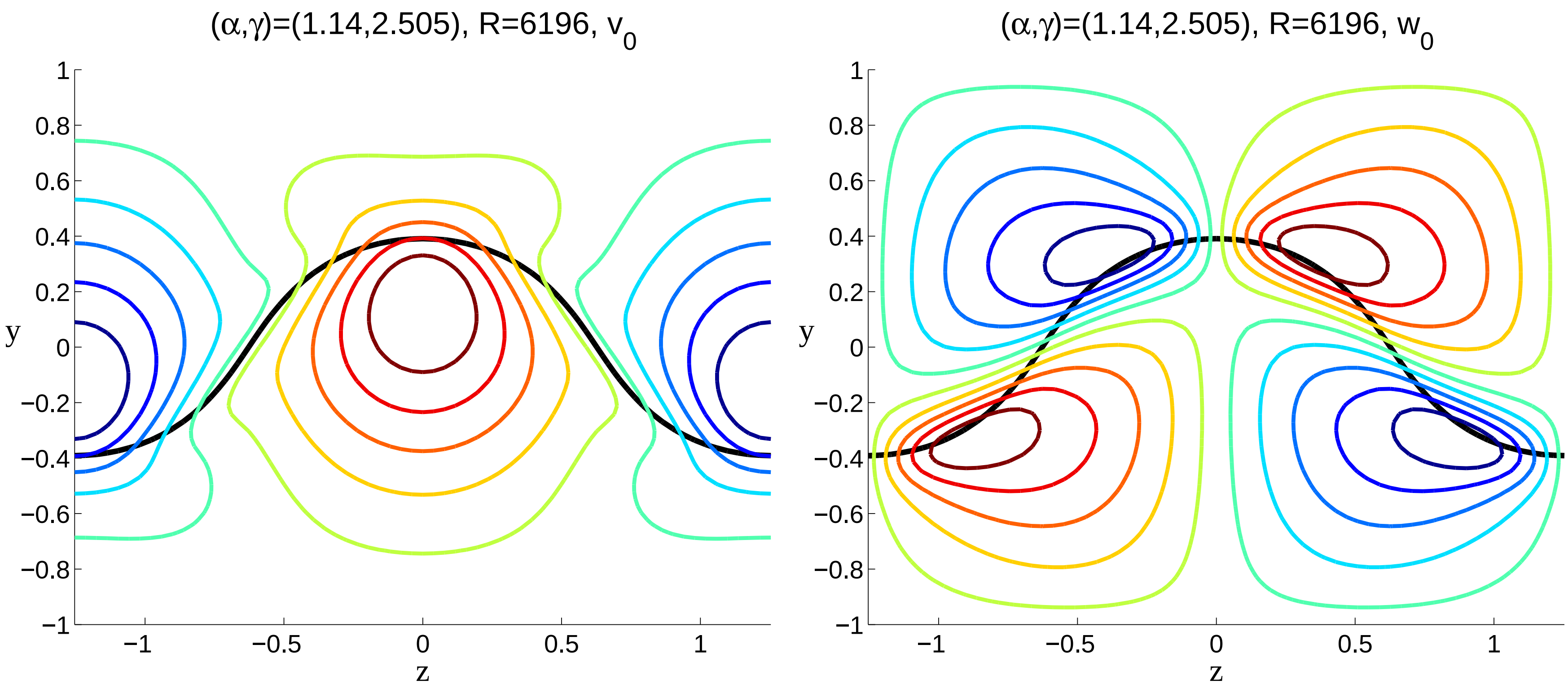


Critical layer! $u_0(y, z) - c = 0$

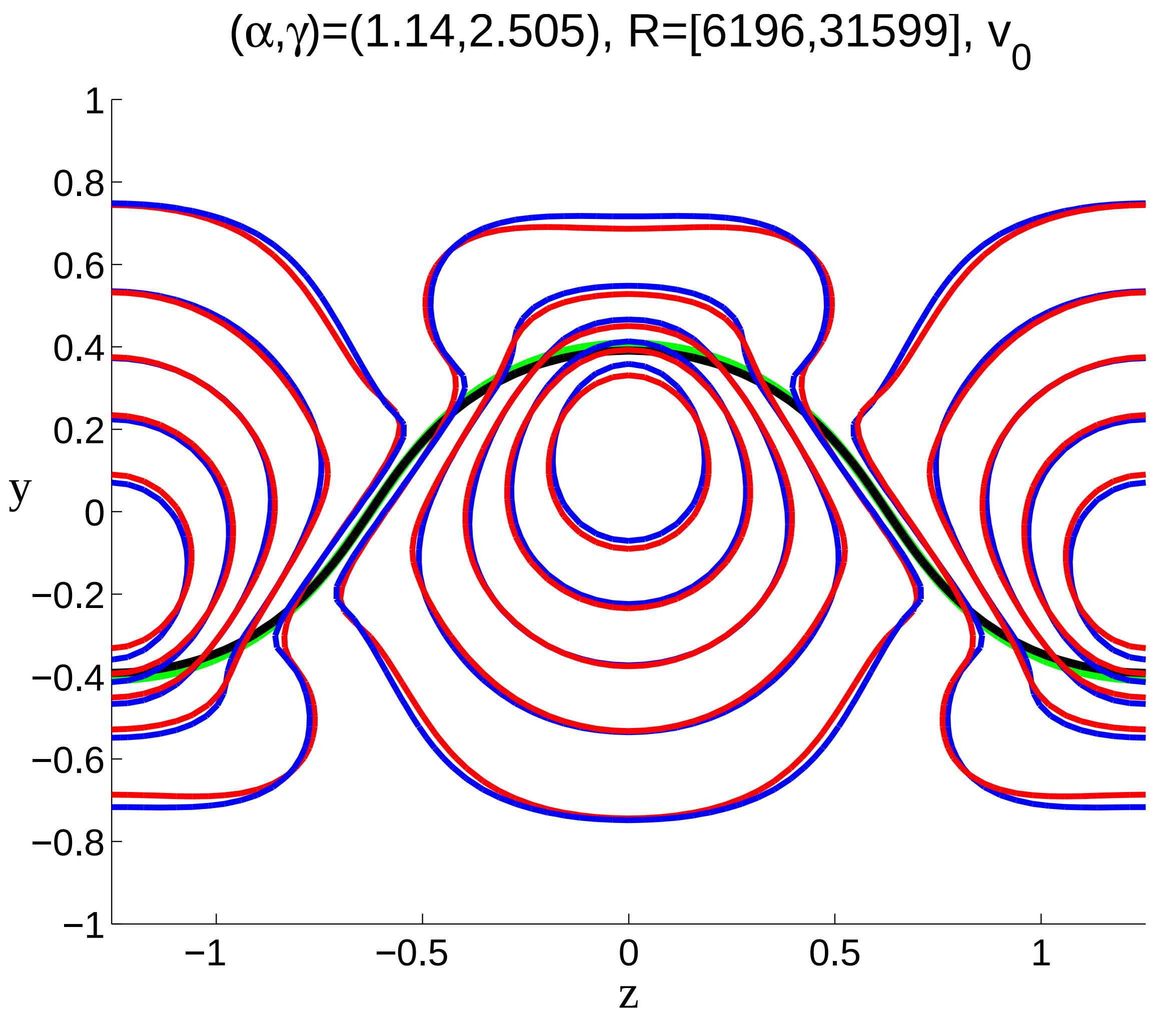
LBS 1st harmonic @ $R = 6196, 31599$



LBS Rolls @ $R = 6200$ (RRC)



Structure of LBS @ $R = 6196, 31599$



SSP exact as $R \to \infty$ (but...)

- $O(1/R)$ rolls $\longrightarrow$ $O(1)$ streaks $\longrightarrow$ streak instability

$$\longleftarrow$$

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- $\approx$ Benney's 'Mean Flow-First Harmonic Theory'

  (inviscid wavepackets, $\epsilon \longrightarrow$ viscous traveling wave, $1/R$)

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  $$\longleftarrow$$
- $\approx$ Benney's 'Mean Flow-First Harmonic Theory'
  (inviscid wavepackets, $\epsilon \longrightarrow$ viscous traveling wave, $1/R$)
- but... 2D Critical Layer $u_0(y,z) - c = 0$
  complicates scaling and asymptotics

SSP asymptotics: coupled 2D modes

$$(\partial_t + \boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{v} + \boldsymbol{\nabla} p = \frac{1}{R}\nabla^2 \boldsymbol{v}, \qquad \boldsymbol{\nabla} \cdot \boldsymbol{v} = 0$$

as $R \to \infty$, there are SF TWS

$$\begin{aligned} \boldsymbol{v}(x,y,z,t) \sim\ & u_0(y, z)\hat{\boldsymbol{x}} \\ & + \Big(v_0(y,z)\hat{\boldsymbol{y}} + w_0(y,z)\hat{\boldsymbol{z}}\Big) \\ & + e^{i\alpha(x-ct)}\hat{\boldsymbol{v}}_1(y,z) + c.c. \end{aligned}$$

SSP asymptotics: leading order eqns

- advection-diffusion of streaky flow $u_0(y, z)$:

$$v_0 \frac{\partial u_0}{\partial y} + w_0 \frac{\partial u_0}{\partial z} = \frac{1}{R} \nabla^2 u_0$$

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- Streamwise rolls: $v_0,\ w_0 \sim 1/R \Rightarrow u_0(y, z)$

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- Streamwise rolls: $v_0(y,z) = \partial_z\Psi_0$, $w_0(y,z) = -\partial_y\Psi_0$

$$\frac{1}{R}\nabla^4\Psi_0 = J(\nabla^2\Psi_0, \Psi_0)$$

Rolls decoupled from streaky flow

$$J(A,B) = \partial_y A\partial_z B - \partial_z A\partial_y B$$

SSP asymptotics: leading order eqns

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$$v_0 \frac{\partial u_0}{\partial y} + w_0 \frac{\partial u_0}{\partial z} = \frac{1}{R}\nabla^2 u_0 - \frac{\partial \overline{u_1 v_1}^x}{\partial y} - \frac{\partial \overline{u_1 w_1}^x}{\partial z}$$

- Streamwise rolls: $v_0(y,z) = \partial_z \Psi_0, \;\; w_0(y,z) = -\partial_y \Psi_0$

$$\frac{1}{R}\nabla^4 \Psi_0 = J(\nabla^2 \Psi_0, \Psi_0) + \frac{\partial^2}{\partial y \partial z}(\overline{v_1^2}^x - \overline{w_1^2}^x) + \left(\frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial y^2}\right)\overline{v_1 w_1}^x$$

1st harmonic feeds on streaks to sustain rolls

$$\boldsymbol{v}_1 \sim 1/R \qquad v_0, w_0 \sim 1/R$$

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- First Harmonic neutrally stable $\boldsymbol{v}_1 = e^{i\alpha x}\hat{\boldsymbol{v}}_1(y,z) + c.c.$

$$(\boldsymbol{v}_0 - c\hat{\boldsymbol{x}}) \cdot \boldsymbol{\nabla} \boldsymbol{v}_1 + \boldsymbol{v}_1 \cdot \boldsymbol{\nabla} \boldsymbol{v}_0 + \boldsymbol{\nabla} p_1 = \frac{1}{R}\nabla^2 \boldsymbol{v}_1, \quad \boldsymbol{\nabla} \cdot \boldsymbol{v}_1 = 0$$

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- The neglected lower branch states:
  ' large' scale, 2D self-sustained critical layer
  drag only 10-20% higher than laminar!
  control?