2 ABSTRACT GROUP THEORY

2-1 Definitions and Nomenclature

By a group we mean a set of *elements* $A, B, C, \ldots$ such that a form of *group multiplication* may be defined which associates a third element with any ordered pair. This multiplication must satisfy the requirements:

1. The product of any two elements is in the set; i.e., the set is *closed* under group multiplication.

2. The *associative law* holds; for example, $A(BC) = (AB)C$.

3. There is a *unit element* E such that $EA = AE = A$.

4. There is in the group an inverse A^{-1} to each element A such that $AA^{-1} = A^{-1}A = E$.

For the present we shall restrict our attention primarily to *finite groups*. These contain a finite number h of group elements, where h is said to be the

order of the group. If group multiplication is commutative, so that $AB = BA$ for all A and B, the group is said to be *Abelian*.

2-2 Illustrative Examples

An example of an Abelian group of infinite order is the set of all positive and negative integers including zero. In this case, ordinary addition serves as the group-multiplication operation, zero serves as the unit element, and $-n$ is the inverse of n. Clearly the set is closed, and the associative law is obeyed.

An example of a non-Abelian group of infinite order is the set of all $n \times n$ matrices with nonvanishing determinants. Here the group-multiplication operation is matrix multiplication, and the unit element is the $n \times n$ unit matrix. The inverse matrix of each matrix may be constructed by the usual methods,[1] since the matrices are required to have nonvanishing determinants.

A physically important example of a finite group is the set of covering operations of a symmetrical object. By a covering operation, we mean a rotation, reflection, or inversion which would bring the object into a form indistinguishable from the original one. For example, all rotations about the center are covering operations of a sphere. In such a group the product AB means the operation obtained by first performing B, then A. The unit operation is no operation at all, or perhaps a rotation through 2π. The inverse of each operation is physically apparent. For example, the inverse of a rotation is a rotation through the same angle in the reverse sense about the same axis.

As a complete example, which we shall often use for illustrative purposes, consider the non-Abelian group of order 6 specified by the following *group-multiplication table:*

	E	A	B	C	D	F
E	E	A	B	C	D	F
A	A	E	D	F	B	C
B	B	F	E	D	C	A
C	C	D	F	E	A	B
D	D	C	A	B	F	E
F	F	B	C	A	E	D

The meaning of this table is that each entry is the product of the element labeling the row times the element labeling the column. For example, $AB = D \neq BA$. This table results, for example, if we take our elements to be the following six matrices, and if ordinary matrix multiplication is

[1] See Appendix A and references cited there.

used as the group-multiplication operation:

$$E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad B = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$

$$C = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \qquad D = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \qquad F = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$$

Verification of the table is left as a simple exercise.

The very same multiplication table could be obtained by considering the group elements $A, \ldots, F$ to represent the proper covering operations of

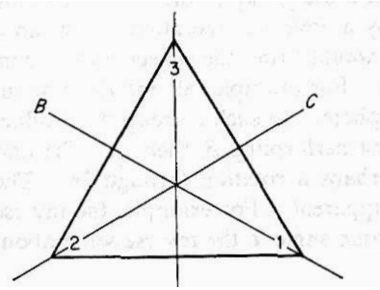

Fig. 2-1. *Symmetry axes of equilateral triangle.*

an equilateral triangle as indicated in Fig. 2-1. The elements A, B, and C are rotations by π about the axes shown. Element D is a clockwise rotation by $2\pi/3$ in the plane of the triangle, and F is a counterclockwise rotation through the same angle. The numbering of the corners destroys the symmetry so that the position of the triangle can be followed through successive operations. If we make the convention that we consider the rotation axes to be kept fixed in space (not rotated with the object), it is easy to verify that the multiplication table given above describes this group as well.

Two groups obeying the same multiplication table are said to be *isomorphic.*

2-3 Rearrangement Theorem

In the multiplication table in the example above, each column or row contains each element once and only once. This rule is true in general and is

called the *rearrangement theorem.* Stated more formally, in the sequence

$$EA_k, A_2A_k, A_3A_k, \ldots, A_hA_k,$$

each group element A_i appears exactly once (in the form A_rA_k). The elements are merely rearranged by multiplying each by A_k.

PROOF: For any A_i and A_k, there exists an element $A_r = A_iA_k^{-1}$ in the group since the group contains inverses and is closed. Since $A_rA_k = A_i$ *for this particular* A_r, A_i *must appear in the sequence at least once.* But there are h elements in the group and h terms in the sequence. Hence there is no opportunity for any element to make more than a single appearance.

2-4 Cyclic Groups

For any group element X, one can form the sequence

$$X, X^2, X^3, \ldots, X^{n-1}, X^n = E$$

This is called the *period* of X, since the sequence would simply repeat this period over and over if it were extended. (Eventually we must find repetition, since the group is assumed to be finite.) The integer n is called the *order* of X, and this period clearly forms a group as it stands, although it need not exhaust all the elements of the group with which we started. Hence it may be said to form a *cyclic group* of order n. If it is indeed only part of a larger group, it is referred to as a cyclic *subgroup*.[1] We note that *all cyclic groups must be* Abelian.

In our standard example of the triangle, the period of D is D, $D^2 = F$, $D^3 = DF = E$. Thus D is of order 3, and D, F, E form a cyclic subgroup of our entire group of order 6.

2-5 Subgroups and Cosets

Let $\mathscr{S} = E, S_2, S_3, \ldots, S_g$ be a *subgroup* of order g of a larger group $\mathscr{G}$ of order h. We then call the set of g elements $EX, S_2X, S_3X, \ldots, S_gX$ a *right coset* $\mathscr{S}X$ if X is not in $\mathscr{S}$. (If X were in $\mathscr{S}$, $\mathscr{S}X$ would simply be the subgroup $\mathscr{S}$ itself, by the rearrangement theorem.) Similarly, we define the set $X\mathscr{S}$ as being a *left coset.* These cosets cannot be subgroups, since they cannot include the identity element. In fact, a coset $\mathscr{S}X$ contains *no* elements in common with the subgroup $\mathscr{S}$.

The proof of this statement is easily given by assuming, on the contrary, that for some element S_k we have $S_kX = S_l$, a member of $\mathscr{S}$. Then $X = S_k^{-1}S_l$, which is in the subgroup, and $\mathscr{S}X$ is not a coset at all, but just $\mathscr{S}$ itself.

[1] Although the concept is introduced here in connection with cyclic groups, subgroups need not be cyclic. Any subset of elements within a group which in itself forms a group is called a subgroup of the larger group.

Next we note that two right (or left) cosets of subgroup $\mathscr{S}$ in $\mathscr{G}$ either are identical or have no elements in common.

PROOF: Consider two cosets $\mathscr{S}X$ and $\mathscr{S}Y$. Assume that there exists a common element $S_kX = S_lY$. Then $XY^{-1} = S_k^{-1}S_l$, which is in $\mathscr{S}$. Therefore $\mathscr{S}XY^{-1} = \mathscr{S}$, by the rearrangement theorem. Postmultiplying both sides by Y leads to $\mathscr{S}X = \mathscr{S}Y$. Thus the two cosets are completely identical if a single common element exists.

If we combine the results of the preceding paragraphs, we can prove the following theorem: *The order g of a subgroup must be an integral divisor of the order h of the entire group.* That is, $h/g = l$, where the integer l is called the *index* of the subgroup $\mathscr{S}$ in $\mathscr{G}$.

PROOF: Each of the h elements of $\mathscr{G}$ must appear either in $\mathscr{S}$ or in a coset $\mathscr{S}X$, for some X. Thus each element must appear in one of the sets $\mathscr{S}, \mathscr{S}X_2, \mathscr{S}X_3, \ldots, \mathscr{S}X_l$, where we have listed all the *distinct* cosets of $\mathscr{S}$ together with $\mathscr{S}$ itself. But we have shown that there are no elements common to any of these collections of g elements. Hence it must be possible to divide the total number of elements h into an integral number of sets of g each, and consequently $h = l \times g$.

As an example, consider the subgroup $\mathscr{S} = A, E$ of our illustrative group of order 6. The right cosets with B and D are identical, namely, $\mathscr{S}B = \mathscr{S}D = B, D$. Also $\mathscr{S}C = \mathscr{S}F = C, F$. We note that, as proved in general, these cosets contain no common elements unless entirely identical and they contain no elements in common with $\mathscr{S}$. Also, the order (2) of the subgroup *is* an integral divisor of the order (6) of the group. To generalize, the order of *any* cyclic subgroup formed by the period of some group element must be a divisor of the order of the group.

2-6 Example Groups of Finite Order

1. *Groups of order* 1. The only example is the group consisting solely of the identity element E.

2. *Groups of order* 2. Again there is only one possibility, the group $(A, A^2 = E)$. This is an Abelian group, and in physical applications A might represent reflection, inversion, or an interchange of two identical particles.

3. *Groups of order* 3. In this case, if we start with two elements A and E, it must be that $A^2 = B \neq E$. Otherwise, if A^2 were to equal E, then (A, E) would form a subgroup of order 2 in a group of order 3, which would violate our theorem. Thus the only possibility is the cyclic group $(A, A^2 = B, A^3 = E)$.

4. *Groups of order* 4. With order 4 we begin to have more than one possible distinct group-multiplication table of given order. The two possibilities here are (1) the cyclic group $(A, A^2, A^3, A^4 = E)$ and (2) the so-called *Vierergruppe* (A, B, C, E) whose multiplication table is:

	E	A	B	C
E	E	A	B	C
A	A	E	C	B
B	B	C	E	A
C	C	B	A	E

Both these groups are Abelian, and in both cases we can pick out subgroups of order 2, as allowed by our theorem. A physical example of the cyclic group of order 4 is provided by the four fold rotations about an axis. On the other hand, the *Vierergruppe* is the rotational-symmetry group of a rectangular solid, if A, B, C are taken to be the rotations by π about the three orthogonal symmetry axes.

5. *Groups of prime order*. These must all be cyclic Abelian groups. Otherwise the period of some element would have to appear as a subgroup whose order was a divisor of a prime number. This general result allows us to note at once that there can be only single groups of order 1, 2, 3, 5, 7, 11, 13, etc.

6. *Permutation groups (of factorial order)*. One group of order $n!$ can always be set up based on all the permutations of n distinguishable things. (Of course, others, such as a cyclic group, can also be found.) A permutation can be specified by a symbol such as

$$\begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ \alpha_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_n \end{pmatrix}$$

where $\alpha_1, \alpha_2, \ldots, \alpha_n = 1, 2, \ldots, n$, except for order. The permutation described by this symbol is one in which the item in position i is shifted to the position indicated in the lower line. Successive permutations form the group-multiplication operation. As an example, our standard example group of order 6 can be viewed as the permutation group of the three numbered corners of the triangle. The permutations may be expressed in the above notation as

$$E = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \qquad A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \qquad D = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \qquad F = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

For example, operator A interchanges corners 1 and 2, whereas D replaces 1 by 3, 2 by 1, and 3 by 2, corresponding to a clockwise rotation by $2\pi/3$.

Applying B followed by A leads to

$$AB = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = D$$

which is consistent with the group-multiplication table worked out previously.

Because of the identity of like particles, permutation of them leaves the Hamiltonian invariant. Accordingly, the permutation group plays an important role in quantum theory.

2-7 Conjugate Elements and Class Structure

An element B is said to be *conjugate* to A if

$$B = XAX^{-1} \qquad \text{or} \qquad A = X^{-1}BX$$

where X is some member of the group. Clearly this is a reciprocal property of the pair of elements. Further, if B and C are both conjugate to A, they are conjugate to each other.

PROOF: Assume that

$$B = XAX^{-1} \qquad \text{and} \qquad C = YAY^{-1}$$

Then $\quad A = Y^{-1}CY$

and $\quad B = XY^{-1}CYX^{-1} = (XY^{-1})C(XY^{-1})^{-1}$

$$= ZCZ^{-1}$$

[In this proof we have used the fact that the inverse of the product of two group elements is the product of the inverses of the elements in inverse order. This is clearly true, since $(RS)(S^{-1}R^{-1}) = R(SS^{-1})R^{-1} = RR^{-1} = E$.]

The properties of conjugate elements given above allow us to collect all mutually conjugate elements into what is called a *class* of elements. The class including A_i is found by forming all products of the form

$$EA_iE^{-1} = A_i,\ A_2A_iA_2^{-1}, \ldots, A_hA_iA_h^{-1}$$

Of course, some elements may be found several times by this procedure. By proceeding in this way, we can divide all the elements of the group among the various distinct classes. Luckily, we may usually avoid this rather tedious method by using physical-symmetry considerations, as shown below. For example, in the group of covering operations of an equilateral triangle, the two rotations by $2\pi/3$ form a class, the three rotations by π form a class, and, as always, *the identity element is in a class by itself.* The latter follows, since $AEA^{-1} = AA^{-1} = E$ for all A. Note that E is the *only class* which is also a *subgroup*, since all other classes must lack the identity element.

In Abelian groups, each element is in a class by itself, since $XAX^{-1} = AXX^{-1} = AE = A$.

If the group elements are represented by matrices, the traces of all elements in a class must be the same. This follows, since in this case the operation of conjugation becomes that of making a similarity transformation, which leaves the trace invariant.[1]

Physical interpretation of class structure. In physical applications the group elements can often be considered to be symmetry operations which are the covering operations of a symmetrical object. In this case, the operation $B = X^{-1}AX$ is the net operation obtained by first rotating the object to some equivalent position by X, next carrying out the operation A, and then undoing the initial rotation by X^{-1}. Thus B must be an operation of the same physical sort as A, such as a rotation through the same angle, but performed about some different (but physically equivalent) axis which is related to the axis of A by the group operation X^{-1}. This is the significance of operators being in the same class.

As a concrete example, consider the covering operations of the equilateral triangle indicated in Fig. 2-1. If we consider the conjugation of A with D, we have $D^{-1}AD = C$. To follow this through in detail, D rotates the triangle clockwise by $2\pi/3$ so that vertex 2 instead of 3 lies on axis A; next the rotation by π about the A axis interchanges 1 and 3; finally $D^{-1} = F$ rotates the triangle back $2\pi/3$ counterclockwise. This sequence leaves precisely the result of a single rotation by π about axis C, which is an axis equivalent to A but rotated $2\pi/3$ counterclockwise by the symmetry operator D^{-1}.

2-8 Normal Divisors and Factor Groups

If a subgroup $\mathscr{S}$ of a larger group $\mathscr{G}$ consists entirely of complete classes, it is called an *invariant subgroup*, or *normal divisor*. By consisting of complete classes, we mean that, if an element A is in $\mathscr{S}$, then all elements $X^{-1}AX$ are in $\mathscr{S}$, even when X runs over elements of $\mathscr{G}$ which are not in $\mathscr{S}$. Such a subgroup is called invariant because by the rearrangement theorem it is unchanged (except for order) by conjugation with any element of $\mathscr{G}$.

To allow a compact discussion, we introduce the notion of a *complex* such as $\mathscr{K} = (K_1, K_2, \ldots, K_n)$, which is a collection of group elements disregarding order. Such a complex can be multiplied by a single element or by another complex. For example,

$$\mathscr{K}X = (K_1X, K_2X, \ldots, K_nX)$$

and $$\mathscr{K}\mathscr{R} = (K_1R_1, K_1R_2, \ldots, K_1R_m, \ldots, K_nR_m)$$

[1] See Appendix A.

Elements are considered to be included only once, regardless of how often they are generated.

We can now state our argument concisely by treating sets of elements as complexes. First, a subgroup is defined by the property of closure, that is, $\mathscr{S}\mathscr{S} = \mathscr{S}$. Second, if $\mathscr{S}$ is an *invariant* subgroup, then $X^{-1}\mathscr{S}X = \mathscr{S}$, for all X in the group $\mathscr{G}$. From this it follows that $\mathscr{S}X = X\mathscr{S}$, or, in words, the left and right cosets of an invariant subgroup are identical.

In Sec. 2-5 we have shown that there are a finite number $(l - 1)$ of distinct cosets for any subgroup $\mathscr{S}$. We may denote each of these as a complex, and if $\mathscr{S}$ is an invariant subgroup, we have, for example, $\mathscr{K}_i = \mathscr{S}K_i = K_i\mathscr{S}$. Note that $\mathscr{S}K_i = \mathscr{S}K_j$ if K_i and K_j are group elements in the same coset, since we are not concerned with the order in which the elements of the complex appear. Together with the subgroup $\mathscr{S}$, this set of $(l - 1)$ distinct complexes can themselves be regarded as the elements of a smaller group (of order $l = h/g$) on a higher level of abstraction. This new group is called the *factor group* of $\mathscr{G}$ with respect to the *normal divisor* (or invariant subgroup) $\mathscr{S}$. In this factor group, $\mathscr{S}$ forms the unit element. We can see this by considering

$$\mathscr{S}\mathscr{K}_i = \mathscr{S}(\mathscr{S}K_i) = (\mathscr{S}\mathscr{S})K_i = \mathscr{S}K_i = \mathscr{K}_i$$

Group multiplication works out as shown in the following example,

$$\mathscr{K}_i\mathscr{K}_j = (\mathscr{S}K_i)(\mathscr{S}K_j) = K_i\mathscr{S}\mathscr{S}K_j = K_i\mathscr{S}K_j = \mathscr{S}(K_iK_j) = (\mathscr{K}_i\mathscr{K}_j)$$

where the last expression refers to the complex which is the coset associated with the product K_iK_j. The concept of factor groups and normal divisors will prove useful in analyzing the structure of groups.

Isomorphy and homomorphy. We have already introduced the concept of isomorphy by noting that two groups having the same multiplication table are called isomorphic. This means that there is a one-to-one correspondence between the elements $A, B, \ldots$ of one group and those $A', B', \ldots$ of the other, such that $AB = C$ implies $A'B' = C'$, and vice versa.

Two groups are said to be *homomorphic* if there exists a correspondence between the elements of the two groups of the sort $A \leftrightarrow A_1', A_2', \ldots$. By this we mean that, if $AB = C$, then the product of any A_i' with any B_j' will be a member of the set C_k'. In general, a homomorphism is a many-to-one correspondence, as indicated here. It specializes to an isomorphism if the correspondence is one-to-one. For example, the group containing the single element E is homomorphic to any other group, since, in view of the fact that each group element is represented by E, group multiplication reduces simply to $EE = E$. A much less trivial example is provided by the homomorphic relation between any group and one of its factor groups (if it has one). The invariant subgroup $\mathscr{S}$ corresponds to all the members of $\mathscr{S}$, and the cosets $\mathscr{K}_i = \mathscr{S}K_i$ correspond to all members of the coset (including K_i and all other group elements having the same coset, which are just the members of $\mathscr{K}_i$). Thus, if $\mathscr{S}$ is of order g, there is a g-to-one correspondence between the original group elements and the elements of the factor group.

2-9 Class Multiplication

In this section we consider a different form of multiplication of collections of group elements in which we do keep track of the number of times an element appears. That is, $\mathscr{R} = \mathscr{K}$ implies that each element appears as often in $\mathscr{R}$ as in $\mathscr{K}$. In this notation

$$X^{-1}\mathscr{C}X = \mathscr{C} \tag{2-1}$$

where $\mathscr{C}$ is any complete *class* of the group and X is any element of the group. (Proof: Each element produced on the left must appear on the right because they are all conjugate to elements in $\mathscr{C}$ and hence are in $\mathscr{C}$ by the definition of a class. But each element on the left is different, because of the uniqueness of group multiplication, as is each on the right. These two statements are consistent only if the two sides of the equation are equal.)

The converse of this theorem is also true: any collection $\mathscr{C}$ obeying (2-1) for all X in the group is comprised wholly of complete classes. (Proof: First subtract all complete classes from both sides and denote any remainder by $\mathscr{R}$. Now consider any element R_i of $\mathscr{R}$ on the left in $X^{-1}\mathscr{R}X = \mathscr{R}$. Since this is assumed true for all X, $\mathscr{R}$ must by definition include the complete class of R. Thus $\mathscr{C}$ must be composed of complete classes.)

If we now apply the theorem (2-1) to the product of two classes, we have

$$\begin{aligned}\mathscr{C}_i\mathscr{C}_j &= X^{-1}\mathscr{C}_iXX^{-1}\mathscr{C}_jX \\ &= X^{-1}(\mathscr{C}_i\mathscr{C}_j)X\end{aligned}$$

for all X. Then, upon applying the converse theorem, it follows that $\mathscr{C}_i\mathscr{C}_j$ consists of complete classes. This may be expressed formally by writing

$$\mathscr{C}_i\mathscr{C}_j = \sum_k c_{ijk}\mathscr{C}_k \tag{2-2}$$

where c_{ijk} is the integer telling how often the complete class $\mathscr{C}_k$ appears in the product $\mathscr{C}_i\mathscr{C}_j$.

An an example, in the symmetry group of the triangle whose class structure we noted earlier, let $\mathscr{C}_i = E$; $\mathscr{C}_2 = A, B, C$; and $\mathscr{C}_3 = D, F$. Then $\mathscr{C}_1\mathscr{C}_2 = \mathscr{C}_2$; $\mathscr{C}_1\mathscr{C}_3 = \mathscr{C}_3$; $\mathscr{C}_2\mathscr{C}_2 = 3\mathscr{C}_1 + 3\mathscr{C}_3$; $\mathscr{C}_2\mathscr{C}_3 = 2\mathscr{C}_2$.

EXERCISES

2-1 Consider the symmetry group of the proper covering operations of a square (D_4). This consists of eight elements:

E = the identity

A, B, C, D = 180° rotations about the corresponding labeled axes in Fig. 2-2 which are considered fixed in *space*, not on the body

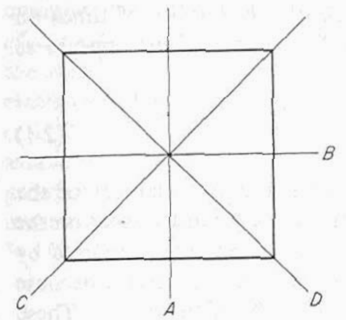

Fig. 2-2. *Symmetry axes of square.*

F, G, H = clockwise rotations in plane of the paper by $\pi/2$, π, and $3\pi/2$, respectively

(*a*) From the geometry, work out the multiplication table of the group; take advantage of the rearrangement theorem to check your result.

(*b*) From the nature of the operations, divide the group elements into classes. If in doubt, check by using the multiplication table [from (*a*)] and the definition of conjugate elements.

(*c*) Write down all the subgroups of the complete group. Note that the orders of the subgroups must be divisors of 8. Which of these subgroups are invariant subgroups (normal divisors)?

(*d*) Work out the cosets of the normal divisors.

(*e*) Work out the group-multiplication tables of the factor groups corresponding to the nontrivial normal divisors of the group.

(*f*) Determine the coefficients c_{ijk} appearing in all class multiplication products.

2-2 List the symmetries of a general rectangle. Work out the multiplication table, and divide the elements into classes.

2-3 Use the multiplication table for the symmetry group of the triangle to verify in several cases the rule for the inverse of a product.

2-4 Consider the group of order $(p - 1)$ obtained by taking as group elements the integers $1, 2, \ldots, (p - 1)$ and as group multiplication ordinary multiplication modulo p, where p is a prime number. (Modulo p means that $m + np$ is considered to be equal to m, where m and n are any integers.)

(*a*) Show that this is a group, and work out the multiplication table when $p = 7$.

(*b*) Prove in general that $A^{p-1} = E$, for all elements A of the group. In this way you have proved Fermat's number-theoretical theorem that $n^p = n(\text{mod } p)$, where n is an integer and p is a prime.

(*c*) Check the theorem for $p = 7$ and $n = 2, 3, 5$.

2-5 Prove that all elements in the same class have the same order when used to generate a cyclic group.

2-6 Show that there is a homomorphism between the cyclic groups of order 4 and 2.

2-7 Prove that a group is Abelian if, and only if, the correspondence of each element to its inverse forms an isomorphism.

2-8 Prove that $c_{ijk} = c_{jik}$ in Eq. (2-2). In other words, prove that $\mathscr{C}_i\mathscr{C}_j = \mathscr{C}_j\mathscr{C}_i$, even if the group is not Abelian.

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