CHARACTER THEORY OF COMPACT LIE GROUPS

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Abstract

This thesis develops the theory of compact Lie groups in order to arrive at the celebrated Weyl character formula. In order to attain this goal harmonic analysis on compact Lie groups, maximal tori, the Weyl group and its realisation as a finite group of isometries of a Euclidean space are studied in great detail. The treatment remains entirely in the realms of real analysis, with no complexification taking place for the general results.

This thesis also introduces a complete orthonormal sequence of rational functions that are related to the Jacobi polynomials. These are attained via the classical Cayley map.

I hereby declare that this submission is my own work and to the best of my knowledge it contains no materials previously published or written by another person, nor material which to a substantial extent has been accepted for the award of any other degree or diploma at UNSW or any other educational institution, except where due acknowledgement is made in the thesis. Any contribution made to the research by others, with whom I have worked at UNSW or elsewhere, is explicitly acknowledged in the thesis.

I also declare that the intellectual content of this thesis is the product of my own work, except to the extent that assistance from others in the project's design and conception or in style, presentation and linguistic expression is acknowledged.

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Part I

Introductory Material

Chapter 1

Lie Theory

Lie theory is central to modern mathematics. It has applications in fields as diverse as algebraic geometry, classical harmonic analysis and mathematical physics. The common theme in mathematics today is geometry, the essence of Lie theory.

The origins of Lie theory were not so lofty, Sophus Lie had the vision to use the geometric ideas that arose out of Klein's Erlanger Programme to solve differential equations. Consequently, the theory developed by Lie and his contemporaries was predominantly local. That is, they studied structures known today as Lie algebras.

In the beginning of the twentieth century, Elie Cartan's thesis classified the semisimple complex Lie algebras. Within another two years, he had classified the real semisimple Lie algebras. This marked a change in the direction of Lie theory research, it moved into global analysis.

Hermann Weyl saw Lie theory as the key to Einstein's General Relativity ([33] pp. vii-viii) and contributed tremendously to the field, the majority of the work in this thesis is due to Weyl. At the same time, Elie Cartan constructed the framework of differential geometry that we use today, in order to understand the global properties of Lie Groups.

Lie groups are the symmetries of continuous geometries. So, in the finite dimensional case, Lie theory is naturally viewed as a generalisation of linear algebra. For this reason, there are good examples of well known groups that are very illustrative of the general theory. Consequently, we will endeavour to provide ample computations to demonstrate what is really occuring. Let us get to work by outlining some of the elementary, but beautiful properties of Lie groups.

1.1 Concepts

When dealing with finite groups, a Hausdorff topology that is compatible with the group actions is given by the discrete topology, yielding little insight into the algebraic structure. In the case of uncountable groups, a Hausdorff topology is what we require to make the global, algebraic structure tractable by our theories of analysis. For example, consider

$$\mathbb{T} = \{z \in \mathbb{C} \,:\, |z| = 1\}.$$

This is clearly a group, and the topology that we would want to endow it with, is the relative topology inherited from $\mathbb{C}$. Observing that $e^{i\theta}e^{i\theta'} = e^{i(\theta+\theta')}$ and $\left(e^{i\theta}\right)^{-1} = e^{-i\theta}$, then multiplication and inversion, viewed as maps from $\mathbb{T}\times\mathbb{T} \to \mathbb{T}$ and $\mathbb{T} \to \mathbb{T}$ respectively, are smooth operations.[1] $\mathbb{T}$ is also naturally viewed as a one-dimensional, smooth manifold. It is also worth remarking that since periodicity is a cyclic phenomenon, then the study of periodic functions reduces to the study of functions on $\mathbb{T}$.

The example above was abelian, which is an unrealistic example of a general group. Now consider

$$SU(2) = \{u \in M_2(\mathbb{C}) \,:\, u^*u = 1, \det u = 1\},$$

[1] For an introduction to the differential geometry of manifolds, see [12] Ch. I.

which is certainly non-abelian. Again, it is clear that multiplication and inversion are smooth operations when given the (strong) operator topology induced from $\mathbb{C}^2$; it is also a compact metric space in this topology. By remarking that any $u \in SU(2)$ can be realised as $u = \left(\begin{smallmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{smallmatrix}\right)$ where $|\alpha|^2 + |\beta|^2 = 1$ and $\alpha,\, \beta \in \mathbb{C}$ then $SU(2)$ is diffeomorphic to S^3, the unit sphere in $\mathbb{R}^4$. So, $SU(2)$ is naturally a real 3-manifold. Note that $SU(2)$ arises naturally in the study of angular momenta of spin-$\frac{1}{2}$ in particle physics. It would seem worthwhile to make the following

Definition 1.1.1 (Lie Group). A **Lie Group**, G, is a group which is also an smooth manifold, with multiplication and inversion smooth maps with respect to this structure.[2]

In the sequel, I will be considering *real* Lie groups only[3]. It is fairly clear that an uncountable, closed group of matrices over $\mathbb{R}$ or $\mathbb{C}$ is a Lie group. We call such objects **matrix groups**. What is interesting, is that there are very important Lie groups that are not matrix groups. I cite the following paradigm example from [7].

Example 1.1.1 (Heisenberg Group). Let

$$N = \left\{ \left(\begin{smallmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{smallmatrix}\right) \,:\, a,\, b,\, c \in \mathbb{R} \right\} \qquad Z = \left\{ \left(\begin{smallmatrix} 1 & 0 & n \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right) \,:\, n \in \mathbb{Z}. \right\}$$

Then N/Z is a Lie group, called the Heisenberg group, that is not a matrix group.

Our setting is now a smooth manifold, and it is useful to have a sensible method to move points around. On a Lie group, there is a natural way of doing this.

Definition 1.1.2 (Translation Operators). Let G be a Lie group and $g \in G$.

1. **Left Translation** is achieved by the map $L_g : G \to G : h \mapsto gh$, from the differentiable structure we know that this map is smooth.

[2]Hilbert's 7th problem (Montgomery-Zippen Theorem) proved that for any topological group, there is at most one differentiable structure on it that endows it with a Lie group structure. Consequently, one may assume that a Lie group has C^1 charts, and it will turn out that they are in fact real-analytic.

[3]Complex (and algbebraic groups over an algebraically closed field) Lie groups are now more commonly dealt within the framework of algebraic geometry. For a brief introduction see [7] and for a more substantial exposition see [4]

2. **Right Translation** is achieved by the map $R_g : G \to G : h \mapsto hg^{-1}$, again this is smooth.

Notice that $L_{gh} = L_g \circ L_h$ and $R_{gh} = R_g R_h$ for all g, $h \in G$. Denote conjugation by $c_g = L_g R_g = R_g L_g$.

Now since $T_1(G)$, the tangent space at the identity of G, is finite dimensional, then we naturally can define inner products on $T_1(G)$. Fixing one, and using left translations we arrive at a left-invariant Riemannian metric on G. We have just proven the

Theorem 1.1.1. *Let G be a Lie group, then $\exists$ a right-invariant (resp. left-invariant) Riemannian metric on G. In particular, it follows that G is Riemannian manifold.*

We will now consider a Lie group G to be a Riemannian manifold endowed with a metric g that is invariant under left translations. The connection will always be the Riemannian connection and so a geodesic $\gamma : (s_1, s_2) \subset \mathbb{R} \to G$ minimises the (arc-length) functional

$$L(\gamma) = \int_{s_1}^{s_2} \left[g_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t))\right]^{1/2} \, \mathrm{d}t.$$

For details, see [12] pp. 47-55.

Our metric is left-invariant and so inner products are preserved under left translations. So, it would seem worthwhile trying to understand the vector space of left invariant vector fields on G, $\mathfrak{g}$. This is characterised as all vector fields V on G such that

$$(L_g)_\star V_1 = V_g \qquad \forall g \in G.$$

From which it is clear that $\mathfrak{g}$ is clearly canonically isomorphic to $T_1(G)$.

Let $\exp : M_n(\mathbb{C}) \to GL(\mathbb{C}^n)$ be the standard matrix exponential. Note that since $t \mapsto \exp(tV)$ defines a smooth curve in $GL(\mathbb{C}^n)$ with tangent vector V at $t = 0$; then to compute $\mathfrak{g}$ for G, a Lie group of matrices, it suffices to find the preimage G under exp.

Example 1.1.2 ($SU(2)$). $\mathfrak{su}(2) = \{X \in M_2(\mathbb{C}) \, : \, X^* = -X, \, \mathrm{tr}\, X = 0\}$.

Example 1.1.3 ($U(n)$). $\mathfrak{u}(n) = \{X \in M_n(\mathbb{C}) \,:\, X^* = -X\}$.

Now, fix $g \in U(n)$ and consider $c_g : h \mapsto ghg^{-1}$. Then define **Ad** as

$$\mathrm{Ad}(g) := (c_g)_\star(1) : \mathfrak{u}(n) \to \mathfrak{u}(n).$$

Let $X \in \mathfrak{u}(n)$ then $t \mapsto \exp(tX)$ is a smooth map at 1 and so

$$\mathrm{Ad}(g)X = \frac{\mathrm{d}}{\mathrm{d}t}\left[g\exp(tX)g^{-1}\right]_{t=0} = gXg^{-1}.$$

So, Ad is conjugation of $\mathfrak{u}(n)$; moreover, it is closed under this operation. We now differentiate $\mathrm{Ad}(\cdot)Y$ at 1 to get the **adjoint**,

$$\mathrm{ad}(\cdot)(Y) := \left[\mathrm{Ad}(\cdot)Y\right]_\star(1) : \mathfrak{u}(n) \to \mathfrak{u}(n).$$

Hence,

$$\begin{aligned}
\mathrm{ad}(X)Y &= \frac{\mathrm{d}}{\mathrm{d}t}\left[\exp(tX)Y\exp(-tX)\right]_{t=0} \\
&= \left[X\exp(tX)Y\exp(-tX) - \exp(tX)YX\exp(-tX)\right]_{t=0} \\
&= XY - YX.
\end{aligned}$$

For the general case, we really need to construct some nice curves to differentiate along in order to use the above maps. Once we do, it will turn out that $\mathfrak{g}$ can be given the structure of a non-associative algebra under the operation $[X, Y] := \mathrm{ad}(X)(Y)$.
In a purely geometrical approach, this corresponds to the Lie derivative of Y with respect to X. For our purposes, this is not a particularly pleasant characterisation of this operation, as we have an algebraic structure on the manifold. The notion that it is capturing conjugation near the identity, is preferred.[4] Let us set up a basic framework for doing such a thing.

[4] So do other authors, see Howe's article [16] for example.

Definition 1.1.3 (One-Parameter Group). A **one-parameter group** of a Lie group G is a smooth homomorphism $\gamma : \mathbb{R} \to G$.

Example 1.1.4. On $SU(2)$ let $H_t = \left(\begin{smallmatrix} it & 0 \\ 0 & -it \end{smallmatrix}\right)$ then $t \mapsto \exp(H_t) = \left(\begin{smallmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{smallmatrix}\right)$ is a one-parameter group. If $SU(2)$ is realised as S^3 then the image of $t \mapsto \exp(H_t)$ is a great circle through ± 1. It can be shown that conjugation by group elements rotates this great circle about the line through ± 1.

The following theorem is precisely the correspondence we are after, see [6] for the complete proof.

Theorem 1.1.2. *Let G be a Lie group and $\gamma : \mathbb{R} \to G$ a one-parameter group. Then the correspondence $\gamma \mapsto \dot{\gamma}(0) \in \mathfrak{g}$ is a canonical bijection between the set of one parameter groups of G and $\mathfrak{g}$.*

Proof (Sketch). Fix $X \in \mathfrak{g}$, and let $\gamma_X : (-\epsilon, \epsilon) \to G$ be the unique the geodesic such that $\dot{\gamma}_X(0) = X$. The existence and uniqueness of such a curve is courtesy of Picard's theorem [21]. We can extend the geodesic to a one parameter group by picking for each $t \in \mathbb{R}$ an $N \in \mathbb{Z}$ such that $\left|\frac{t}{N}\right| < \epsilon$, from which we define

$$\gamma_X(t) = \gamma_X\left(\frac{t}{N}\right)^N .$$

It is easy to check that this is a well-defined homomorphism. $\square$

We now have the theoretically convenient

Definition 1.1.4 (Exponential). Let $\exp : \mathfrak{g} \to G : X \mapsto \gamma^X(1)$, where γ^X is the unique one-parameter group tangent to X at 1. Observe, $\exp([t+s]X) = \exp(tX)\exp(sX)$.

Example 1.1.5. For a matrix group the Lie exp is the standard exp of matrices since $\frac{\mathrm{d}}{\mathrm{d}t}\exp(tX) = X\exp(tX)$.

Unfortunately, we have the following

Example 1.1.6. Consider $SL_2(\mathbb{R})$, this is a noncompact Lie group; with $\mathfrak{sl}(2,\mathbb{R})$ equal to the vector space of trace zero, real matrices. With the usual exponential of matrices, it is easy to see that $-I \notin \exp(\mathfrak{sl}(2,\mathbb{R}))$.

However, in the case that G is compact, exp will turn out to be surjective. There is a substantial amount of work to be done to prove this though.

So, we are now in a position to define Ad, ad on an arbitrary Lie group analogous to our matrix example. Call $[X, Y] = \mathrm{ad}(X)Y$ the **Lie Bracket** of X and Y; an operation which $\mathfrak{g}$ is closed under. The Lie bracket is also

- bilinear,
- skewsymmetric, and
- satisfies the *Jacobi identity*: $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$.

Definition 1.1.5 (Lie Algebra). Given a Lie group G, the **Lie algebra $\mathfrak{g}$** of G is the set of left invariant vector fields on G, equipped with the Lie bracket $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$.

Example 1.1.7 ($\mathfrak{su}(2)$-triple). If we give $\mathfrak{su}(2)$ the basis

$$H = \left(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right), \quad X = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right), \quad Y = \left(\begin{smallmatrix} 0 & i \\ i & 0 \end{smallmatrix}\right),$$

then an easy computation shows that $[H, X] = 2Y$, $[H, Y] = -2X$, $[X, Y] = H$.

Remark 1.1.1. Rearranging the Jacobi identity we get

$$[X, [Y, Z]] = [[X, Y], Z] + [Y, [X, Z]],$$

which is reminiscent of the product rule for differentiation of functions.

For a good introduction on the theory of Lie algebras constructed in this way, we recommend [10]. We end with a quote from Howe[16].

> The basic object mediating between Lie groups and Lie algebras is the one-parameter group. Just as an abstract group is a coherent system of cyclic groups, a Lie group is a (very) coherent system of one-parameter groups.

1.2 Lie Subgroups and Homomorphisms

There is substantial theory required to deal with this concept in a rigorous manner, which I will avoid as it would take us too far afield, and the techniques are not of use anywhere else in the thesis. For a a comprehensive reference I would recommend [12] Ch. II §2. I will cite some of the important results, but with no proof. This definition is due to Chevalléy, and also appears in [12] p. 112.

Definition 1.2.1 (Lie Subgroups and Subalgebras). If G is a Lie group and H a submanifold, then G is a **Lie subgroup** if H is a subgroup of G and a topological group (in the relative topology). A **Lie subalgebra** $\mathfrak{h}$ of $\mathfrak{g}$ is vector subspace that is closed under the Lie bracket.

$$\boxed{\textbf{Unless stated otherwise: a subgroup means a Lie subgroup}}$$

$$\boxed{\textbf{Unless stated otherwise: a subalgebra means a Lie subalgebra}}$$

This definition is chosen specifically to obtain the

Theorem 1.2.1 (Analytic Subgroups). *Let G be a Lie group. If H is a Lie subgroup of G, then the Lie algebra $\mathfrak{h}$ is a subalgebra of $\mathfrak{g}$. Each subalgebra of $\mathfrak{g}$ is the Lie algebra of exactly one connected Lie subgroup.*

Proof. See [12] pp.112-114. □

The other notion fundamental to group theory is in the following

Definition 1.2.2 (Homomorphism). A **homomorphism** of the Lie groups G and H is a smooth group homomorphism of G onto H. A homomorphism of Lie algebras is a linear map that preserves the Lie bracket i.e. $\phi([x,y]) = [\phi(x), \phi(y)]$.

The continuity is required so that the kernels and images of Lie group homomorphisms are Lie groups; see [12] p. 116 for details.

1.3 Integration

Recalling Frobenius' theory of finite group (over fields of characteristic 0) representation theory we had the **regular representation** on the group algebra, $\mathbb{C}G$, where the group acted by left or right translations. The beauty of this object was that it was canonically associated to G and, from the representation theory point of view, all the irreducibles appeared in it. However, in the case that G is a Lie group, $\mathbb{C}G$ is rather unwieldy. So, since G is a topological group too, it is more natural to consider the regular representation to be on $C_c(G)$, the space of continuous functions on G with compact support with the action given by

$$G \times C_c(G) \ni (g, f) \mapsto L_g f := \{h \mapsto f(gh)\}.$$

Unfortunately, $C_c(G)$ is a fair way from being a Hilbert space, which is a place we normally like representations to occur.[5] However, from the general theory of locally compact Hausdorff spaces, if we had a Radon measure, μ, on G, we would have $C_c(G)$ uniformly dense in the Hilbert space $L^2_\mu(G)$ (see [14] p. 140).

Therefore, our problem for the moment is constructing a Radon measure on G such that if $f \in L^2(G)$, then $L_g f \in L^2(G)$. It would also be nice if this measure had some good uniqueness properties. It turns out that we have the following

[5]It does have a good structure as a normed linear space under $\|f\| = \sup_{x\in G} |f(x)|$. Moreover, if G is compact, then $C_c(G) = C(G)$ and, is a Banach space in this norm; see [21] p. 57 (the example generalises easily) for details.

Theorem 1.3.1 (Haar-Weil). *Let G be a Lie group, then:*

1. *$\exists$ a Radon measure $\mu : \mathscr{B}(G) \to [0, \infty]$ such that if $E \in \mathscr{B}(G)$ then $\mu(gE) = \mu(E)$ $\forall g \in G$, called a **left Haar measure** on G. Similarly, there is a right Haar measure on G; and*
2. *if ν, $\mu : \mathscr{B}(G) \to [0, \infty]$ are two left (resp. right) Haar measures on G, then $\exists c > 0$ such that $\mu = c\nu$.*

Proof. It is routine but nevertheless quite involved to catch all the details in the contruction of the measure, see [20] ch. VIII. □

Remark 1.3.1. It is not hard to see that right Haar measures exist– we just pick a positive definite tensor of type $(0, 2)$ on $T_1(G)$ and then use translations to move it around the manifold. Even so, there are some details in this construction that need to be checked.

Observe that if $\mu : \mathscr{B}(G) \to [0, \infty]$ is a left Haar measure, then $\mu_g := \mu \circ R_g$ is another left Haar measure. By Theorem 1.3.1, it follows that $\mu_g = \Delta(g)\mu$. Also, if $\mu_g = \mu$ for all g, then μ would be a right Haar measure too. That is, every left Haar measure is a right Haar measure. There is an important case where this always happens:

Proposition 1.3.1. *If G is a compact Lie group, then $\Delta \equiv 1$.*

Proof. Since G is compact and a Haar measure is a Radon measure, then $0 < \mu(G) < \infty$; this also implies Δ is integrable since it is continuous. Hence, for $h \in G$

$$\int_G \mathrm{d}\mu(g) = \int_G \mathrm{d}\mu(hg) = \int_G \Delta(h) \, \mathrm{d}\mu(g) = \Delta(h) \int_G \mathrm{d}\mu(g).$$

This proves the result. □

When discussing measures on Lie groups, we will always mean the left Haar measure. If G is compact, we will always mean the left Haar measure (and so a right Haar measure) normalised so that $\mu(G) = 1$.

Chapter 2

Representations

2.1 Introduction

It is fairly clear that all matrix groups are Lie groups, and are very well understood in their own right. The question then arises, how similar are Lie groups to matrix groups? The representation theory of Lie groups aims to answer this question. There are analytic motivations for this too. Consider the following

Example 2.1.1 ($\mathbb{T}$). For $n \in \mathbb{Z}$, define $\chi_n : \mathbb{T} \to \mathbb{T} : z \mapsto z^n$. A classic result is that $\widehat{\mathbb{T}} = \{\chi_n\}_{n\in\mathbb{Z}}$ is a complete orthonormal system in $L^2(\mathbb{T})$ (for the analytic details, see [30] p. 119). It is also clear that $\widehat{\mathbb{T}}$ is contained in the set of continuous homomorphisms $\chi : \mathbb{T} \to \mathbb{C}^\times$. We now show the reverse inclusion.

Suppose χ is as above, and set $z_0 = \chi(e^i)$. By the homomorphism property: $\chi(e^{i/2})\chi(e^{i/2}) = \chi(e^i) = z_0$. This implies $\chi(e^{i/2}) = z_0^{1/2}$ and so $\chi(e^{is}) = z_0^s$ for all dyadic rationals s. Continuity implies $\chi(e^{i\theta}) = z_0^\theta$ for all θ and so since $1 = \chi(e^{2\pi i}) = z_0^{2\pi}$ and so $z_0 = e^{in}$ for some $n \in \mathbb{Z}$; which proves the result. So we have a correspondence between homomorphisms and a complete orthonormal system.

Given $f \in L^2(\mathbb{T})$, we also have the Fourier transform $\{\hat{f}(n)\}_{n\in\mathbb{Z}}$ of f, where

$$\hat{f}(n) = \frac{1}{2\pi}\int_{\mathbb{T}} f(z)\overline{\chi}_n(z)\frac{\mathrm{d}z}{z} = \langle f, \chi_n\rangle_{\mathbb{T}}\,.$$

Also,

$$f = \sum_{n\in\mathbb{Z}} \hat{f}(n)\chi_n \quad a.e.$$

If we let V_n be the space

$$V_n = \mathrm{span}_{\mathbb{C}}(\chi_n),$$

then we have the orthogonal direct sum

$$L^2(\mathbb{T}) = \bigoplus_{n\in\mathbb{Z}} V_n.$$

Note that $\mathbb{T}$ acts on V_n by $z \cdot f = z^n f$, for $z \in \mathbb{T}$ and $f \in V_n$.

Unfortunately, we have the following

Example 2.1.2. Suppose that G is non-abelian, then there is a conjugacy class containing at least two elements. So, let $g \neq g'$ be conjugate in G and so $g = hg'h^{-1}$. If $\chi : G \to \mathbb{C}^\times$ is a continuous homomorphism then $\chi(g) = \chi(hg'h^{-1}) = \chi(h)\chi(g')\chi(h)^{-1} = \chi(g')$.

Hence, in a nonabelian group G the continuous homomorphisms $\chi : G \to \mathbb{C}^\times$ fail to separate points, and so cannot be dense in $L^2(G)$. The appropriate generalisation turns out to be the

Definition 2.1.1 (Representation). A **representation** of a Lie group G on the topological space vector space V is a Lie group homomorphism of G onto $GL(V)$.

Unless stated otherwise: all representations are assumed to be finite dimensional.

Example 2.1.3. $U(n)$ has a natural representation on $\mathbb{C}^n$ given by inclusion.

Notation 2.1.1. If G admits a representation on V, we will write $\pi_V : G \to GL(V)$ to mean the relevant continuous group homomorphism.

Remark 2.1.1. We will often refer to a representation of G on V as the G-module V, with the action given by the representation. That is, $g \cdot v = \pi_V(g)v \; \forall v \in V$. For some detailed discussions on the general theory of modules we recommend [23].

Example 2.1.4 **(**Ad**).** Let G be a Lie group, then $g \mapsto \mathrm{Ad}(g) \in GL(\mathfrak{g})$ is a smooth homomorphism. This representation is canonically associated to G, it is called the Ad representation.

We have a corresponding notion occuring in the Lie algebra.

Definition 2.1.2 (Representation-Lie Algebra). A **representation** of Lie algebras is a homomorphism of Lie algebras $\phi : \mathfrak{g} \to \mathfrak{gl}(V)$.

Notice that if $\pi_V : G \to GL(V)$ is a representation, then $(\pi_V)_*(1_G) : \mathfrak{g} \to \mathfrak{gl}(V)$ is a homomorphism of Lie algebras. We refer to this as the representation **induced** from π_V.

Example 2.1.5 **(**ad**).** $\mathfrak{g} \ni X \mapsto \mathrm{ad}(X) \in \mathfrak{gl}(\mathfrak{g})$ is a homomorphism of Lie algebras, called the adjoint. Again, this is canonically associated to $\mathfrak{g}$.

Example 2.1.6 (Contragredient). Given a representation $\pi_V : G \to GL(V)$, there is a possibility of constructing another representation from it, namely a representation on V^* given by $\pi_{V^*}(g)\alpha(v) = \alpha(\pi_V(g)^* v)$.

Of interest too, is how we can construct new G-modules out of existing ones. The standard constructions are through the tensor product and the direct sum. The new G-module structure is given by

$$\pi_{V\otimes W}(g)(v \otimes w) = (\pi_V(g)v) \otimes (\pi_W(g)w),$$

$$\pi_{V\oplus W}(g)(v \oplus w) = (\pi_V(g)v) \oplus (\pi_W(g)w).$$

For details on the above construction, we refer the reader to [23]. On the Lie algebra, the representative structure above descends to

$$(\pi_{V\otimes W})_*(1_G)(v\otimes w) = (\pi_V(g)v)\otimes w + v\otimes(\pi_W(g)w),$$
$$(\pi_{V\otimes W})_*(1_G)(v\oplus w) = (\pi_V(g)v)\oplus(\pi_W(g)w).$$

When examining two G-modules, it is of interest to know how similar they are. The appropriate object of study is the set of G-equivariant homomorphisms between V and W, $\mathrm{Hom}_G(V,W)$. This is the set of maps $\Phi : V \to W$ such that $\pi_W \circ \Phi = \Phi \circ \pi_V$.

Lemma 2.1.1 (Schur). *Let V be an irreducible G-module, then $\mathrm{Hom}_G(V)$ is a division algebra. Moreover, if V is over $\mathbb{C}$, then $\dim \mathrm{Hom}_G(V) = 1$.*

Proof. Straightforward, see [9] pp. 212-213 for example. □

2.2 Complete Reducibility

When studying matrices, one often tries to examine its invariant subspaces, it would seem natural to do the same with a representation; in fact, of most interest (in this branch of the theory) are the smallest ones, and when we can build up other G-modules from these.

Definition 2.2.1. A G-module V is **irreducible** if the only invariant subspaces of V under the action of G are (0) and V.
If every G-module V can be written as a direct sum of irreducible G-modules, then we say G is **completely reducible**.

Suppose that V is a finite dimensional G-module, then clearly an inner product (resp. hermitian form) exists on V. Suppose that G is compact and let μ be the normalised Haar measure on G, then the map $\langle\cdot,\cdot\rangle_V : V\times V \to \Bbbk$:

$$\langle v, w\rangle_V = \int_G (\pi_V(g)v, \pi_V(g)w)\,\mathrm{d}\mu(g)$$

is clearly an inner product (resp. hermitian form) on V that is invariant under the action of G, since G is compact. In fact we have just proven the

Lemma 2.2.1 (Weyl's Trick). *Let G be a compact Lie group and suppose that V is a G-module. Then, there exists a G-invariant inner product (hermitian form if the space is complex) on V.*

Remark 2.2.1. This concept holds in greater generality, see [20] p. 382 for instance.

Notation 2.2.1. If G is compact, for a G-module V, we let $\langle \cdot, \cdot \rangle_V$ denote a fixed G-invariant inner product (resp. hermitian form) on V.

Corollary 2.2.1. *Let V be the $\mathfrak{g}$-module induced from G, then $\langle \cdot, \cdot \rangle_V$ is $\mathfrak{g}$-invariant. That is, $\langle g \cdot v, w \rangle_V = - \langle v, g \cdot w \rangle$.*

Proof. Notice that $\langle \pi_V(g)v, w \rangle_V = \langle v, \pi_V(g)^{-1}w \rangle_V$ for all $g \in G$ and v, $w \in V$. Thus, differentiating the above at the identity yields $\langle (\pi_V)_\star(1_G)(X)v, w \rangle_V = \langle v, (\pi_V)_\star(1_G)(X)w \rangle_V$ $\forall X \in \mathfrak{g}$. □

We now have the all important

Theorem 2.2.1. *Let G be a compact Lie group, then G is completely reducible.*

Proof. Let V be G-module, if V is irreducible then we are done. If not, pick $V_1 \leq V$ such that V_1 is a G-submodule, it suffices to prove that $V = V_1 \oplus V_1^\perp$ (as G-modules). Let $v \in V_1$, $g \in G$ and $w \in V_1^\perp$, then

$$\langle v, g \cdot w \rangle_V = \langle g^* \cdot v, w \rangle_V = \left\langle g^{-1} \cdot v, w \right\rangle_V = 0.$$

□

2.3 Characters

As one would rightly assume, a general representation is a very complicated object. It would now seem appropriate to introduce the following

Definition 2.3.1 (Character). Let $\pi_V : G \to GL(V)$ be a representation, then the **character** of V is $\chi_V = \operatorname{tr} \pi_V$.

The trace map used in the above definition is the standard one from linear algebra, in particular it is indepedent of the choice of basis for V. The other important fact is that characters are *constant on conjugacy classes*.

A better, coordinate free, definition for constructing the trace map was completed by Bourbaki, we repeat [2] pp. 46-47. We start this definition by noticing that we have a canonical isomorphism

$$\phi : V^* \otimes V \to \operatorname{Hom}_{\Bbbk}(V) : \lambda \otimes v \mapsto \{\xi \mapsto \lambda(\xi)v\}.$$

We also have the evaluation map

$$\epsilon : V^* \otimes V \to \Bbbk : \lambda \otimes v \mapsto \lambda(v).$$

So, for $\pi \in \operatorname{Hom}_{\mathbb{C}}(V)$, we define $\operatorname{tr} \pi = \epsilon \circ \phi^{-1}(\pi)$.

Example 2.3.1. Pick some basis $\{e_i\}_{i=1}^n$ of V and let $\{e_i^*\}_{i=1}^n$ be the dual basis. Then $\phi(e_j^* \otimes e_i)(e_k) = e_j^*(e_k)e_i = \delta_{jk}e_i$ and so $\phi(e_j^* \otimes e_i) = E_{ij}$, where E_{ij} is the matrix with zeroes everywhere except for the (i, j)-th position with respect to the above basis. Moreover, since $\epsilon(e_j^* \otimes e_i) = \delta_{ij}$, then if $A = \sum_{i,j=1}^n a_{ij}E_{ij} \in \operatorname{Hom}_{\Bbbk}(V)$, then

$$\epsilon \circ \phi^{-1}(A) = \sum_{i,j=1}^n a_{ij}\delta_{ij} = \sum_{i=1}^n a_{ii} = \operatorname{tr} A.$$

So, the two versions agree.

Remark 2.3.1. $\operatorname{tr} \mathrm{id}_V = \dim V$, and so $\chi_V(1) = \dim V$.

Remark 2.3.2. Characters are continuous class functions.

Bourbaki's definition is significantly easier to work with in proofs. In fact, the next proposition is essentially obvious from these definitions.

Proposition 2.3.1. *Let V, W be G-modules, then we have the following properties.*

1. $\chi_{V\oplus W} = \chi_V + \chi_W$;
2. $\chi_{V\otimes W} = \chi_V \chi_W$; *and*
3. $\chi_{V^*} = \overline{\chi_V}$ *if V is over $\mathbb{C}$.*

We also have the

Proposition 2.3.2. *Let V be a G-module, then*

$$\int_G \chi_V(g)\,\mathrm{d}g = \dim V.$$

Proof(Sketch). Note that by the linearity of the trace we have:

$$\int_G \chi_V(g)\,\mathrm{d}g = \int_G \operatorname{tr} \pi_V(g)\,\mathrm{d}g = \operatorname{tr} \int_G \pi_V(g)\,\mathrm{d}g.$$

But $K := \int_G \pi_V(g)\,\mathrm{d}g$ belongs to $GL(V)$ and

$$\left(\int_G \pi_V(g)\,\mathrm{d}g\right)^2 = \int_G \int_G \pi_V(gh)\,\mathrm{d}g\,\mathrm{d}h = \int_G \pi_V(g)\,\mathrm{d}g.$$

So, $\operatorname{tr} K$ is the dimension of the subspace that is G is invariant. If V is irreducible, then $\operatorname{tr} K = \dim V$. Otherwise we use the decomposition as a direct sum of irreducibles to obtain the result. □

Corollary 2.3.1. *Let V, W be irreducible G-modules over $\mathbb{C}$, then*

$$\int_G \chi_V(g)\overline{\chi_W}(g)\,\mathrm{d}g = \dim \operatorname{Hom}_G(W, V).$$

Proof. Just notice that $X := W^* \otimes V \cong \operatorname{Hom}_G(W, V)$ is a G-module and $\chi_X = \overline{\chi_W}\chi_V$. □

In the next chapter we finally do some serious computations which illustrate the theory discussed in the previous chapters.

Chapter 3

Representations of $SU(2)$

Employing relatively elementary methods, a complete set of irreducible representations of $SU(2)$ is constructed, realised on spaces of homogeneous polynomials. The treatment is related to [15] pp. 127-142, but is really attributed to the infinitesimal methods employed by Cartan ([5] p. 18). In particular, the characters and the matrix entries of the continuous, irreducible, unitary representations are computed explicitly.

3.1 Homogeneous Polynomials

Since $SU(2)$ acts naturally on $\mathbb{C}^2$, there is an induced representation on the space of complex polynomials in two variables given by

$$SU(2) \times \mathbb{C}[z_1, z_2] \ni (g, p) \mapsto \{(z_1, z_2) \mapsto p((z_1, z_2)g)\}.$$

Explicitly, we can write for $g = \left(\begin{smallmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{smallmatrix}\right) \in SU(2)$:

$$(g \cdot p)(z_1, z_2) = p(\alpha z_1 - \overline{\beta} z_2, \beta z_1 + \overline{\alpha} z_2).$$

So, for each non-negative half-integer ℓ we set

$$\mathcal{H}_\ell = \mathbb{C}[z_1, z_2]_{2\ell},$$

the complex vector space of homogeneous polynomials over $\mathbb{C}^2$.[1] It is now convenient to introduce the following basis for $\mathcal{H}_\ell$

$$\xi_k = z_1^{\ell+k} z_2^{\ell-k} \qquad k = -\ell,\, -\ell+1,\, \ldots,\, 0,\, \ldots,\, \ell-1,\, \ell.$$

From the explicit characterisation of the representation it is clear that $\mathcal{H}_\ell$ is stable under the $SU(2)$ action, we denote the restriction to $\mathcal{H}_\ell$ by $T_\ell : SU(2) \to GL(\mathcal{H}_\ell)$.

What we aim to prove is that the $\mathcal{H}_\ell$ are *all* of the continuous, irreducible representations of $SU(2)$. First, let us construct the matrix coefficients of the representations. We define an inner product as $\langle \xi_\mu, \xi_\nu \rangle = \delta_{\mu\nu}$, and the matrix entries of the representation as

$$t_\ell^{ij}(g) = \langle T_\ell(g)\xi_j, \xi_i \rangle\,, \qquad i,\, j \in \{-\ell, -\ell+1, \ldots, \ell\}.$$

Now,

$$\begin{aligned}
T_\ell \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} \xi_j &= (\alpha z_1 - \overline{\beta} z_2)^{\ell+j} (\beta z_1 + \overline{\alpha} z_2)^{\ell-j} \\
&= \left(\sum_{k=0}^{\ell+j} (-1)^k \binom{\ell+j}{k} \alpha^k \overline{\beta}^{\ell+j-k} z_1^k z_2^{\ell+j-k} \right) \times \left(\sum_{m=0}^{\ell-j} \binom{\ell-j}{m} \beta^m \overline{\alpha}^{\ell-j-m} z_1^m z_2^{\ell-j-m} \right) \\
&= \sum_{i=-\ell}^{\ell} \left(\sum_{s=-\ell}^{i} (-1)^{j-s} \binom{\ell+j}{\ell+s} \binom{\ell-j}{i-s} \alpha^{\ell+s} \beta^{i-s} \overline{\beta}^{j-s} \overline{\alpha}^{\ell-j-i+s} \right) \xi_i.
\end{aligned}$$

Hence, it is clear that

$$t_\ell^{ij} \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} = \sum_{s=-\ell}^{i} (-1)^{j-s} \binom{\ell+j}{\ell+s} \binom{\ell-j}{i-s} \alpha^{\ell+s} \beta^{i-s} \overline{\beta}^{j-s} \overline{\alpha}^{\ell-j-i+s}.$$

[1] This choice of indexation is classical, arising out of the indexing of spin numbers in quantum mechanics.

Remark 3.1.1. The matrix entries are not holomorphic functions in the variables α, β. This hints at the fact that the representation theory for $SU(2)$ is very much a phenomenon occurring in the category of smooth functions.

Now, let

$$T = \left\{ \left(\begin{smallmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{smallmatrix} \right) \,:\, \theta \in \mathbb{R} \right\} \subset SU(2)$$

and notice that every element of $SU(2)$ is conjugate to an element of T. So, to compute the character of $\mathcal{H}_\ell$ we need only consider its restriction to T. Indeed, the action is explicitly given by

$$T_\ell \left(\begin{smallmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{smallmatrix} \right) \xi_k = e^{2ki\theta} \xi_k.$$

So, it follows now that the character $\chi_\ell : SU(2) \to \mathbb{C}$ of $\mathcal{H}_\ell$ is:

$$\boxed{\chi_\ell \left(\begin{smallmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{smallmatrix} \right) = \sum_{n=-\ell}^{\ell} e^{2ni\theta} = \frac{e^{(2\ell+1)\theta} - e^{-(2\ell+1)\theta}}{e^{i\theta} - e^{-i\theta}} = \frac{\sin(2\ell+1)\theta}{\sin\theta}},$$

extended to be a class function on $SU(2)$.

A remark worth making now is that we have just diagonalised the T-action of $SU(2)$ on $\mathcal{H}_\ell$ and written $\mathcal{H}_\ell$ as a direct sum of (complex) T-modules. For $k \in \{-\ell, \dots, \ell\}$ we have the induced $\mathfrak{t}$-module action:

$$(T_\ell(1))_* \left(\begin{smallmatrix} it & 0 \\ 0 & -it \end{smallmatrix} \right) \xi_k = i(2kt)\xi_k.$$

Thus, ξ_k is a eigenvector for the $\mathfrak{t}$ action with eigenvalue, $i\alpha_k$, where $\alpha_k(\cdot) = 2k\cdot$.

Definition 3.1.1 (Weight). A **weight** of $SU(2)$ is an $\alpha \in \mathfrak{t}^*$ such that $\alpha \left(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix} \right) \in \mathbb{Z}$.

Consequently, we call $\operatorname{span}\{\xi_k\}$ a weight space of weight α_k. Also, we have an action, called the W-action, on T that swaps the eigenvalues. The induced W-action on the α_k

takes $\alpha_k \mapsto \alpha_{-k}$.

There is also a unique, "highest" weight for $\mathcal{H}_\ell$, namely ℓ, and the representations are indexed by these highest weights. Another, more subtle remark is that both the numerator and the denominator of the character computed above are W-antisymmetric. As will be shown in this thesis, $SU(2)$ is an inspiring and illustrative group for a great deal of the theory on compact Lie groups.

We now introduce the operators $\{H_\ell, X_\ell, Y_\ell\}$ with the actions:

$$\begin{aligned}
H_\ell(\xi_k) &= \frac{\mathrm{d}}{\mathrm{d}t} T_\ell \left(\begin{smallmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{smallmatrix} \right) \xi_k \Big|_{t=0} = 2ki\xi_k, \\
X_\ell(\xi_k) &= \frac{\mathrm{d}}{\mathrm{d}t} T_\ell \left(\begin{smallmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{smallmatrix} \right) \xi_k \Big|_{t=0} = (\ell - k)\xi_{k+1} - (\ell + k)\xi_{k-1}, \\
Y_\ell(\xi_k) &= \frac{\mathrm{d}}{\mathrm{d}t} T_\ell \left(\begin{smallmatrix} \cos t & i\sin t \\ i\sin t & \cos t \end{smallmatrix} \right) \xi_k \Big|_{t=0} = i(\ell - k)\xi_{k+1} + i(\ell + k)\xi_{k-1}.
\end{aligned}$$

Indeed, since

$$\begin{aligned}
\exp \begin{pmatrix} it & 0 \\ 0 & -it \end{pmatrix} &= \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix}, \\
\exp \begin{pmatrix} 0 & t \\ -t & 0 \end{pmatrix} &= \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}, \\
\exp \begin{pmatrix} 0 & it \\ it & 0 \end{pmatrix} &= \begin{pmatrix} \cos t & i\sin t \\ i\sin t & \cos t \end{pmatrix},
\end{aligned}$$

then $\{H_\ell, X_\ell, Y_\ell\}$ forms an $\mathfrak{su}(2)$ triple and so is a representation for $\mathfrak{su}(2)_\mathbb{C} := \mathfrak{su}(2) \otimes \mathbb{C}$, since $\mathcal{H}_\ell$ is a complex vector space. In fact, by its very construction it is the $\mathfrak{su}(2)_\mathbb{C}$-module induced from the representation of $SU(2)$ on $\mathcal{H}_\ell$.

Remark 3.1.2. Note that by taking the exponential of the complex span of the operators $\{H_\ell, X_\ell, Y_\ell\}$ we would have extended the smooth action of $SU(2)$ on $\mathcal{H}_\ell$ to a holomorphic action of $SL(2,\mathbb{C})$ on $\mathcal{H}_\ell$. Roughly speaking, this is because $\mathfrak{su}(2)_{\mathbb{C}} = \mathfrak{sl}(2,\mathbb{C})$.

To prove the irreducibility of $\mathcal{H}_\ell$ as an $SU(2)$-module, it suffices to prove that it is an irreducible $\mathfrak{su}(2)_{\mathbb{C}}$-module, since (matrix) exponentiation and differentiation are mutually inverse bijections between the $\mathfrak{su}(2)$-modules and $SU(2)$-modules. It is obvious that irreducibility is preserved in this correspondence. In general, the following operators are more convenient to work with.

Notation 3.1.1.

$$\begin{aligned}
\widehat{H}_\ell &= -iH_\ell; \\
\widehat{X}_\ell &= \frac{1}{2}(X_\ell - iY_\ell); \text{ and} \\
\widehat{Y}_\ell &= -\frac{1}{2}(X_\ell + iY_\ell).
\end{aligned}$$

We now have an obvious

Lemma 3.1.1 (String Basis).

$$\begin{aligned}
\hat{H}_\ell(\xi_k) &= 2k\xi_k; \\
\hat{X}_\ell(\xi_k) &= (\ell - k)\xi_{k+1}; \textit{ and} \\
\hat{Y}_\ell(\xi_k) &= (\ell + k)\xi_{k-1}.
\end{aligned}$$

Theorem 3.1.1. *$\{\mathcal{H}_\ell\}_\ell$ is a complete list of the continuous, irreducible, finite dimensional representations of $SU(2)$.*

Proof. By Lemma 3.1.1 it is clear that ξ_ℓ generates $\mathcal{H}_\ell$. So, it suffices to prove that, given any non-zero $v \in \mathcal{H}_\ell$, then there is a sequence of actions that take it to ξ_ℓ.

Notice that since $\{\xi_k\}$ is a basis of $\mathcal{H}_\ell$, we can write $v = \sum_i a_i \xi_i$ for some $a_i \in \mathbb{C}$. Let $j = \min\{i \,:\, a_i \neq 0\}$, then

$$\xi_\ell = \frac{(\ell - j - 1)!}{a_j (2\ell)!} \widehat{X}_\ell^{\ell - j - 1}(v),$$

as required.

To prove that any irreducible $SU(2)$-module is isomorphic to one of the $\mathcal{H}_\ell$ it suffices to show that any complex $\mathfrak{su}(2)_\mathbb{C}$-module, V, arising from a real $\mathfrak{su}(2)$-module, exhibits a string basis as in Lemma 3.1.1.

For a weight α write

$$V_\alpha = \{v \in V \,:\, H(v) = \alpha(H) v \,\forall H \in \mathfrak{t}_\mathbb{C}\}.$$

From linear algebra, $V = \bigoplus_\alpha V_\alpha$. Now, let $k \in \mathbb{Z}^+$, then

$$H\left(X^k(v)\right) = [H, X](X^{k-1}(v)) + XHX^{k-1}(v).$$

But

$$HX(v) = [H, X](v) + XH(v) = 2X(v) + XH(v),$$

so if $v \in V_\alpha$, then

$$HX(v) = (\alpha(H) + 2)X(v) \implies H(X^k(v)) = (\alpha(H) + 2k)X^k(v).$$

In other words, $X^k(v)$ is an eigenvector of H with eigenvalue $\alpha(H) + 2k$. Indeed, since $\dim V < \infty$, then $X^k(v) = 0$ for some $k \in \mathbb{Z}^+$ with $k \leq \dim V$ (which we choose to be minimal). Set $n' = \frac{\dim V}{2}$, and $\zeta_{n'} = X^{k-1}(v)$ and define (inductively) for $k = -n', -n' + 1, \ldots, n' - 1$ the ζ_k by

$$\zeta_{k-1} = \frac{1}{n' + k} Y(\zeta_k).$$

Indeed, by irreducibility the $\mathfrak{su}(2)_{\mathbb{C}}$-module generated by the $\{\zeta_k\}$ is either (0) or V and, since $\zeta_{n'} \neq 0$, it must be V. It is obvious the $\mathfrak{su}(2)_{\mathbb{C}}$-module isomorphism $\zeta_k \mapsto \xi_k \in \mathcal{H}_{n'}$ is well-defined, and we are done. $\square$

We call the vector $\zeta_{n'}$ constructed above a **highest weight vector** for V. Note that every $SU(2)$-module created by a highest weight vector is irreducible. The converse also holds. The following (standard) argument appears courtesy of [6] p. 86.

Corollary 3.1.1. *If V is a continuous, irreducible, unitary representation of $SU(2)$, then $\dim V < \infty$.*

Proof. Suppose that $\dim V = \infty$, the fundamental observation is that $\operatorname{span}\{\cos n\theta\}_{n\in\mathbb{N}} = \operatorname{span}\{\chi_{k/2}\}_{k\in\mathbb{N}}$ (which is easy to check) and so, by the density of $\operatorname{span}\{\cos n\theta\}_{n\in\mathbb{N}}$ in the even functions on $L^2(T)$ ([30] p. 119 can be adapted to prove this), it follows from Corollary 2.3.1 that $\chi_V = 0$ if $\dim V = \infty$–a contradiction. $\square$

Remark 3.1.3. Again, since $\{\cos n\theta\}_{n\in\mathbb{N}}$ is dense in the space of even functions on $[-\pi, \pi]$, then $\overline{\operatorname{span}\{\chi_\ell\}} = L^2(T)^W$. This is the conclusion of the Peter-Weyl Theorem, which will be discussed in the next chapter. A deeper conclusion of the Peter-Weyl Theorem is that $\overline{\operatorname{span}\{t_\ell^{ij}\}} = L^2(SU(2))$. This fact can be seen directly at the moment by using the Stone-Weierstraß theorem.

3.2 Haar Measure

Note that since $SU(2)$ can be viewed as S^3, multiplication can be viewed as an orthogonal transformation of S^3. So, an invariant measure on $SU(2)$ would be the normalised surface measure on S^3. In other words

$$\int_{SU(2)} f(g)\,\mathrm{d}g = \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{0}^{\pi}\int_{0}^{\pi} f(\theta,\varphi_1,\varphi_2)\sin^2\varphi_1\sin\varphi_2\,\mathrm{d}\varphi_1\,\mathrm{d}\varphi_2\,\mathrm{d}\theta.$$

3.3 Decomposition of the Tensor Product into Irreducibles

In the Standard Model of particle physics, fundamental particles are identified with an irreducible representation of a certain symmetry group of the particle. For the case of a spin $\frac{1}{2}$ particle, for example, an electron, the symmetry group of the angular momenta is $SU(2)$. Moreover, the interaction between two particles is represented by their tensor product; and the decomposition of the tensor product into irreducibles corresponds to the fundamental particles emitted by the interaction. The case of $SU(2)$ is classical, and will be proven now. The more general case can be dealt with once the Weyl Character formula has been proven.

Theorem 3.3.1.

$$\mathcal{H}_{\ell'}\otimes\mathcal{H}_{\ell} \cong \bigoplus_{j=|\ell'-\ell|}^{\ell+\ell'} \mathcal{H}_j$$

Proof. Note first that for $j \in \{-\ell',\dots,\ell'\}$ and $k \in \{-\ell,\dots,\ell\}$ we have

$$H(\xi_j\otimes\xi_k) = (2j+2k)(\xi_j\otimes\xi_k).$$

Then without loss, $\ell' \geq \ell$ and define for each $n \in \{0,\dots,\ell'+\ell-(\ell'-\ell)\}$:

$$\nu_n = \sum_{m=0}^{n}(-1)^m\binom{n}{m}\xi_{\ell'-m}\otimes\xi_{\ell-n+m}.$$

Then,

$$
\begin{aligned}
H(\nu_n) &= \sum_{m=0}^{n} (-1)^m \binom{n}{m} H\left(\xi_{\ell'-m} \otimes \xi_{\ell-n+m}\right) \\
&= \sum_{m=0}^{n} (-1)^m \binom{n}{m} 2([\ell'-m] + [\ell-n+m])(\xi_{\ell'-m} \otimes \xi_{\ell-n+m}) \\
\therefore \quad H(\nu_n) &= 2(\ell'+\ell-n)\nu_n. \\
X(\nu_k) &= \sum_{m=0}^{n} (-1)^m \binom{n}{m} \left(X(\xi_{\ell'-m}) \otimes \xi_{\ell-n+m} + \xi_{\ell'-m} \otimes X(\xi_{\ell-n+m})\right) \\
&= \sum_{m=1}^{n} (-1)^m \binom{n}{m} m(\xi_{\ell'-m+1} \otimes \xi_{\ell-n+m}) \\
&\quad + \sum_{m=0}^{n-1} (-1)^m \binom{n}{m} (n-m)(\xi_{\ell'-m} \otimes \xi_{\ell-n+m+1}) \\
&= n \sum_{m=1}^{n} (-1)^m \binom{n-1}{m-1} (\xi_{\ell'-m+1} \otimes \xi_{\ell-n+m}) \\
&\quad + n \sum_{m=0}^{n-1} (-1)^m \binom{n-1}{m} (\xi_{\ell'-m} \otimes \xi_{\ell-n+m+1}) \\
&= 0.
\end{aligned}
$$

So, ν_n is a highest weight vector of weight $\ell+\ell'-n$. Note that the sum of the dimensions of the highest weight submodules we have calculated is

$$
\begin{aligned}
\sum_{n=0}^{2\ell} [2(\ell+\ell'-n)+1] &= (4\ell+2)(\ell+\ell') - 2\ell(\ell+1) + 4\ell + 2 \\
&= (2\ell+1)(2\ell'+1) \\
&= \dim \mathcal{H}_\ell \otimes \mathcal{H}_{\ell'}.
\end{aligned}
$$

We are done. $\square$

Trivially, the following trigonometric identity is obtained.

Corollary 3.3.1. *Let ℓ, ℓ' be nonnegative half integers, then*

$$\sin(2\ell' + 1)\theta \times \sin(2\ell + 1)\theta = \sin\theta \sum_{j=|\ell'-\ell|}^{\ell'+\ell} \sin(2j + 1)\theta.$$

In the proof above we only calculated the highest weight vectors, as this was all that was required for the decomposition. However, if one would like to do computations on the decomposed tensor product (as a physicist would) one would certainly be interested in the rest of the vectors in the highest weight modules. More precisely, one would seek linear combinations of the $\xi_r \otimes \xi_s$ that are orthonormal bases for $\mathcal{H}_\ell \otimes \mathcal{H}_{\ell'}$ and for each of the highest weight modules. The coefficients are generally called Clebsch-Gordan coefficients. For some recent work on the matter see [1] pp. 25-89.

Part II

Elementary Structure Theory

Chapter 4

The Peter-Weyl Theorem

The Peter-Weyl Theorem opens the door to abstract harmonic analysis. It really tells us that there are enough representations to separate points of the group and that $C(G)$ can be decomposed like the group algebra $\mathbb{C}G$, in finite group representation theory. Moreover, it tells us that every Lie group is a closed subgroup of $U(n)$ for some n and that every continuous, unitary, irreducible representation of a compact Lie group is finite dimensional. In addition, it also informs us that the exponential is surjective. This is certainly a theorem we would wish to understand in its entirety. Surprisingly, for the case of a compact group, the result is not especially difficult to prove, with the bulk theory embedded in the well-understood field of Fredholm operators[1].

4.1 Representative Functions

If $\pi_V : G \to GL(V)$ is a finite dimensional, unitary representation and $\mathcal{B} = \{\xi\}_{i=1}^{\dim V}$ is an orthonormal basis for V, then $\langle \xi_j, \pi_V(g^{-1})\xi_i \rangle_V$ corresponds to the (i,j)-th matrix entry of $\pi_V(g)$ with respect to the basis $\mathcal{B}$. In an attempt to be as canonical as possible, the following definition is used to generalise the concept of a matrix coefficient.

Definition 4.1.1 (Representative Function). A **representative function** of G is any function of the form $g \mapsto \lambda(\pi_V(g^{-1})u)$, where V is some G-module, $\lambda \in V^*$ and $u \in V$.

[1] The non-compact case, which we do not prove is significantly more complicated. The problems are essentially because the L^2 space associated to it is non-separable. See [31] for how this problem is dealt with.

Now, since $g \mapsto \pi_V(g)$ is continuous, then clearly any representative function is continuous. Call the linear span of all the matrix coefficients the **space of matrix coefficients** or **representative functions** of G, $\mathbb{M}(G)$. This is a subspace of $C(G)$.

The implication of the Peter-Weyl Theorem is that $\mathbb{M}(G)$ is uniformly dense in $C(G)$, and so there are enough representations to separate the points of G.[2] This is the fact that enables us to do harmonic analysis on compact Lie groups.[3]

Now, we let G act on $C(G)$ by left translations and note that this is a continuous representation on the infinite dimensional space $C(G)$, we will call this the **left regular representation**.[4] Similarly, we have a right regular representation. Indeed, we now have a natural representation of $G \times G$ on $C(G)$ given by the left part acting by left translations and the right part acting by right translations.

Remark 4.1.1. Note that this representation is somewhat more canonical since left and right translations commute. Also, the diagonal of the $G \times G$-action is conjugation, so $C(G)^\Delta$ corresponds to the continuous class functions on G, where $\Delta := \{(g, g) \in G \times G\}$.

4.2 Decomposition of the Regular Representations

One of the many highlights of the Frobenius-Schur representation theory for finite groups over $\mathbb{C}$ was the following decomposition ([10] p. 37) of algebras:

$$\mathbb{C}G = \bigoplus_{V \in \widehat{G}} V^* \otimes V,$$

where $\widehat{G}$ was a set of representatives of the equivalence classes of irreducible representations of G on complex vector spaces. It will turn out that we will obtain an identical

[2]By Urysohn's lemma, since G compact and Hausdorff, $C(G)$ separates the points of G. By density, $\mathbb{M}(G)$ has enough members to separate points of G as $C(G)$.

[3]This section only relies on the compact Hausdorff property that G possesses, so we can do harmonic analysis on compact topological groups too.

[4]In the sense of strong convergence.

decomposition for our compact Lie group G with $\mathbb{C}G$ replaced by $C(G)$. The following remarks appear in [29] and are the fundamental correspondence we seek.

If V is a G-module, then $V^* \otimes V \cong \operatorname{Hom} V$ can be endowed with another two G-module structures:

$$G \times (V^* \otimes V) \ni (g, \lambda \otimes v) \mapsto \pi_V^R(g)(\lambda \otimes v) := \lambda \otimes \pi_V(g)v \in V^* \otimes V; \text{ and}$$
$$G \times (V^* \otimes V) \ni (g, \lambda \otimes v) \mapsto \pi_V^L(g)(\lambda \otimes v) := \pi_V^R(g^{-1})(\lambda \otimes v) \in V^* \otimes V.$$

There is also an induced $(G \times G)$-action on $V^* \otimes V$ with the left and right copies of G acting via the left and right representations above.

Proposition 4.2.1. *If V is a complex G-module, then there is a homomorphism of G and $(G \times G)$-modules given by:*

$$V^* \otimes V \ni \lambda \otimes v \mapsto \{g \mapsto \pi_V(g)_V^L(\lambda \otimes v)\} \in C(G); \textit{ and}$$
$$V^* \otimes V \ni \lambda \otimes v \mapsto \{g \mapsto \pi_V(g)_V^R(\lambda \otimes v)\} \in C(G).$$

With the $(G \times G)$-module homomorphism induced from the above two. Moreover, if V is irreducible the G-module homomorphisms are faithful.

Proof. We just have to check the left action, the proofs of the right action and the two-sided action will be identical. Let g, $h \in G$, $\lambda \in V^*$, $v \in V$. Then,

$$\begin{aligned} L_g\{h \mapsto \lambda(\pi_V(h^{-1})v)\} &= \{h \mapsto \lambda(\pi_V(g^{-1}h^{-1})v)\} \\ &= \{h \mapsto \lambda(\pi_V(g^{-1})\pi_V(h^{-1})v)\} \\ &= \pi_V^L(g)(\lambda \otimes v). \end{aligned}$$

It suffices to show that the left action is faithful whenever V is irreducible. Note that if $v \neq 0$ and $\lambda(\pi_V(h^{-1})v) = \lambda(v)$ for all $h \in G$, then by irreducibility, $\pi_V(G)\,\mathrm{span}_{\mathbb{C}}(v) = V$ and so $\lambda(w) = 0$ for all $w \in V$ and so $\lambda = 0$. It follows that the left action is faithful. $\square$

Vogan [29] adopts the following

Definition 4.2.1 (Matrix Coefficient Maps). The mappings in Proposition 4.2.1 are called the **matrix coefficient maps**.

Following [7] p. 93 we make the following

Notation 4.2.1. For a G-module V, let

$$V^{\mathrm{fin}} = \{v \in V \,:\, \dim \pi_G(\mathrm{span}(v)) < \infty\}.$$

That is, all those vectors that belong to a finite dimensional, G-invariant subspace of V. Clearly, V^{fin} is a G-submodule of V.

Still following [7] p. 93 we obtain the following

Lemma 4.2.1.

$$\boxed{C(G)^{\mathrm{fin}} = \mathbb{M}\,(G).}$$

Proof. Let V be a finite dimensional G-submodule of $C(G)$, and let $f \in V$. Remarking that $f(g) = (L_g f)(1)$, since L_g has finite range on V, its action can be written as a matrix. This certainly implies that $f(g)$ can be written as a finite linear combination of matrix coefficients. The reverse inclusion is clear from Lemma 4.2.1. $\square$

The following definition is made naïvely. for the case of compact Lie groups we shall see later that this is not contentious.[5]

[5]It is undoubtedly the case that in considering non-compact Lie groups, we are forced into the theory of Plancherel measures. In the semisimple case we resort to the Harish-Chandra theory to resolve our conundrums, for a comprehensive introduction see [31] and [32]. For the nilpotent case, we make use of Kirillov's theory, see [19] for example. Plancherel theory is an active area of research.

Definition 4.2.2 (Dual). Let $\widehat{G}$ denote the set of equivalence classes of complex, continuous, irreducible, unitary, finite dimensional representations of G. $\widehat{G}$ is referred to as the **dual**.

Example 4.2.1 ($SU(2)$). $\widehat{SU(2)} = \{\mathcal{H}_{k/2}\}_{k\in\mathbb{N}}$.

Notation 4.2.2. If $\boldsymbol{\theta} \in \mathbb{R}^n$ it will be convenient to denote $\boldsymbol{e^{i\theta}} := (e^{i\theta_1}, \ldots, e^{i\theta_n}) \in \mathbb{T}^n$.

Example 4.2.2 ($\mathbb{T}^n$). Let $\widehat{\Lambda} = \{\alpha \in (\mathbb{R}^n)^* : \alpha(\mathbb{Z}^n) \subset 2\pi\mathbb{Z}\}$ then note that if we define $\chi_\alpha : \mathbb{T}^n \to \mathbb{C}^\times : \boldsymbol{e^{i\theta}} \mapsto e^{i\alpha(\boldsymbol{\theta})}$ for $\alpha \in \widehat{\Lambda}$, then an easy computation demonstrates that $\widehat{\mathbb{T}^n} = \{\chi_\alpha\}_{\alpha\in\widehat{\Lambda}}$ (see [33] p. 198 for the computation).

In the last example we had $\widehat{\mathbb{T}}^n \cong (\widehat{\mathbb{T}})^{\otimes n}$; this is precisely the content of the

Lemma 4.2.2. *Let G, H be compact Lie groups, $V \in \widehat{G}$, $W \in \widehat{H}$, then $V \otimes W \in \widehat{G \times H}$. Moreover, any element of $\widehat{G \times H}$ arises in this way.*

Proof. Observe:

$$\operatorname{Hom}_{G\times H}(V \otimes W) \cong [(V \otimes W)^* \otimes (V \otimes W)]^{G\times H} \cong (V^* \otimes V)^G \otimes (W^* \otimes W)^H.$$

By Schur's Lemma, $V \otimes W$ is irreducible.

Suppose that V' is an irreducible $(G\times H)$-module, let $v \in V'\backslash\{0\}$, then set $V := (G, 1_H)\operatorname{span}_{\mathbb{C}} v$, $W := (1_G, H)\operatorname{span}_{\mathbb{C}}(v)$, and it is routine to verify the rest. ∎

Example 4.2.3 ($U(n)$). Observe that $U(n) \cong SU(n) \times \mathbb{T}$ via $g \mapsto (g/\det g, \det g)$. So, if $V_{U(n)} \cong V_{SU(n)} \otimes V_{\mathbb{T}}$, then $\chi_{V_{U(n)}} = \chi_{V_{SU(n)}}\chi_{V_{\mathbb{T}}}$.

Proposition 4.2.2.

$$\boxed{\mathbb{M}(G) \cong \bigoplus_{V\in\widehat{G}} V^* \otimes V,}$$

as an orthogonal direct sum of $(G \times G)$-modules. The isomorphism is given by the matrix coefficient map.

Proof. Everything is clear from Theorem 2.2.1 (complete reducibility), Proposition 4.2.1, Lemma 4.2.2 and Lemma 4.2.1. ☐

Let us examine what the inner product looks like on each summand in the above decomposition of $\mathbb{M}(G) \leq L^2(G)$. Observe that irreducibility and Schur's Lemma implies that there is a unique hermitian form on V up to scalar multiple. So, there is some constant K_V such that

$$\langle \lambda_1 \otimes v_1, \lambda_2 \otimes v_2 \rangle_{V^* \otimes V} = K_V \langle v_1, v_2 \rangle_V \langle \lambda_1, \lambda_2 \rangle_{V^*}.$$

But we also have by a similar argument:

$$\langle \lambda_1 \otimes v_1, \lambda_2 \otimes v_2 \rangle_{V^* \otimes V} = K'_V \int_G \pi_V^L(g)(\lambda_1 \otimes v_1) \overline{\pi_V^L(g)(\lambda_2 \otimes v_2)} \, \mathrm{d}\mu(g).$$

By computing the ratio K_V / K'_V, we arrive at the (classical) result

Corollary 4.2.1 (Schur Orthogonality). *Let V, $W \in \widehat{G}$, ξ, $\nu \in V$ and ξ', $\nu' \in W$. Then*

$$\int_G \langle \pi_V(g)\xi, \nu \rangle_V \overline{\langle \pi_W(g)\xi', \nu' \rangle_W} \, \mathrm{d}\mu(g) = \begin{cases} \frac{\langle \xi, \nu \rangle_V \overline{\langle \xi', \nu' \rangle_W}}{\dim V} & \textit{if } V \cong W, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. The proof is straightforward and identical to the case for finite groups, see [7] pp. 98-99 for the details.

☐

Once we show the density of $\mathbb{M}(G)$ in $C(G)$ (and so in $L^2(G)$) we will be able to do harmonic analysis on G because then the representative functions of irreducible representations, scaled by $(\dim V)^{-1/2}$, with respect to an orthonormal basis on each representation, will be a complete orthonormal system on $L^2(G)$. A similar argument demonstrates that the characters of the irreducible representations form a complete orthonormal system on the set of $L^2(G)$ class functions. For a comprehensive account, see [14] and [15].

4.3 Main Result

Consider the following

Example 4.3.1 ($\mathbb{T}$). Let $G = \mathbb{T}$, and consider V_n as defined in Example 2.1.1. Then, if $f \in L^2(G)$, $\int_G f(g)\overline{\chi_n(g)}\,\mathrm{d}g$ generates the n-th Fourier coefficient of f.

What is really occuring in the above example is projection. Considering Proposition 4.2.2, the appropriate generalisation is the

Definition 4.3.1 (Fourier Transform). For a G-module V, and $f \in C(G)$, define the **Fourier transform** as the unique operator that satisfies $T_f : V \to V$ as

$$T_f(\xi) = \int_G \pi_V(g) f(g) \xi \,\mathrm{d}\mu(g).$$[6]

We now endow $C(G)$ with a ring structure, by using **convolution** of functions:

$$(f_1 * f_2)(g) = \int_G f_1(h) f_2(h^{-1}g)\,\mathrm{d}\mu(h).$$

Moreover, the Fourier transform makes V into a $C(G)$-module when we use convolution. In particular, if we let $V = C(G)$, then it becomes an algebra.[7] Unfortunately we have the following

Example 4.3.2. Let $f_2 \in C(G)$ and suppose that $f_1 * f_2 = f_1$ for all $f_1 \in C(G)$, then

$$0 = \left| \int_G f_1(h) f_2(h^{-1}g) - f_1(g)\,\mathrm{d}\mu(h) \right| \geq \int_G ||f_1(h) f_2(h^{-1}g)| - |f_1(g)||\,\mathrm{d}\mu(h) \geq 0.$$

[6] To (rigorously) construct an operator out of the above definition we should make use of the Bochner integral, see [31] pp. 378-379.

[7] It actually has the structure of a C^*-algebra, but this would take us too far afield. However, it is worth remarking that $\overline{\mathbb{M}(G)} = C_0(G)$ is equivalent to the injectivity of the Fourier transform on $C(G)$. These results hold in the case that G is only a locally compact Hausdorff topological group, and are contained in the Gel'fand-Raîkov Theorem ([14] p. 343).

Hence, $|f_1(h)f_2(h^{-1}g)| = |f_1(g)|$ for all h, $g \in G$ and $f_1 \in C(G)$. If we set $h = 1$, then $|f_1(1)||f_2(g)| = |f_1(g)|$ for all $g \in G$ and $f_1 \in C(G)$; this is impossible, as $C(G)$ separates points.

Following from the above, $C(G)$ is an algebra without identity. The following lemmata intend to fix this dilemma analytically.

Lemma 4.3.1. *For each neighbourhood U of $1 \in G$ with $U = U^{-1}$, there is a function f_U with $\operatorname{supp} f_U \subset U$, $f_U(x) = f_U(x^{-1})$ and has $\int_G f_U(g)\, \mathrm{d}\mu(g) = 1$.*

Proof. Use Urysohn's lemma to construct $f_U \in C_c(G) = C(G)$ with $0 \leq f_U \leq 1_U$. Then, let $f_U(x) := f_U(x)f_U(x^{-1})$ and we can rescale as necessary to ensure that it integrates to 1. $\square$

Lemma 4.3.2. *Let $f \in C(G)$, then $\forall \epsilon > 0$ $\exists U \subset G$, open, such that $U = U^{-1}$, $1 \in U$ and*

$$|f(y) - f(x)| < \epsilon \qquad \forall x^{-1}y \in U.$$

Proof. By continuity, $\exists U' \subset G$ open such that $|f(y) - f(x)| < \epsilon$ $\forall x, y \in U'$. So, since multiplication is smooth and invertible, then for $z \in U'$, $V_z := z^{-1}U' \cap U'^{-1}z$ is open, nonempty (since 1 is in both), and is a neighbourhood of the identity with $V_z^{-1} = V_z$. Set $U = \bigcup_{z \in U} V_z$, then U is open, contains 1 and $U = U^{-1}$. Moreover, if $x^{-1}y \in U$, then $y \in xU \subset U'$, $x \in yU \subset U'$ and so $|f(y) - f(x)| < \epsilon$. $\square$

We now have the object we are after, this proof is close to [2] pp. 54-55.

Lemma 4.3.3. *Let $f \in C(G)$, then $\forall \epsilon > 0$ $\exists U \subset G$ open with $U = U^{-1}$ such that $\|f * f_U - f\| \leq \epsilon$.*

Proof. Using Lemma 4.3.2, pick U such that $|f(y) - f(x)| < \epsilon$ whenever $x^{-1}y \in U$. Then,

$$|f_U(x^{-1}y)f(y) - f_U(x^{-1}y)f(x)| < \epsilon f_U(x^{-1}y) \qquad \forall x, y \in G,$$

where f_U is as in Lemma 4.3.1. Integrating over y yields

$$|f(x) - (f * f_U)(x)| < \epsilon \qquad \forall x \in G.$$

Taking the supremum gives the desired result. $\square$

In fact, we are almost at our destination, and now follow the somewhat standard treatment given in [2] pp. 55-56. For each neighbourhood U of 1 with $U = U^{-1}$ define

$$K_U : G \times G \to \mathbb{R} : (x, y) \mapsto f_U(x^{-1}y).$$

Then $K_U(gx, gy) = f_U(x^{-1}g^{-1}gy) = f_U(x^{-1}y) = K_U(x, y)$ and $K_U(x, y) = f_U(x^{-1}y) = f_U(y^{-1}x) = K_U(y, x)$. Moreover, $K_U \in C(G \times G)$ and so the operator

$$(f * f_U)(g) = \int_G f(h)K_U(h, g)\, \mathrm{d}\mu(y)$$

is a Fredholm operator with a continuous, symmetric and G-invariant kernel. This implies that it is self-adjoint and completely continuous on $L^2(G)$ (we extend by density) enabling us to apply the Hilbert-Schmidt Theorem ([21] p. 248).

Now, if $V_{\lambda,U} := V_{\lambda,U} \cap C(G)$ (again, by density) is an eigenspace for $(f * f_U)$ with eigenvalue λ, and if $f \in V_{\lambda,U}$, then $(f * f_U) = \lambda f$. In particular, since K_U is G invariant, then for g', $g \in G$

$$L_g(f * f_U)(g') = \int_G f(h)K_U(h, gg')\, \mathrm{d}\mu(h) = \int_G f(gh)K_U(h, g')\, \mathrm{d}\mu(h) = ((L_g f) * f_U)(g').$$

So, $V_{\lambda,U}$ is L_g invariant and, since $\dim V_{\lambda,U} < \infty$ (Hilbert-Schmidt gives us this), then by Lemma 4.2.1 we have proved that any $f \in C(G)$ can be uniformly approximated by elements of $\mathbb{M}(G)$. That is, we have the

Theorem 4.3.1 (Peter-Weyl).

$$\boxed{\overline{\mathbb{M}(G)} = C(G)}$$

4.4 Some Implications

The first breakthrough is contained in the

Corollary 4.4.1. *If V is an irreducible representation of G, then* $\dim V < \infty$.

Proof. Note that the characters of the finite dimensional representations are dense in the space of L^2 class functions on G. So, if W is not finite dimensional and is irreducible, then

$$\int_G \chi_V(g)\overline{\chi_W(g)}\,\mathrm{d}\mu(g) = 0,$$

for all finite dimensional G-modules, V. By density, $\chi_W = 0$. □

The second breakthrough comes by noticing that G satisfies the descending chain condition. Namely, if $U_1 \supset U_2 \supset \cdots$ is a sequence of closed subgroups of G, then it must eventually be constant, since each proper inclusion decreases the dimension by 1. Also, Theorem 4.3.1 implies that $\mathbb{M}(G)$ is a dense subalgebra of $C(G)$, so it *must* separate points of G. Hence, for each $g \in G$ $\exists f \in \mathbb{M}(G)$ such that $f(g) \neq f(1)$. It follows that since $\mathbb{M}(G)$ is the linear span and product of matrix entries of representations, then there is some finite-dimensional representation π with f as an entry. We have now almost proven the

Corollary 4.4.2. *G admits a faithful, finite dimensional representation. In other words, G is isomorphic to a closed subgroup of $U(n)$ for some n.*

Proof. Consider the representation $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$, if $\ker \mathrm{Ad} = \{1\}$ then we are done. Otherwise, set $U_0 := \ker \mathrm{Ad}$ and pick $g_0 \in U_0 \setminus \{1\}$. By the above remarks there exists a representation π_1 such that $g_0 \notin \ker \pi_1$. We can repeat this process until we arrive at a representation with trivial kernel, and the process must stop after a finite number of steps because of the descending chain condition. □

It would now seem worthwhile to use $U(n)$ as an example of the general compact Lie group. There are, of course, a number of more examples, but most lack the easy description and generality that $U(n)$ possesses.[8]

Now, following [34] p. 291, we remark that since $N : \mathfrak{u}(n) \times \mathfrak{u}(n) \to \mathbb{R} : (X, Y) \mapsto \mathrm{tr}(XY^*)$ is an inner product such that if $g \in U(n)$ and X, $Y \in \mathfrak{u}(n)$, then

$$N(gX, gY) = \mathrm{tr}(gXY^*g^*) = \mathrm{tr}(gg^*XY^*) = \mathrm{tr}(XY^*) = N(X, Y).$$

Similarly, we obtain $N(Xg^{-1}, Yg^{-1}) = N(X, Y)$. So, N can be made into a left and right invariant Riemmanian metric on $U(n)$. In particular, this holds for any closed subgroup of $U(n)$ and hence *any* compact Lie group. It follows now that $U(n)$ (and so any closed subgroup) can be given the structure of a compact metric space with the metric given by infimum's of the lengths of the curves connecting points. In fact, we are now in a position to prove:

Theorem 4.4.1. *Let G be a compact, connected Lie group. If $g \in G$, then g belongs to a one-parameter subgroup of G.*

Proof. Let γ be a geodesic that connects g and 1, for the existence of such a curve see [12] p. 56. Since γ is a geodesic, there is a curve $g_1 \subset \gamma$ such that $g_1 : [0, 1] \to G$, $g_1(0) = 1$

[8]See [12] ch. X for a development of the classification theorem, and [6] ch. 6 for some computations. The classical groups and their invariants are comprehensively computed in the classic [33].

and g_1 is the unique geodesic between 1 and $g_1(1) = g_1$.

Now, fix $t_0 \in (0, 1)$, then $g_1' : [0, t_0/2] \to G : t \mapsto g_1(t)$ is the unique geodesic connecting 1 and $g_1(t_0/2)$ (by appealing to the principle of optimality). Moreover, $g_1'' : [0, t_0/2] \to G : t \mapsto g_1(t_0/2)g_1(t)$ is the unique geodesic connecting $g_1(t_0/2)$ and $g_1(t_0)$. Hence, again by the principle of optimality $g_1(t_0) = g_1(t_0/2)g_1(t_0/2)$ for all $t \in [0, 1]$.

Indeed, arguing inductively, we now have that the identity $g(s+t) = g(s)g(t)$ holds for all dyadic rationals in s, $t \in [0, 1]$, which are dense in $[0, 1]$; by continuity of g_1, it follows that we now must have $g_1(s + t) = g_1(s)g_1(t)$ for all t, $s \in [0, 1]$ such that $s + t \in [0, 1]$. We now extend $g_1 : [0, 1] \to G$ to $\tilde{g} : \mathbb{R} \to G$ via

$$\tilde{g}(t) = \begin{cases} g_1\left(\frac{t}{\lceil t \rceil}\right)^{\lceil t \rceil} & \text{if } t > 0, \\ g_1\left(\frac{t}{\lfloor t \rfloor}\right)^{\lfloor t \rfloor} & \text{if } t < 0. \end{cases}$$

Note that $\tilde{g}$ clearly is a well-defined, one-parameter group of G (see [2] p. 10 for the computation) and so it suffices to show that $g \in \tilde{g}(\mathbb{R})$.

Now, choose T so large that the length of the geodesic $\tilde{g}([0, T])$ is at least as long as the length of γ. Let $\gamma^* = \tilde{g}([0, T]) \cap \gamma$, then γ^* is a closed set and so consequently the segment of $\tilde{g}$ belonging to γ^* containing 1 arises as a closed interval $[0, t_0]$.
Set $g_0 = \tilde{g}(t_0)$, if $g_0 = g$ then we are done. Otherwise, since the metric is left-invariant then $h = g_0^{-1}\gamma$ is a geodesic through 1. Let $h_0 \in G$ be sufficiently close to 1 so that by the previous arguments, h_0 belongs to some part of a one-parameter subgroup. It now follows that for τ sufficiently small, then $\tilde{g}(t) = g_0 h(\tau)$, in particular $\tilde{g}(t_0) = g_0$ and so $h(\tau) = \tilde{g}(t - t_0)$. Moreover, it is now clear that h agrees with $\tilde{g}$ up to an additive parameter,

namely t_0. However, this implies that t_0 is not a boundary point; this is a contradiction. So $\tilde{g}(t_0) = g$. $\square$

So, if $g \in G$, then $\exists X \in \mathfrak{g}$ such that $g = \gamma_X(s)$ for some $s \in \mathbb{R}$ and so $g = \gamma_{sX}(1) = \exp(sX)$ by Theorem 1.1.2. That is, we have the

Corollary 4.4.3. *If G is a compact Lie group, then* $\exp : \mathfrak{g} \to G$ *is surjective. .*

This corollary is very convenient for (theoretical) computations as we always have a nice smooth curve to differentiate along. This will definitely be used a couple of times in this thesis.

Chapter 5

Maximal Tori and the Weyl Group

Recall that on $SU(2)$ the character theory was understood by remarking that any $g \in SU(2)$ was conjugate to an element of the form $\left(\begin{smallmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{smallmatrix}\right)$. Also note that the set of these elements is naturally identified with $\mathbb{T}$.

More generally, observe that since any $u \in U(n)$ is a normal operator then it is unitarily conjugate to an element of the form

$$\begin{pmatrix} e^{i\theta_1} & & \\ & \ddots & \\ & & e^{i\theta_n} .^1 \end{pmatrix}.$$

The group of matrices above are naturally associated with an element of $\mathbb{T}^n$. In particular, since characters are class functions, it would suffice to understand how it behaves on diagonal matrices. It would seem that these compact, abelian Lie groups are deserving of some special attention.

5.1 Tori

We now consider in detail a priviliged class of compact, abelian Lie groups called tori. Consider $\mathbb{T} \subset \mathbb{C}$, defined as

$$\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}.$$

It is obvious that $\mathbb{T}$ is a compact Lie group.

Definition 5.1.1 (Torus). A **torus** is a compact Lie group that is isomorphic to $\mathbb{T}^n$ for some $n \in \mathbb{Z}^+$.

The following proposition appears in [6] p. 158.

Proposition 5.1.1. $\mathrm{Aut}\,(\mathbb{T}^n) \cong GL(\mathbb{Z}^n)$.

Proof. Consider $\mathbb{T}^n \cong (\mathbb{R}/\mathbb{Z})^n$, then for $\phi \in \mathrm{Aut}\,(\mathbb{T}^n)$ we have $\phi(1) = 1$ and so $\phi(\mathbb{Z}^n) = \mathbb{Z}^n$. Hence, ϕ induces an automorphism of $\mathbb{Z}^n$; the correspondence is clearly a homomorphism. To show the map is onto, we just observe that if $\phi \in GL(\mathbb{Z}^n)$, then $\tilde{\phi}(\boldsymbol{e}^{i\boldsymbol{\theta}}) := \boldsymbol{e}^{i\phi(\boldsymbol{\theta})}$ is clearly an automorphism of $\mathbb{T}^n$. $\square$

Unless stated otherwise: G is a compact, connected Lie Group.

When diagonalising a matrix, or at least attempting to, we are really trying to obtain the simplest description of its action. Subsequently, the following definition is the appropriate generalisation.[2]

Definition 5.1.2 (Maximal Torus). Let T be a torus of G. We say that T is a **maximal torus** if $T' \subsetneq T$ and T' is a torus, then $\dim T' < \dim T$.

Now, since G is compact and connected ,then it contains one-parameter groups, so follows that G contains a torus. In fact, since $\dim G < \infty$ and since each torus $\mathbb{T}^k$ is compact, then $\dim \mathbb{T}^k < \dim \mathbb{T}^{k+1}$, and so any compact Lie group G contains a maximal torus.

Unless stated otherwise: T denotes a fixed maximal torus in G.

Now, if T' is another diagonal subgroup in $U(n)$, then the fact that it is conjugate to T really relies on some (finite-dimensional) functional analysis. It turns out that there is a Lie theoretic proof, which is one of the principal aims of this chapter. We cannot prove it yet, but we will return after we have discussed maximality.

[2]The non-connected case is very hard, for an introduction see [6] pp. 176-182 for an introduction.

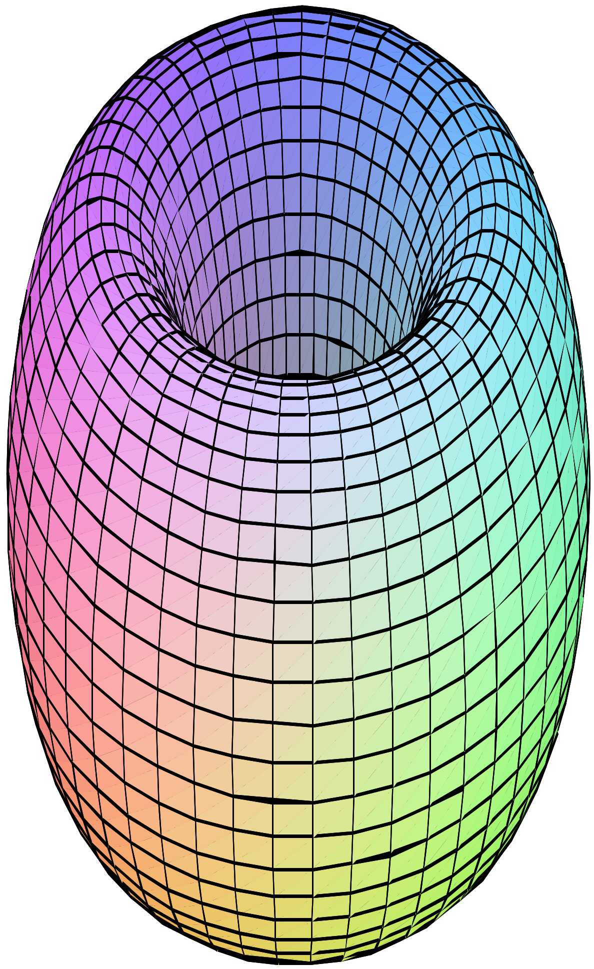

Figure 5.1: $\mathbb{T}^2$ embedded in $\mathbb{R}^3$.

5.2 Maximality

It is often important to know in mathematics, how "big" or "small" some structure is, relative to some measure. So, we now look at how big the maximal tori are, relative to other abelian Lie groups.

Definition 5.2.1 (Maximal Abelian). $H \leq G$ is **maximal abelian** if A is abelian and $H \subsetneq A \subset T$, then $A = G$.

We now have the obvious

Lemma 5.2.1. *If $\mathfrak{g}$ is an abelian Lie algebra over $\Bbbk$, then $\mathfrak{g} \cong \Bbbk^n$, where $n = \dim_{\Bbbk} \mathfrak{g}$.*

Theorem 5.2.1 (Classification of Compact Abelian Lie groups). *If H is a compact abelian Lie group, then $H \cong \mathbb{T}^n \times C$ for some n and some abelian group C. In particular, the maximal torus of a torus is itself.*

Proof. Consider the Lie algebra of the identity component H_1 of H, this is abelian since H is and so it is isomorphic to $\mathbb{R}^n$. Let $\{x_1, \ldots, x_n\}$ be a basis for this Lie algebra, then by the surjectivity of exp onto H_1 and the commutivity of the Lie algebra we must have that if $g \in H_1$, then $g = \exp(t_1 x_1) \cdots \exp(t_n x_n)$ for some $t_1, \ldots, t_n \in \mathbb{R}$. By compactness of H_1, $\ker \exp \cong \mathbb{Z}^n$ and so $H_1 \cong \mathbb{R}^n/\mathbb{Z}^n \cong (\mathbb{R}/\mathbb{Z})^n \cong \mathbb{T}^n$.
In particular, H/H_1 is a finite (since it is compact and discrete) abelian group and $H \cong H_1 \times H/H_1$ (everything commutes). □

The classification theorem provides the result we desire.

Corollary 5.2.1. *A maximal torus is maximal abelian.*

Proof. Let T be a maximal torus and A abelian such that $T \subset A \subset G$. If G is abelian, then $A = T = G$ (Theorem 5.2.1) and so we are done. If G is not abelian, then $G \neq A$, and with A a compact abelian Lie group it must be a torus too (by the Lemma), but this contradicts the maximality of the dimension of T. □

There is a counterexample worth mentioning for this theory.

Example 5.2.1. A maximal abelian subgroup is not necessarily a torus. Consider $\{I, -I\} \leq SO(2)$, this is maximal abelian but is definitely not a torus.

From these deliberations, we are now given the following

Corollary 5.2.2. *If $ntn^{-1} = t$ for all $t \in T$, then $n \in T$.*

Proof. If G is abelian, then the result is clear from Theorem 5.2.1 . Otherwise, the closure of the subgroup generated by n and T is abelian and so must be T by Corollary 5.2.1. □

5.3 The Conjugation Theorem

We now have most of the mathematical machinery required to prove that all maximal tori are conjugate. In fact, it suffices to remark briefly on the concept of a **topological generator**. A topological generator in a Lie group is an element $t \in G$ such that $\{t^n\}_{n \in \mathbb{N}}$ is dense in G.

Example 5.3.1. $e^{i\sqrt{2}}$ is a topological generator of $\mathbb{T}$. Similarly $(e^{i\sqrt{2}}, e^{i\sqrt{3}}, \ldots, e^{i\sqrt{p_n}})$, where p_n is the nth prime, is a topological generator for $\mathbb{T}^n$.

So, for a torus we have that t is a topological generator if the arguments of the components of t are algebraically independent over $\mathbb{Q}$. We conclude that the set of topological generators in $\mathbb{T}^n$ form a dense subset. The next theorem appears in [34] p. 293.

Theorem 5.3.1. *Let $g \in G$, then g belongs to a conjugate of T.*

Proof. By Corollary 4.4.3 we now know $\exists X \in \mathfrak{g}$ such that $g = \exp(X)$. Set $\phi(h) = \langle X, \mathrm{Ad}(h)Y \rangle_{\mathfrak{g}}$, where $Y \in \mathfrak{g}$ is arbitrary; and note that ϕ is a smooth map from G to $\mathbb{R}$. Pick $v \in \mathfrak{g}$, then $\psi(s) = h\exp(sv)$ is a smooth curve in a neighbourhood of h, with tangent vector v at h. Hence,

$$(\phi)_\star(h)v = \frac{\mathrm{d}}{\mathrm{d}s}\left[\langle X, \mathrm{Ad}(\exp(sv))Y\rangle_{\mathfrak{g}}\right]_{s=0} = \left\langle \mathrm{Ad}(h^{-1})X, \mathrm{ad}(v)Y \right\rangle_{\mathfrak{g}} = -\left\langle \mathrm{ad}(\mathrm{Ad}(h^{-1})X)Y, v \right\rangle_{\mathfrak{g}}.$$

But ϕ is continuous and G is compact, so ϕ has a maximum, $g_0 \in G$. Moreover, since ϕ is differentiable, we also require that $(\phi)_\star(g_0) = 0$. In particular, $\langle \mathrm{ad}(\mathrm{Ad}(h^{-1})X)Y, v\rangle_{\mathfrak{g}} = 0$ $\forall v \in \mathfrak{g}$ implies that $\mathrm{ad}(\mathrm{Ad}(h^{-1}(X))Y \equiv 0$.

Observing that Y is still free, we can choose it so that $y = \exp Y$ is a topological generator for T. In fact, $\mathrm{ad}(X)Y = 0$ implies $\mathrm{Ad}(yh^{-1})X = X$ and so $\mathrm{Ad}(y^k h^{-1})X = X$ for all $k \in \mathbb{N}$. By density and continuity we now have $\mathrm{Ad}(th^{-1})X = X$ for all $t \in T$. Taking exponentials we find that $h^{-1}gh$ commutes with every element of T. By Corollary 5.2.2, g belongs to a conjugate of T. □

Corollary 5.3.1. *All maximal tori are conjugate. In particular, all maximal tori have the same dimension.*

Proof. Let T_1, T_2 be two maximal tori and $u \in T_1$ a generator. Then $\exists g \in G$ such that $u \in gT_2g^{-1}$. That is, $u_1 = g^{-1}ug \in T_2$, which is clearly a generator for T_2. Hence $T_1 = gT_2g^{-1}$ as required. □

The following definition is now justified.

Definition 5.3.1 (Rank). The **rank** of G is the dimension of T.

Corollary 5.3.2. *If $g \in G$ then g belongs to a maximal torus.*

So, we have just proven that the general case is not that different from $U(n)$; we have a sensible notion of being able to "diagonalise" an element of a compact Lie group G. Its eigenvalues corresponding to its coordinates in a maximal torus that it belongs to.

Remark 5.3.1. Further more, the abstraction of maximal tori and orbits can be used to prove some interesting results relating to the eigenvalue decomposition of matrices, see for example [8].

We are now given a characterisation of the center of G, in terms of the maximal tori.

Corollary 5.3.3.

$$Z(G) = \bigcap_{g \in G} gTg^{-1}.$$

We will see a characterisation of the center of G that is over a *finite* intersection in Chapter 6. We have one final corollary from the conjugation theorem (the proof is related to [2] p. 93).

Corollary 5.3.4. *Suppose g, $h \in G$ commute, then there is a maximal torus T, containing g and h.*

Proof. Consider a one parameter group H containing h and let $X := \overline{\langle g, H \rangle}$, then this is a compact abelian Lie group. Noting that gX_1 generates X/X_1 implies that $X/X_1 \cong \mathbb{Z}/m\mathbb{Z}$ for some m and we make the observation that X/X_1 can be realised as the m-th roots of unity in some torus T'. Since X_1 is connected then it is torus too (c.f. Theorem 5.2.1). Hence, $X \cong X_1 \times X/X_1 \leq X_1 \times \mathbb{T}$ and so X is contained in a torus, thus in a maximal torus of G. $\square$

5.4 Representations of Tori

By the previous section, to understand the characters of a compact Lie group G, then it suffices to understand their restriction to some maximal torus T. Since $\mathfrak{g}$ is really the only representation canonically associated to G, then it would be worthwhile to understand it better. Note that $\mathfrak{g}$ is a real vector space, so we now desire to understand the real representations of T, as $\mathfrak{g}$ becomes a T-module by restriction of Ad.

Definition 5.4.1 (Integer and Weight Lattices). Let $\Lambda = \ker \exp|_{\mathfrak{t}}$, we refer to Λ as the **integer lattice** since it is naturally isomorphic to $\mathbb{Z}^n$ where $n = \dim T$.
Let $\widehat{\Lambda} = \{\alpha \in \mathfrak{t}^* : \alpha(\Lambda) \subset 2\pi\mathbb{Z}\}$. We call $\widehat{\Lambda}$ the **weight lattice**.

Notation 5.4.1. For each $\alpha \in \widehat{\Lambda} \setminus \{0\}$ let $T_\alpha^{\mathbb{R}}$ be the two dimensional real T-module with the action of T on $T_\alpha^{\mathbb{R}}$ given by $\exp(t) \mapsto \left(\begin{smallmatrix} \cos\alpha(t) & \sin\alpha(t) \\ -\sin\alpha(t) & \cos\alpha(t) \end{smallmatrix}\right)$. Also, let $T_\alpha^{\mathbb{C}}$ be the one dimensional complex T-module with the action given by $t \to t^\alpha := \{\boldsymbol{e}^{i\boldsymbol{\theta}} \mapsto \boldsymbol{e}^{i\alpha(\boldsymbol{\theta})}\}$. Let $T_0^{\Bbbk}$ be the trivial T-module over $\Bbbk = \mathbb{R}$, or $\mathbb{C}$.

Theorem 5.4.1. *The irreducible T-modules over $\mathbb{R}$ are the $T_\alpha^{\mathbb{R}}$ and over $\mathbb{C}$ are the $T_\alpha^{\mathbb{C}}$.*

Proof. This is merely linear algebra, see [18] for example. $\square$

We now compute some instructive examples.

Example 5.4.1 ($SU(2)$). Consider the following basis for $\mathfrak{su}(2)$.

$$h = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad v = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad w = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

Then,

$$\begin{aligned} \mathrm{Ad}\left(\begin{smallmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{smallmatrix}\right) v &= \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix} \\ &= \begin{pmatrix} 0 & e^{2i\theta} \\ -e^{-2i\theta} & 0 \end{pmatrix} \\ \therefore \quad \mathrm{Ad}\left(\begin{smallmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{smallmatrix}\right) v &= \cos 2\theta v + \sin 2\theta w. \end{aligned}$$

Similarly,

$$\begin{aligned} \mathrm{Ad}\left(\begin{smallmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{smallmatrix}\right) w &= -\sin 2\theta v + \cos 2\theta w \\ \mathrm{Ad}\left(\begin{smallmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{smallmatrix}\right) h &= h. \end{aligned}$$

So, let $\alpha \in \mathbb{R}^*$ be such that $\alpha(x) = 2x$, then $\mathfrak{su}(2) = T_0 \oplus T_\alpha^{\mathbb{R}}$.

Notation 5.4.2. Let E_{rs} be the matrix with 1 in the (r, s)-th position and zeroes elsewhere.

Example 5.4.2 ($U(n)$). For $1 \leq r < s \leq n$ let

$$v_{rs} = E_{rs} - E_{rs};$$
$$w_{rs} = iE_{rs} + iE_{sr}; \text{ and}$$
$$h_r = iE_{rr}.$$

Then $\{h_r\}_{r=1}^n \cup \{v_{rs}, w_{rs}\}_{1 \leq r < s \leq n}$ is a basis for $\mathfrak{u}(n)$. Let $\alpha_{rs} \in (\mathbb{R}^n)^*$ be defined as

$$\alpha_{rs}(\boldsymbol{\theta}) = \theta_r - \theta_s.$$

A similar computation to the previous example demonstrates that in this basis

$$\mathrm{Ad}\begin{pmatrix} e^{\imath\theta_1} & & \\ & \ddots & \\ & & e^{i\theta_n} \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}}_{n} \oplus \bigoplus_{1 \leq r < s \leq n} \begin{pmatrix} \cos\alpha_{rs}(\boldsymbol{\theta}) & \sin\alpha_{rs}(\boldsymbol{\theta}) \\ -\sin\alpha_{rs}(\boldsymbol{\theta}) & \cos\alpha_{rs}(\boldsymbol{\theta}) \end{pmatrix}.$$

Hence,

$$\mathfrak{u}(n) = nT_0^{\mathbb{R}} \oplus \bigoplus_{1 \leq r < s \leq n} T_{\alpha_{rs}}^{\mathbb{R}}.$$

Corollary 5.4.1. *Let $\mathfrak{t}_\alpha^{\Bbbk}$ denote the induced $T_\alpha^{\Bbbk}$-module. Then, the irreducible $\mathfrak{t}$-modules over $\mathbb{R}$ are the $\mathfrak{t}_\alpha^{\mathbb{R}}$ and over $\mathbb{C}$ are the $\mathfrak{t}_\alpha^{\mathbb{C}}$ where $\alpha \in \hat{\Lambda}$.*

Explicitly, the two dimensional real $\mathfrak{t}$-module has an action given by

$$\mathfrak{t} \ni t \mapsto \begin{pmatrix} 0 & \alpha(t) \\ -\alpha(t) & 0 \end{pmatrix}$$

Corollary 5.4.2. *The induced $\mathfrak{t}$-module $\mathfrak{t}_\alpha^{\mathbb{R}}$ has $\mathrm{ad}|_{\mathfrak{t}}(\mathrm{ad}|_{\mathfrak{t}}(t)v) = -\alpha(t)^2 v \; \forall v$ belonging to a module equivlent to $\mathfrak{t}_\alpha^{\mathbb{R}}$.*

Remark 5.4.1. Note that $T_\alpha^{\mathbb{R}} \cong T_{-\alpha}^{\mathbb{R}}$ as T-modules (they are taken to one another by conjugation with $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$). In particular, when indexing the real representations of T it will be customary to choose a sign of the weight. It is clear that up to isomorphism, the choice of sign is immaterial; similarly, this is so when discussing representations of $\mathfrak{t}$.

5.5 The Weyl Group

In the previous sections we studied how maximal tori are related to each other and their representation theory. It is would now seem natural to consider how they are related to themselves. The processs of diagonalising some unitary matrix is unique up to permutation of eigenvalues. This is the abstraction that is developed now. The appropriate notion is that permuting eigenvalues preserves the conjugacy class.

We are moved to consider the **normalizer** of T in G. That is, the set

$$N(T) = \{g \in G \, : \, gTg^{-1} = T\}.$$

Note that clearly $N(T)$ is a closed subgroup of G, hence it is a Lie subgroup of G. Moreover, T is a normal subgroup of $N(T)$ so it is natural to consider the quotient.

Definition 5.5.1 (Weyl Group). The **Weyl Group** of G relative to T is the group $W(G,T) = N(T)/T$.

Example 5.5.1 ($SU(2)$). The Weyl group of $SU(2)$ is $\mathbb{Z}_2$, which acts by changing the sign of the eigenvalues.

Example 5.5.2 ($U(n)$). The Weyl group of $U(n)$ is simply $\mathfrak{S}_n$, the symmetric group on n-letters.

It is always important to know whether or not a structure possesses some uniqueness properties. As one would hope, we have the following

Proposition 5.5.1. *If $W(G,T)$, $W(G,T')$ are two Weyl groups of G, then $W(G,T) \cong W(G,T')$.*

Proof. Note that, since all maximal tori are conjugate (c.f. Theorem 5.3.1), then $T = gT'g^{-1}$ for some $g \in G$. Indeed, it follows that $N(T) = gN(T')g^{-1} \cong N(T')$ and, since $T \cong T'$, we have $N(T)/T \cong N(T')/T'$. $\square$

Since G, T are fixed we refer to $W(G,T)$ as W. A remark worth making now is contained in the

Lemma 5.5.1. *The representation $W \ni nT \to \mathrm{Ad}(n) \in GL(\mathfrak{t})$ is faithful.*

Proof. Immediate from Corollary 5.2.2. $\square$

The following result (or one of its equivalent assertions) is normally proven using the Lefshetz fixed point theorem ([2] p. 95); this is a very deep, topological result. The other, standard method proves that (in this case) $N(T)_0 = Z(T) = T$ ([6] p. 159). However, the following theorem can be proven directly from our treatment.

Theorem 5.5.1. *W is finite.*

Proof. Consider the following commutative diagram.

$$\begin{tikzcd} N(T)/T \arrow[r, "\mathrm{Ad}"] \arrow[d, "\psi"'] & \mathrm{Aut}\,(\mathfrak{t}) \\ \mathrm{Aut}\,(T) \arrow[r, "\phi"] & GL(\mathbb{Z}^{\mathrm{rank}\,G}) \arrow[u, "\imath"'] \end{tikzcd} ,$$

where $m = \mathrm{rank}\,G$, $\psi : N(T)/T \to \mathrm{Aut}\,(T) : nT \mapsto n^{-1} \cdot n$, $\imath : GL(m, \mathbb{Z}) \hookrightarrow \mathrm{Aut}\,(\mathfrak{t})$ is inclusion, and ϕ is the isomorphism in Proposition 5.1.1.

Lemma 5.5.1 implies ψ is a monomorphism and Proposition 5.1.1 informs us that ϕ is an isomorphism; putting these together demonstrates $N(T)/T$ is discrete.

Also, the Ad representation is unitary with respect to $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ and so $N(T)/T$ has compact image in $\mathrm{Aut}\,(\mathfrak{t})$. Moreover, $\mathrm{Ad} = \imath \circ \phi \circ \psi$ is injective and continuous and so $N(T)/T$ is compact. Indeed, this implies that $W = N(T)/T$ is compact and discrete, and so finite. $\square$

This theorem is far from obvious, and it is not surprising that it has some far reaching consequences. The result has essentially reduced understanding the non-uniqueness of conjugation of an element of G into a maximal torus into a very combinatorial problem. We explore this idea in great depth in the next two chapters.

Part III

The Weyl Character Formula

Chapter 6

Roots

In an earlier example we calculated that

$$\mathfrak{u}(n) \cong \mathbb{R}^n \oplus \bigoplus_{1 \leq r < s \leq n} T^{\mathbb{R}}_{\alpha_{rs}}$$

as T-modules, where $\alpha_{rs}(\boldsymbol{\theta}) = \theta_r - \theta_s$. It would seem apparent that since $\mathfrak{u}(n)$ is a real representation of $U(n)$, which is canonically associated to it, then the linear forms $\alpha_{rs} : \mathbb{R}^n \to \mathbb{R}$ would be important to the character theory.

However, there is some arbitrariness in our choice, since $T^{\mathbb{R}}_{\alpha_{rs}} \cong T^{\mathbb{R}}_{-\alpha_{rs}}$ as T-modules (c.f. Remark 5.4.1), so it would seem that the canonical object of study should be $\Phi := \{\alpha_{rs} \,:\, 1 \leq r, s \leq n\}$ for $U(n)$.

The real reason for this is that they are capturing conjugation by an element of the torus at an infinitesimal level; this concept now generalises easily.

6.1 Decomposition of Ad

Consider $\mathrm{Ad} : G \to GL(\mathfrak{g})$, this restricts to a real representation of T on $\mathfrak{g}$. By Theorem 2.2.1 and Theorem 5.4.1

$$\mathfrak{g} \cong n_0 T^{\mathbb{R}}_0 \oplus \bigoplus_{\alpha \in \widehat{\Lambda} \setminus \{0\}} n_\alpha T^{\mathbb{R}}_\alpha.$$

To tidy this decomposition up a little, we adopt the following

Notation 6.1.1. For $\alpha \in \widehat{\Lambda}$, let $\mathfrak{g}_\alpha := (n_\alpha + n_{-\alpha})T^{\mathbb{R}}_\alpha$.

Then

$$\mathfrak{g} \cong \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha,$$

where $\Phi^+ \subset \widehat{\Lambda}$ is some choice (c.f. Remark 5.4.1) of weights so that the decomposition works.

Lemma 6.1.1. $\mathfrak{g}_0 = \mathfrak{t}$.

Proof. Clearly, $\mathfrak{t} \subset \mathfrak{g}_0$, so it suffices to show the reverse inclusion. Let $X \in \mathfrak{g}_0$ and consider $T' := \overline{\langle T, \exp(X) \rangle}$, then T' is closed and abelian and so must be T by Corollary 5.2.1. This is only possible if $\exp(X) \in T$ and so $X \in \mathfrak{t}$. $\square$

Example 6.1.1 ($U(n)$). $\Phi^+ = \{\alpha_{12}, \alpha_{13}, \alpha_{23}\}$ would suffice for $n = 3$. In general, we could have $\Phi^+ = \{\alpha_{rs} : 1 \leq r < s \leq n\}$.

Example 6.1.2 ($SU(n)$). $\Phi^+ = \{\alpha_{rs} : 1 \leq r < s \leq n\}$. An important remark is that since $\mathfrak{su}(n)$ consists of trace zero skew-hermitian matrices, then these are actually linear functionals on $\mathbb{R}^n / \mathbb{R} \cdot (1, 1, \ldots, 1)$ ([2] p. 265).

Following the line of argument in the beginning of this chapter we could conclude that the following definition is fundamental.

Definition 6.1.1 (Roots). Let $\Phi := \{\alpha \in \widehat{\Lambda} \setminus \{0\} : n_\alpha \neq 0\}$, we call these the **roots** of $\mathfrak{g}$.

Remark 6.1.1. $\Phi = \Phi^+ \cup -\Phi^+$ as a disjoint union. A systematic description of these types of decomposition will be completed later.

For the moment, the remark is handy for computations. The following two examples are now immediate.

Example 6.1.3 ($U(3)$). $\Phi = \{\alpha_{12}, \alpha_{13}, \alpha_{23}, \alpha_{21}, \alpha_{31}, \alpha_{32}\}$.

Now, observe that $\mathfrak{S}_n$ acts on $\mathbb{R}^n (\cong \mathfrak{t})$ via the permution of coordinates, it follows that there is an induced action on $(\mathbb{R}^n)^*$ given by $w \cdot \alpha = \alpha \circ w^{-1}$.

Example 6.1.4 ($U(3)$). $(12) \cdot \alpha_{12} = \alpha_{21} = -\alpha_{12}$.

In general, it is natural for $W(G,T)$ to act via conjugation on T so $W(G,T)$ can act via Ad on $\mathfrak{t}$. This is summarised in the following

Lemma 6.1.2. *For $w \in W(G,T)$ write $w = nT$ where $n \in N(T)$.*

1. *W acts on T as $w \cdot t = ntn^{-1}$;*
2. *W acts on $\alpha \in \mathfrak{t}$ as $w \cdot \alpha = \mathrm{Ad}(n)\alpha$;*
3. *W acts on $\lambda \in \Lambda$ as $w \cdot \lambda = \mathrm{Ad}(n)\lambda$; and*
4. *W acts on $\alpha \in \widehat{\Lambda}$ as $w^\star \alpha = \alpha \circ \mathrm{Ad}(n^{-1})$.*

Proof. The first two are obviously well defined actions since T and $\mathfrak{t}$ are abelian. Note too that if $\lambda \in \Lambda$, then $\exp(\mathrm{Ad}(n)\lambda) = n \exp(\lambda) n^{-1} = 1$; that is, $\mathrm{Ad}(n)\lambda \in \Lambda$. This proves (3) and (4). $\square$

Notation 6.1.2. We will be using the "$\cdot$" notation for the *standard* actions of W on a number of mathematical objects; the context of the action will always be clear. In fact, these actions are really all the same since $\langle \mathrm{Ad}(w)X, Y\rangle_{\mathfrak{g}} = \langle X, \mathrm{Ad}(w^{-1})Y\rangle_{\mathfrak{g}}$ for all X, $Y \in \mathfrak{g}$ and recalling that any element in $\mathfrak{t}^*$ arises as $\langle X, \cdot\rangle_{\mathfrak{g}}$ for some $X \in \mathfrak{g}$.

From this slight digression into generality, we return to remark that if $w \in W$, then $w \cdot := \mathrm{Ad}(w) \in GL(\mathfrak{g})$. Consequently,

$$\mathfrak{g} = w^{-1} \cdot \mathfrak{g} \cong \bigoplus_{\alpha \in \widehat{\Lambda}} n_\alpha w^{-1} \cdot T_\alpha^{\mathbb{R}} = \bigoplus_{\alpha} n_\alpha T^{\mathbb{R}}_{(w^{-1})^* \alpha} = \bigoplus_{\alpha'} n_{w \cdot \alpha'} T_\alpha^{\mathbb{R}}.$$

Noting that each $T_\alpha^{\mathbb{R}}$ is irreducible over $\mathbb{R}$, then we have proven the following

Proposition 6.1.1.

$$\boxed{n_{w\cdot\alpha} = n_\alpha \quad \forall w \in W \quad \forall \alpha \in \widehat{\Lambda}}$$

An important corollary now follows immediately, once remarking that Φ must be finite.

Corollary 6.1.1. *$W(G,T)$ acts via permutation on Φ.*

The last two observations provide a deep connection between combinatorics, geometry and Lie theory. In fact, the proof of the Weyl character formula essentially amounts to exploiting these connections in a slightly more general context. For the case of $U(n)$, we almost would be adequately equipped with Lie theory to prove the Weyl character formula (see [33] pp. 194-200); for the general case we are required to understand the Weyl group in a more geometric and combinatorial fashion.

What has not been mentioned yet is the uniqueness, if any, that Φ possesses. Intuitively, it should be unique up to some change of basis as it is are capturing conjugation, the canonical inner automorphism of any group. The precise explanation for this is that if T' is another maximal torus in G, then $T' = gTg^{-1}$ by Corollary 5.3.1 . So, if Φ' is a set of roots associated to T', then for any $\alpha \in \Phi$, $\alpha' = \alpha \circ \mathrm{Ad}_g \in \Phi'$ and conversely. Hence, there is an inner automorphism of G connecting Φ and Φ'.

6.2 Lie Triples

Let us start this section with a motivating example, consider the following elements of $\mathfrak{u}(3)$.

$$H = \begin{pmatrix} i & & \\ & i & \\ & & \end{pmatrix}, \quad X = \begin{pmatrix} & 1 & \\ -1 & & \\ & & \end{pmatrix}, \quad Y = \begin{pmatrix} & i & \\ i & & \\ & & \end{pmatrix}.$$

This is just $\mathfrak{su}(2) \hookrightarrow \mathfrak{u}(3)$, with real Lie algebra generated isomorphic to $\mathfrak{su}(2)$. In general, some index gymnastics demonstrates that for $1 \le r < s \le n$,

$$[h_r - h_s, v_{rs}] = 2w_{rs}, \quad [h_r - h_s, w_{rs}] = -2v_{rs}, \quad [v_{rs}, w_{rs}] = h_r - h_s.$$

So, $\{h_r - h_s, v_{rs}, w_{rs}\}$ generates a real Lie algebra isomorphic $\mathfrak{su}(2)$. Consequently, this binds $\mathfrak{u}(n)$ to $\mathfrak{su}(2)$. Taking the exponential of these produces copies of $SU(2)$ in $U(n)$.

We now proceed locate, and describe some rank 1 Lie groups in any compact, connected Lie group G. The treatment here is non-standard, where the standard methodology is to complexify $\mathfrak{g}$ and then compute the eigenvalues (which is what the roots correspond to). Whilst Lie algebraically, this technique is not problematic, it would essentially involve the study of the theory of complex semisimple Lie algebras, a step we believe to be unnecessary.[1]

Geometrically, the group is complexified too and corresponds to a complex manifold with the representations of interest holomorphic, instead of unitary. This means that from our compact, real manifold we are pushed into the realms of algebraic geometry, with a non-compact space. One would hope that we would not need this complex mathematical machinery for understanding the character theory of compact Lie groups since we are only concerned with torus elements.[2] In fact, recovering a real Lie algebra from a complex semisimple one is tedious at best; see [11] ch. 9 for the required computations.

[1]See [26] for a good treatment of this theory

[2]The tools of algebraic geometry do prove to be *very* fruitful. One of the most eloquent models of the finite dimensional representation theory of complex, semisimple Lie groups is the Borel-Weil Theorem (see [31] ch. 3); from which we obtain a model of the representation theory of compact Lie groups ([20] pp. 292-298). When studying the actual representation spaces, it is very natural to consider the complexification as G/T is naturally endowed with the structure of a complex manifold (it actually becomes a space of flags, see [10] pp. 382-383.), which is necessary to understand when dealing with representations.

Now, since $\mathfrak{t}^*$ has an inner product, then there is a canonical isomorphism between $\mathfrak{t}$ and $\mathfrak{t}^*$ given by $\mathfrak{t} \ni \alpha \mapsto \langle \alpha, \cdot \rangle_{\mathfrak{g}} \in \mathfrak{t}^*$.

Notation 6.2.1. For an element $\alpha \in \mathfrak{t}^*$ let H_α be the unique element of $\mathfrak{t}$ such that $\alpha(\cdot) = \langle H_\alpha, \cdot \rangle_{\mathfrak{g}}$.

Example 6.2.1 ($U(n)$). $H_{\alpha_{12}} = (1, -1, 0, \ldots, 0)$.

Definition 6.2.1 (Coroot). Let $\alpha \in \Phi$, then the **coroot** of α is the unique element $H^\alpha \in \operatorname{span}(H_\alpha)$ such that $\alpha(H^\alpha) = 2$. The **coroot lattice** is the $\mathbb{Z}$-span of the set of coroots.

Example 6.2.2 ($U(n)$). $H^{\alpha_{12}} = (1, -1, 0, \ldots, 0)$.

Trivially, we are now able to induce a non-degenerate, symmetric, bilinear form on $\mathfrak{t}^*$. For λ, $\mu \in \mathfrak{t}^*$ define

$$(\lambda, \mu) = \langle H_\lambda, H_\mu \rangle_{\mathfrak{g}} .$$

Notation 6.2.2. For each $\alpha \in \Phi$ define

$$\alpha^\vee = \frac{2\alpha}{(\alpha, \alpha)}.$$

and so $H^\alpha = H_{\alpha^\vee}$.

Fix $X^\alpha \in \mathfrak{g}_\alpha$ with $\langle X^\alpha, X^\alpha \rangle_{\mathfrak{g}} = \frac{1}{\alpha(H_\alpha)}$ and let $Y^\alpha = \frac{1}{2} \operatorname{ad}(H^\alpha) X^\alpha$. Some small observations are contained in the following lemmata, which is the author's treatment of the theory.

Lemma 6.2.1. *Let* $\alpha \in \Phi$, $H \in \ker \alpha$ *then* $\operatorname{ad}(H)|_{\mathfrak{g}_\alpha} = 0$.

Proof. By remarking that if $\alpha(H) = 0$, then for each $t \in \mathbb{R}$, $\operatorname{Ad}(\exp(tH)|_{\mathfrak{g}_\alpha} = 1$, by considering the root space decomposition. Taking the derivative of both sides at $t = 0$ yields $\operatorname{ad}(H)|_{\mathfrak{g}_\alpha} = 0$. $\square$

Lemma 6.2.2.

1. $\langle X^\alpha, Y^\alpha \rangle_{\mathfrak{g}} = 0$.

2. $[X^\alpha, Y^\alpha] \in \mathrm{span}(H_\alpha)$.

3. $\alpha([X^\alpha, Y^\alpha]) = 2$, *so* $[X^\alpha, Y^\alpha] = H^\alpha$.

Proof.

1.

$$\langle X^\alpha, Y^\alpha \rangle_{\mathfrak{g}} = \frac{1}{2} \langle X^\alpha, \mathrm{ad}(H^\alpha) X^\alpha \rangle_{\mathfrak{g}} = \frac{1}{2} \langle \mathrm{ad}(X^\alpha) X^\alpha, H^\alpha \rangle_{\mathfrak{g}} = 0.$$

2. Write $\mathfrak{t} = \mathrm{span}(H_\alpha) \oplus \ker \alpha$, then for $Z = Z_1 + Z_2 \in \mathfrak{t}$ with $Z_1 \in \mathrm{span}(H_\alpha)$, $Z_2 \in \ker \alpha$ we have by the Jacobi identity and Lemma 6.2.1

$$\begin{aligned} [Z, [X^\alpha, Y^\alpha]] &= [[Z, X^\alpha], Y^\alpha] + [X^\alpha, [Z, Y^\alpha]] \\ &= [[Z_1, X^\alpha], Y^\alpha] + [X^\alpha, [Z_1, Y^\alpha]]. \end{aligned}$$

Now, write $Z_1 = \mu H^\alpha$, then

$$[[Z_1, X^\alpha], Y^\alpha] + [X^\alpha, [Z_1, Y^\alpha]] = [2\mu Y^\alpha, Y^\alpha] - [X^\alpha, 2\mu X^\alpha] = 0.$$

Since $Z \in \mathfrak{t}$ was arbitrary, from Lemma 6.1.1, we conclude that $[X^\alpha, Y^\alpha] \in \mathfrak{t}$. Now, pick $Z_2 \in \ker \alpha$, then by Lemma 6.2.1

$$\langle [X^\alpha, Y^\alpha], Z_2 \rangle_{\mathfrak{g}} = \langle Y^\alpha, [Z_2, X^\alpha] \rangle_{\mathfrak{g}} = 0.$$

Hence, $[X^\alpha, Y^\alpha] \in \mathrm{span}(H_\alpha)$.

3.

$$\begin{aligned}
\langle [X^\alpha, Y^\alpha], H_\alpha \rangle_{\mathfrak{g}} &= - \langle [H_\alpha, Y^\alpha], X^\alpha \rangle_{\mathfrak{g}} \\
&= -\frac{\alpha(H_\alpha)}{2} \langle [H^\alpha, Y^\alpha], X^\alpha \rangle_{\mathfrak{g}} \\
&= 2\alpha(H_\alpha) \langle X^\alpha, X^\alpha \rangle = 2
\end{aligned}$$

□

Notation 6.2.3. Let $\mathfrak{s}^\alpha$ be the real Lie subalgebra of $\mathfrak{g}$ generated by $\{H^\alpha, X^\alpha, Y^\alpha\}$. Also, let $\mathfrak{t}^\alpha = (\mathfrak{s}^\alpha)_0$.

We finally have the result we were seeking.

Proposition 6.2.1.

$$\boxed{\mathfrak{s}^\alpha \cong \mathfrak{su}(2)}$$

Proof. By Corollary 5.4.2, $[H^\alpha, [H^\alpha, X^\alpha]] = -\alpha^2(H^\alpha)X^\alpha$ and so we have the following commutation relations

$$\begin{aligned}
[H^\alpha, X^\alpha] &= 2Y^\alpha, \\
[Y^\alpha, H^\alpha] &= 2X^\alpha.
\end{aligned}$$

The last commutation relation is by virtue of Lemma 6.2.2. □

Now, for $U(n)$ we saw that $\dim \mathfrak{g}_{\alpha_{rs}} = 2$ and moreoever, $\dim \mathfrak{g}_{c\alpha_{rs}} > 0$ if and only if $c \in \{-1, 0, 1\}$. In particular, the Lie algebra $\mathfrak{s}^{\alpha_{rs}}$ is unique.

For the general case, it is not clear that at all that the rank 1 Lie algebra constructed above has any uniqueness properties. In the next theorem, we show that the same proof holds in a the general context. The proof is modelled on [13] pp. 66-67, but is modified for our treatment.

Theorem 6.2.1. $\dim \mathfrak{g}_\alpha = 2$ *for all* $\alpha \in \Phi$. *Moreover,* $c\alpha \in \Phi$ *if and only if* $c \in \{-1, 0, 1\}$.

Proof. Now, let $\mathfrak{g}'_\alpha = \mathfrak{t}^\alpha \oplus \bigoplus_{c \in \mathbb{R}^\times} \mathfrak{g}_{c\alpha}$, then clearly $\mathfrak{g}'_\alpha$ is a $\mathfrak{s}^\alpha$-module (via $\mathrm{ad}|_{\mathfrak{s}^\alpha}$). So, suppose that $c\alpha \in \Phi$, then it is easy to see that $H^{c\alpha} = \frac{1}{c} H^\alpha$. Hence,

$$\mathrm{ad}(H^\alpha) X^{c\alpha} = (2c) Y^{c\alpha}; \text{ and}$$

$$\mathrm{ad}(H^\alpha) Y^{c\alpha} = -(2c) X^{c\alpha}.$$

So, since $\mathfrak{s}^\alpha \cong \mathfrak{su}(2)$, then it necessarily follows that $2c \in \mathbb{Z}$. Note too that $\mathfrak{g}'_\alpha / \mathfrak{s}^\alpha$ is an $\mathfrak{s}^\alpha$-module complementary to $\mathfrak{s}^\alpha$ in $\mathfrak{g}'_\alpha$. However, by remarking that 0 is not a weight of $\mathfrak{g}'_\alpha / \mathfrak{s}^\alpha$ and the actions of X^α, Y^α raise and lower weights in multiples of 2, then the weights of $\mathfrak{g}'_\alpha$ must be odd.

Indeed, 4 is not a weight of $\mathfrak{g}'_\alpha / \mathfrak{s}^\alpha$ and so by the previous remarks it follows that $2\alpha \notin \Phi$. It is now clear that $\alpha/2 \notin \Phi$ since otherwise we could deduce that $\alpha = 2(1/2\alpha) \notin \Phi$. Hence, 1 is not a weight of $\mathfrak{g}'_\alpha / \mathfrak{s}_\alpha$. By reminding ourselves again that X^α, Y^α raise and lower weights by multiples of 2, it is now immediate that $\mathfrak{g}'_\alpha / \mathfrak{s}^\alpha = 0$. The results are now clear, since then $\mathfrak{s}^\alpha = \mathfrak{t}^\alpha \oplus \mathfrak{g}_\alpha \cong \mathfrak{t}^\alpha \oplus \mathfrak{g}_{-\alpha}$. □

Corollary 6.2.1.

$$\dim G = \operatorname{rank} G + |\Phi|.$$

So, we have the uniqueness we desired.

Notation 6.2.4. Let $S^\alpha = \overline{\langle \exp(\mathfrak{s}^\alpha) \rangle} \leq G$.

On the group level we have the

Proposition 6.2.2.

$$S^{\alpha} \cong SU(2) \text{ or } SO(3).$$

In particular, the Weyl groups for both $\mathbb{Z}_2$.

Proof. Omitted. The standard proof of this is topological, see [2] pp. 107-108 for a good exposition. ☐

Again, we must remark how amazing this result actually is–starting from an arbitrary compact Lie group we have managed to locate some canonical rank 1 groups contained in it.

6.3 Regularity

For an element of $U(n)$ with distinct eigenvalues, $\mathfrak{S}_n$ acts particularly well on it. In particular, the action is proper and so the conjugacy class containing such an element intersects the maximal torus $|\mathfrak{S}_n|$ times. In fact, it is clear that the set of elements with a repeated eigenvalue form a manifold of lower dimension than $U(n)$. So since the character theory revolves around understanding conjugation properly, and since they are also continuous, it would suffice to understand conjugation on elements with distinct eigenvalues.

More generally, we regard G as a homogenous space by its action on itself as conjugation, we then decompose G into a disjoint union of orbits. For $U(n)$, observe that an element has a repeated eigenvalue if and only if it belongs to more than one maximal tori. This is the concept that we generalise.

Definition 6.3.1 (Regular Point). Let $g \in G$, then say g is **regular** if it belongs to exactly one maximal torus, and say g is **singular** otherwise. Let G_{reg} denote the set of all regular points.

Remark 6.3.1. This definition is independent of T. For convenience, we now set $T_{\text{reg}} := T \cap G_{\text{reg}}$.

Regular points are related to conjugation so it would seem natural that they would be connected to the roots. Let us take this idea a little further.

Now, if $t \in T \subset U(n)$ has eigenvalues $\{t_1, \ldots, t_n\}$ (as a multiset) and $t_1 = t_2$ then the action of t on $\mathfrak{g}_{\alpha_{12}}$ is trivial. It would seem that it would be worthwhile understanding this phenomenon more generally. So, for each $\alpha \in \Phi$ set

$$U^\alpha = \ker\left((\operatorname{Ad}|_T)|_{\mathfrak{g}_\alpha}\right) \leq G,$$

which is precisely the elements of T that act trivially on $\mathfrak{g}_\alpha$ via $\operatorname{Ad}|_T$, so the union of all such groups is the set of singular elements of T. Consequently, we have the following lemma from [6] p. 189.

Lemma 6.3.1.

$$T \setminus T_{\text{reg}} = \bigcup_{\alpha \in \Phi} U^\alpha$$

By Schur's lemma, $Z(U(n))$ is precisely the set of elements of the form λI for some $\lambda \in \mathbb{T}$. That is, $Z(U(n))$ consists of all elements with all eigenvalues equal. The Lie theoretic context is contained in the

Proposition 6.3.1.

$$Z(G) = \bigcap_{\alpha \in \Phi} U^\alpha.$$

Proof. The right hand side is the set of elements that conjugate trivially on $\mathfrak{g}$. Using the surjectivity of $\exp : \mathfrak{g} \to G$ (Corollary 4.4.3), we see that this is the set of elements that conjugate trivially on G; i.e. $Z(G)$. $\square$

In general, it is fairly obvious that the set of singular elements forms a manifold of at least codimension 1 in G. However, it is not obvious that the following holds.

Proposition 6.3.2.

$$\dim(G \setminus G_{\text{reg}}) \leq \dim G - 3.$$

Proof. Observe that if $t \in T \setminus T_{\text{reg}}$, then by Lemma 6.3.1 t belongs to U^α for some $\alpha \in \Phi$. In fact, it follows now that $\dim Z(t) \geq \operatorname{rank} G + \dim \mathfrak{g}_\alpha = \operatorname{rank} G + 2$.

Observe that since U^α is of codimension 1 in T, then U^α posseses a topological generator ([6] p. 190). For the remainder of the proof, we follow [2] pp. 99-100. Let $t \in U^\alpha$ be such an element and consider $\psi_\alpha : G/Z(t) \times U^\alpha \to G \setminus G_{\text{reg}} : (gZ(t), u) \mapsto gug^{-1}$. This is well-defined since $Z(t)$ centres everything in U^α, it is smooth too.

Thus, since $G \setminus G_{\text{reg}} = \bigcup_{\alpha \in \Phi} \operatorname{im} \psi_\alpha$, then by the finiteness of the union, $\dim G \setminus G_{\text{reg}} = \dim \psi_\alpha \leq \dim G/Z(t) + \dim U^\alpha = \dim G - \dim Z(t) + \text{rank } G - 1 = \dim G - 3$. $\square$

Let us relate these concepts to the Lie algebra. Given that it is a vector space, one would hope that the computations would be less cumbersome.

Definition 6.3.2 (Regular Points-Lie Algebra). Say $g \in \mathfrak{g}$ is **regular** if $\exp(g) \in G$ is regular, and call g **singular** otherwise. Denote the set of regular points in $\mathfrak{g}$ by $\mathfrak{g}_{\text{reg}}$.

The true reason why this definition is natural is that $t \in \mathfrak{t}_{\text{reg}} = \mathfrak{t} \cap \mathfrak{g}_{\text{reg}}$ if and only if t lies in the Lie algebra of only one maximal torus.

We $\alpha \in \Phi$ set

$$\mathfrak{u}^\alpha = T_1(U^\alpha),$$

We call $\mathfrak{u}^\alpha$ the **root hyperplane** of α. We now have the corresponding results:

Corollary 6.3.1.

$$\mathfrak{t} \setminus \mathfrak{t}_{\text{reg}} = \bigcup_{\alpha \in \Phi} \mathfrak{u}^{\alpha}.$$

Corollary 6.3.2.

$$Z(\mathfrak{g}) = \bigcap_{\alpha \in \Phi} \mathfrak{u}^{\alpha}.$$

By considering duality, the following is result is clear.

Corollary 6.3.3.

$$\operatorname{span} \Phi = (\mathfrak{t}/Z(\mathfrak{g}))^*.$$

Remark 6.3.2. It is because of this last corollary that we generally study semisimple Lie groups. One of the (many) miracles of Lie theory is that the simplest part of the group (i.e. its center) tends to complicate the representation theory substantially. For a comprehensive treatment of the theory of real reductive Lie groups, consult Vogan's tome [28].

Moreover, for each $\alpha \in \Phi$, since $\mathfrak{u}^{\alpha}$ is a hyperplane in $\mathfrak{t}$ (of codimension 1), then

$$\bigcup_{\alpha \in \Phi} \mathfrak{u}^{\alpha}$$

partitions $\mathfrak{t}$ into a finite number of connected, convex regions, called **Weyl chambers**. Given a Weyl chamber C, it is often convenient to refer to $\overline{C} \cap \mathfrak{u}^{\alpha}$ as the **walls** of C.

Example 6.3.1 ($U(n)$). The standard Weyl chamber for $U(n)$ would be

$$C = \{\boldsymbol{\theta} \in \mathfrak{t} : \theta_1 > \theta_2 > \cdots > \theta_n > 0\}.$$

Note that the closure of this chamber is clearly

$$\overline{C} = \{\boldsymbol{\theta} \in \mathfrak{t} : \theta_1 \geq \theta_2 \geq \cdots \geq \theta_n \geq 0\}.$$

So that all of the walls of this chamber are now of the form

$$\overline{C} \cap \mathfrak{u}^{\alpha_{r,r+1}} = \{\boldsymbol{\theta} \in \mathfrak{t} \,:\, \theta_1 \geq \cdots \geq \theta_r = \theta_{r+1} \geq \cdots \geq \theta_n \geq 0\}.$$

$$\boxed{\textbf{Unless stated otherwise: } C \textbf{ is a fixed Weyl Chamber.}}$$

The Weyl chambers will be of central concern in the next chapter, where their relationships to each other is described completely. Observe that while the actual Weyl chambers are independent of T, their configuration is not. However, since there is an automorphism of $\mathfrak{t}$ relating two different sets of roots, then there is an automorphism of $\mathfrak{t}$ relating two Weyl chambers. The remarkable result is that this automorphism is actually an element of the Weyl group.

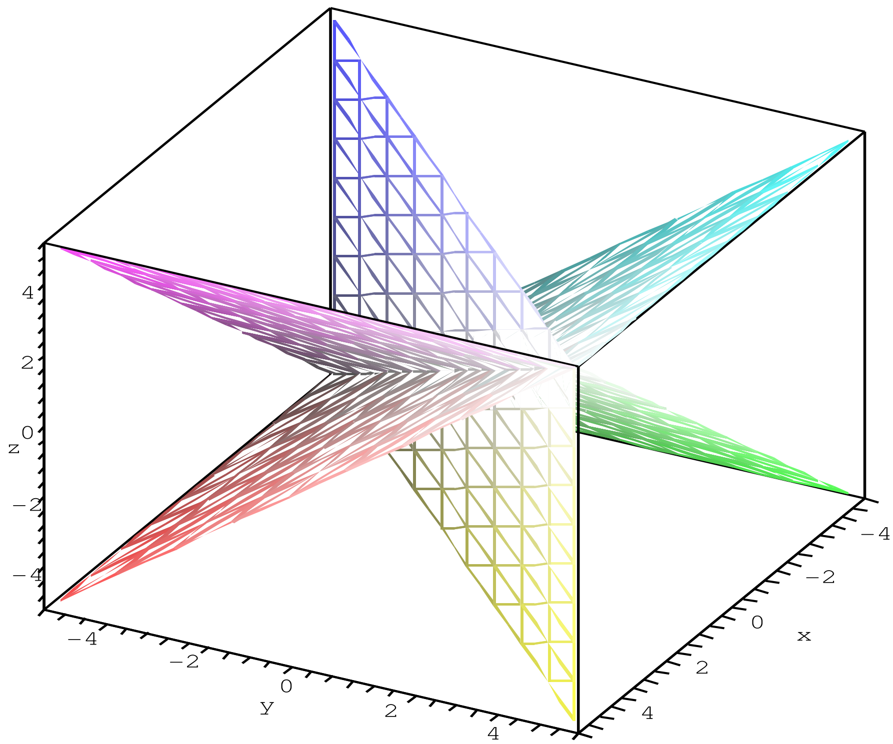

Figure 6.1: The Stiefel Diagram for $U(3)$.

Chapter 7

The Stiefel Diagram

The Weyl chambers discussed at the end of the previous chapter arose as convex regions in $\mathfrak{t}$ whose walls were the root hyperplanes. Let us connect these with the Weyl group; for the moment we work in $\mathfrak{t}^*$. Consider $(12) \in \mathfrak{S}_3$ then

$$(12)^*\alpha_{12} = \alpha_{21} = -\alpha_{12};$$
$$(12)^*\alpha_{23} = \alpha_{13}; \text{ and}$$
$$(12)^*\alpha_{13} = \alpha_{23}.$$

So, the transposition (12) is a reflection in $\ker \alpha_{12}$ and leaves the remaining set of positive roots invariant. We also know that transpositions generate $\mathfrak{S}_3$. The geometric realisation of this is that the Weyl group's action can be generated from reflections. This argument can clearly be extended to cover $U(n)$.

The principal aim of this chapter is prove that the Weyl group of an arbitrary compact Lie group can be generated by reflections in the root hyperplanes. A priori, it is far from obvious that a general Weyl group would be generated in this way.

7.1 Reflections

The example with $U(3)$ showed that when studying the Weyl group it could be realised as a finite group of isometries of a Euclidean space.[1] The group was generated by reflections in the root hyperplanes, precisely the Lie algebraic shadow of where conjugation is a degenerate action in the group. It would now seem apparent that the object of study should be the set of singular points, and the symmetries that it admits.[2] We now consider the general case.

Definition 7.1.1 (Stiefel Diagram). The **Stiefel Diagram** is the set of singular points in $\mathfrak{t}$.

Since $\mathfrak{t}$ and $\mathfrak{t}^*$ are canonically isomorphic via a W-invariant inner product then we will refer to the Stiefel diagram in both spaces.[3]. Note too that W has a faithful, unitary representation and so must be isomorphic to a finite group of isometries of the Euclidean space $\mathfrak{t}^*$. The first step in describing this group is contained in the next proposition.

Proposition 7.1.1. *Let $\alpha \in \Phi$, then $\exists \sigma_\alpha \in W$ such that $\sigma_\alpha \cdot \alpha = -\alpha$ and $\sigma_\alpha \cdot v = v$ for all $v \in \ker \alpha$. We refer to σ_α as a* ***reflection****.*

Proof. Consider S^α, then the Weyl group $W^\alpha := W(S^\alpha, T^\alpha)$ is $\mathbb{Z}_2$, generated by the involution $\sigma_\alpha : T^\alpha \to T^\alpha : t \mapsto t^{-1}$.

Now, notice that $\sigma_\alpha = nT^\alpha$ for some $n \in N(T^\alpha)$ with $ntn^{-1} = t^{-1}$ for all $t \in T^\alpha$. Indeed, this implies that $n \cdot t^\alpha = t^{-\alpha}$, but α is a (real) linear form on $\mathfrak{t}$ and so it follows that if $u \in U^\alpha$, then $n \cdot u^\alpha = u^\alpha$.

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[1] This provides a good motivation for the study of Coxeter groups. See [17] for an excellent exposition of the theory.

[2] This parallels the origins of Lie theory, the Erlanger Programme.

[3] If a distinction is required then we refer to it as the *dual* stiefel diagram.

In fact, since (as a vector space) $\mathfrak{t} = \mathfrak{u}^\alpha \oplus \mathfrak{t}/\mathfrak{u}^\alpha$, then using the exponential restricted to $\mathfrak{t}$, we have that $n \in N(T)$. So, σ_α is a well-defined member of $W(G, T)$. In particular, $\sigma_\alpha(\alpha) = -\alpha$ and $\sigma_\alpha(\beta) = \beta$ for all $\beta \in \mathfrak{u}^\alpha$, as required. □

Now let $\alpha \in \Phi$, then since $(\cdot, \cdot)$ is W-invariant it follows that σ_α is an involutive isometry of $\mathfrak{t}^*$, so it *must* be a reflection. Hence, we can write the action explicitly as

$$\sigma_\alpha(\theta) = \theta - (\alpha^\vee, \theta)\alpha \qquad \forall \theta \in \mathfrak{t}^*.$$

For explicitness, we note that the corresponding formula on $\mathfrak{t}$ is

$$\sigma_\alpha(\tau) = \tau - \alpha^\vee(\tau) H_\alpha \qquad \forall \tau \in \mathfrak{t}.$$

By remarking that W acts via permutation on Φ, we now have an immediate corollary, which was far from obvious.

Corollary 7.1.1. *$\beta - (\alpha^\vee, \beta)\alpha \in \Phi$ $\forall \alpha$, $\beta \in \Phi$.*

7.2 Cartan Integers

Now, for α, $\beta \in \Phi$ let

$$n_{\alpha\beta} = (\alpha^\vee, \beta).$$

Lemma 7.2.1.

$$n_{\alpha\beta} \in \mathbb{Z} \qquad \forall \alpha,\ \beta \in \Phi.$$

*Consequently, we call the $n_{\alpha\beta}$ the **Cartan Integers** of the root system Φ.*

Proof. We remark that $\beta(H^\alpha) \in \mathbb{Z}$ for each α, $\beta \in \Phi$ since α is a weight of $\mathfrak{s}^\alpha \cong \mathfrak{su}(2)$ and $\mathfrak{g}$ is an $\mathfrak{s}^\alpha$-module by restriction of ad. □

From the observation:

$$n_{\alpha\beta}n_{\beta\alpha} = \frac{2(\alpha,\beta)}{(\alpha,\alpha)}\frac{2(\beta,\alpha)}{(\beta,\beta)} = 4\frac{(\alpha,\beta)^2}{(\alpha,\alpha)(\beta,\beta)} \in \mathbb{Z},$$

and the Cauchy-Schwarz inequality we have

$$n_{\alpha\beta}n_{\beta\alpha} \in \{0, 1, 2, 3, 4\}.$$

It is now apparent that sets of roots of compact Lie groups are very special objects, and that there are not that many of them. However, central result for this section is the following

Lemma 7.2.2. *Let α, $\beta \in \Phi$ be non-proportional and $(\alpha, \beta) > 0$, then $\alpha - \beta \in \Phi$. Hence, if $(\alpha, \beta) < 0$, then $\alpha + \beta \in \Phi$.*

Proof. Note that since $\alpha - \beta \in \Phi$ iff $\beta - \alpha \in \Phi$, so we may as well assume that $\|\alpha\| \geq \|\beta\|$. In particular, it follows that $0 < n_{\alpha\beta} = \frac{2(\alpha,\beta)}{(\alpha,\alpha)} \leq \frac{2(\alpha,\beta)}{(\beta,\beta)} \leq n_{\beta\alpha}$. But then $0 < n_{\alpha\beta}^2 \leq n_{\alpha\beta}n_{\beta\alpha} \leq 4$, implying $n_{\alpha\beta}$, $n_{\beta\alpha} \in \{1, 2\}$. However, if $n_{\beta\alpha} = 2$, then $(\alpha, \beta) = (\beta, \beta)$ and so $(\alpha - \beta, \beta) = 0$, which contradicts the non-portionality of the roots.
Hence, $\sigma_\beta(\alpha) = \alpha - \beta \in \Phi$. The last statement is done by replacing β with $-\beta$ and applying the first statement. ☐

This lemma is really quite nice as it is so unexpected, without noticing the Cartan integers it would have been very hard to prove directly.

7.3 Weyl Chambers and Simple Roots

For $U(n)$ and $r = 1, \ldots, n-1$ define $\omega_r(\boldsymbol{\theta}) = \theta_r - \theta_{r+1}$, then $\omega_r \in \Phi^+$ and $\{\omega_s\}_{s=1}^{n-1}$ is a linearly independent set. Moreover, if $r < s$ then $\alpha_{rs} = \sum_{j=r}^{s-1} \omega_j$ and so $\Phi^+ \subset \mathrm{span}_{\mathbb{N}}\{\omega_j\}_{j=1}^{n-1}$. Observe that ω_r cannot be written as a positive sum of positive roots. This situation generalises, but we have to work reasonably hard.

Pick a chamber, C and let

$$\Phi^+(C) = \{\alpha \in \Phi \,:\, \alpha(t) > 0,\, \forall t \in C\}.$$

We call this the set of **positive roots** associated to C. The reason for this nomenclature is that

$$\mathfrak{g} \cong \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi^+(C)} \mathfrak{g}_\alpha,$$

which will be obvious later. Now, given a root $\alpha \in \Phi$, say that α is **decomposable** if it can be written as $\alpha = n\beta + m\gamma$ where $n,\, m \in \mathbb{Z}^+$ and $\beta,\, \gamma \in \Phi^+(C)$. If α is indecomposable, then it will be referred to as **simple**.

It is obvious by the finiteness of Φ, and the previous results that we can arrive at a set of simple roots $A = \{\alpha_1,\, \ldots,\, \alpha_l\}$. It turns out that A is much better behaved than one would expect, and has quite a remarkable geometric and combinatorial connection to the Stiefel diagram.

Lemma 7.3.1.

1. $\Phi^+(C) \subset \mathrm{span}_{\mathbb{N}}(A)$.
2. *Let* $\alpha,\, \beta \in A$ *then* $(\alpha, \beta) \leq 0$.

Proof.

1. Let $\alpha \in \Phi^+(C)$, if α is simple, then we are done. Otherwise, $\exists \beta_1,\, \beta_2 \in \Phi^+(C)$ and $n_1,\, n_2 \in \mathbb{Z}^+$ such that $\alpha = n_1\beta_1 + n_2\beta_2$. Repeating the above procedure on β_1, and then on β_2 yields the result by the finiteness of Φ.
2. If $(\alpha, \beta) > 0$, then by Lemma 7.2.2 it follows that $\alpha - \beta \in \Phi$. Note that either $\alpha - \beta \in \Phi^+(C)$ or $\beta - \alpha \in \Phi^+(C)$ and so one of $\beta = \alpha + (\beta - \alpha)$ or $\alpha = \beta + (\alpha - \beta)$ would lead to a decomposition of α or β that are contrary to the assumption of simplicity.

□

For sake of convenience, we introduce a little more notation; namely let

$$\Phi^-(C) = -\Phi^+(C).$$

Equipped with Lemma 7.3.1, and our notation we arrive at the desired results.

Proposition 7.3.1.

1. *$\Phi = \Phi^+(C) \cup \Phi^-(C)$ as a disjoint union.*
2. *A is a linearly independent set. In particular, it is a basis for $(\mathfrak{t}/Z(\mathfrak{g}))^*$.*
3. *Given C, A is the unique set of simple roots with these properties. We refer to the set of simple roots as $\mathscr{A}(C)$.*

Proof.

1. Now, suppose that $\alpha \in \Phi^+(C) \cap \Phi^-(C)$, then $\alpha(t) > 0$ and $\alpha(t) < 0$ on C for all $t \in C$; which is clearly impossible.

   Now, suppose that $\beta \in \Phi$, it suffices to prove that either $\beta \in \Phi^+(C)$ or $\beta \in \Phi^-(C)$. Hence, it suffices to prove that if $t \in C$ and $\beta(t) > 0$, then $\beta \in \Phi^+(C)$ or if $\beta(t) \leq 0$, then $\beta \in \Phi^-(C)$. We can recast this again as β does not change sign on C.
   It is convenient to remark now that $\beta \neq 0$ on C since otherwise there would be a singular point in C, which is impossible. So, let u be another point in C then by the connectedness of C there is a continuous curve connecting t and u. Indeed, if $\beta(t)$ and $\beta(u)$ are of different sign then by continuity β it must have a zero in C, contradicting the previous remark.
2. (I approximately follow [3] pp. 116-117) Suppose that $\lambda_1, \ldots, \lambda_l \in \mathbb{R}$ and

$$\sum_{i=1}^{l} \lambda_i \alpha_i = 0.$$

Let I_+, $I_- \subset [1..l]$ be such that $\forall i \in I_+$, $\lambda_i \in \mathbb{R}^+$ and $\forall j \in I_-$ we have $\lambda_j \in -\mathbb{R}^+$ (we can obviously ignore the zero terms). Notice that Lemma 7.3.1 implies

$$\left(\sum_{i \in I_+} \lambda_i \alpha_i, \sum_{j \in I_-} (-\lambda_j) \alpha_j\right) = \sum_{i \in I_+, j \in I_-} \lambda_i(-\lambda_j)(\alpha_i, \alpha_j) \leq 0.$$

If the above expression is 0, then since all terms are of the same sign then for the sum to be zero, it follows that for all $i \in I_+$ and $j \in I_-$ we have $\lambda_i \lambda_j (\alpha_i, \alpha_j) = 0$. Indeed, since the λ_i are all non-zero we must have that $(\alpha_i, \alpha_j) = 0$. Hence, in this case

$$\left(\sum_{i \in I_+} \lambda_i \alpha_i, \sum_{j \in I_+} \lambda_j \alpha_j\right) = 0.$$

So, $\sum_{i \in I_+} \lambda_i \alpha_i = 0$ and similarly for the sum over I_-. Indeed, we must have for $\xi \in \Phi^+(C)$

$$0 = \left(\xi, \sum_{i \in I_+} \lambda_i \alpha_i\right) = \sum_{i \in I_+} \lambda_i (\xi, \alpha_i).$$

But $(\xi, \alpha_i) > 0$ and since the λ_i are all of the same sign they must all be zero. Similarly for the case of I_-.

Otherwise, it is stricly less than 0 and so by Lemma 7.2.2 it follows that $0 = \sum_{i=1}^n \lambda_i \alpha_i \in \Phi$, which is impossible. The final remark is now clear in light of Corollary 6.3.3.

3. (The author's proof) Let $B = \{\beta_1, \ldots, \beta_l\}$ be a set of simple roots on $\Phi^+(C)$. Note that since A, B are $\mathbb{Z}$ bases for Φ, then there is an automorphism of $(\mathfrak{t}/Z(\mathfrak{g}))^*$, Y, that takes the basis A to B. Indeed, it follows that since A, $B \subset \Phi^+(C) \subset \text{span}_{\mathbb{Z}^+}(A)$ then the matrix for Y has entries in $\mathbb{Z}^+$. However, the elements of A, B are all simple and so Y must be a permutation matrix. That is, $A = B$.

$\square$

After a large result like the above, it is not surprising a few interesting corollaries follow.

Corollary 7.3.1.

$$\mathfrak{g} \cong \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi^+(C)} \mathfrak{g}_\alpha.$$

So we finally have obtained a systematic decomposition of $\mathrm{Ad}\,|_T$.

Corollary 7.3.2.

$$\Phi \subset \operatorname{span}_{\mathbb{Z}} \mathscr{A}(C).$$

Corollary 7.3.3. *Let $\alpha \in \mathscr{A}(C)$ then $\Phi^+(C) - \{\alpha\}$ is σ_α-stable.*

Proof. Note that if $\beta \in \mathscr{A}(C) - \{\alpha\}$, then $\sigma_\alpha(\beta) = \beta - (\alpha^\vee, \beta)\alpha$. But notice that $(\alpha, \beta) \leq 0$ so $\sigma_\alpha(\beta)(t) > 0$ for all $t \in C$ (since α, $\beta \in \Phi^+(C)$). That is, $\sigma_\alpha(\beta) \in \Phi^+(C)$. $\square$

There is a converse to the last proposition too. To make the statement precise, we need to introduce another

Definition 7.3.1 (Fundamental Weyl Chamber). Let A be a system of simple roots, define

$$\mathscr{C}(A) = \{t \in \mathfrak{t} \,:\, \alpha(t) > 0, \,\forall \alpha \in A\}.$$

Call $\mathscr{C}(A)$ the **fundamental Weyl chamber generated by** A.

This really was the natural choice, we just have to check that our naming is proper.

Proposition 7.3.2. *Let A be a set of simple roots, then $\mathscr{C}(A)$ is a Weyl chamber.*

Proof. It suffices to prove that $\mathscr{C}(A)$ is a convex region containing only regular points of $\mathfrak{t}$. To do this, we simply note that if t, t' belong to $\mathscr{C}(A)$, then obviously for each $\lambda \in [0,1]$ $\alpha(\lambda t + [1-\lambda]t') = \lambda\alpha(t) + (1-\lambda)\alpha(t') > 0$ since $\alpha(t)$, $\alpha(t') > 0$ and so $\lambda\alpha(t) + (1-\lambda)\alpha(t') \in \mathscr{C}(A)$, and so $\mathscr{C}(A)$ is convex. $\square$

This is certainly enough to obtain the

Corollary 7.3.4. *There is a bijection between dominant Weyl chambers and simple root systems. These maps are given explicitly by $\mathscr{A}$ and $\mathscr{C}$.*

With the previous corollary in mind, it would seem worthwhile to explicitly characterise all simple root systems, enabling us to explicitly describe the Weyl chambers. A preliminary observation is the following

Lemma 7.3.2. *Let $w \in W$, then $w(C)$ is a Weyl chamber. Hence, $w \cdot \mathscr{A}(C)$ is a set of simple roots.*

Proof. It suffices to prove that $w(C)$ is convex and contains only regular points. Let $c \in C$, then $\alpha(c) \neq 0$ for all $\alpha \in \Phi$. Since W permutes Φ it follows that $0 \neq w^{-1} \cdot \alpha(c) = \alpha(w \cdot c)$, and so $w \cdot c$ is regular. The convexity follows by the linearity of w. □

Consider $U(3)$ again, then the set of roots $\{\omega_1, \omega_2\}$ is simple; and so corresponds to a fundamental Weyl chamber. Since each of these simple roots is positive on this chamber, and there are only two walls of the chamber, then all the walls arise as the kernels of the simple roots associated to the chamber. In general we have the

Proposition 7.3.3. *Let C be some Weyl chamber and consider some $\alpha \in \mathscr{A}(C)$, then $\mathfrak{u}^\alpha \cap \overline{C}$ is a wall of C. Moreover, any wall of C arises in this way.*

Proof. Now, $\sigma_\alpha(C)$ is another Weyl chamber with a simple root system containing $-\alpha$. That is, $\mathfrak{u}^\alpha \cap \overline{C}$ is a wall of C since α must be of different sign in C and $\sigma_\alpha(C)$. Conversely, given a wall $K = \mathfrak{u}^\alpha \cap \overline{C}$ for some $\alpha \in \Phi^+(C)$, then since $\Phi^+(C) \subset \operatorname{span}_{\mathbb{N}} \mathscr{A}(C)$ and the decomposition of α into simple roots implies that they are all zero on the wall K, this implies that K arises as $\mathfrak{u}^\beta \cap \overline{C}$ where $\beta \in \mathscr{A}(C)$. □

We are now really starting to understand the Stiefel diagram geometrically, and its connection the the simple root systems. The subsequent theorem is really telling us that the Stiefel diagram is very well behaved. The lemma and notation are courtesy of [3], p. 124.

Notation 7.3.1. For Weyl chambers C, C' let $\ell(C, C') = |\Phi^+(C) \cap \Phi^-(C')|$.

Lemma 7.3.3. *If C, C' are distinct Weyl chambers then, there exists another chamber C'', adjacent to C such that $\ell(C'', C') = \ell(C, C') - 1$.*

Proof. If $\ell(C, C') = 1$, then the result is trivial by Proposition 7.3.3. So, we assume now $\ell(C, C') > 1$ and so $\mathscr{A}(C) - \Phi^+(C')$ is non-empty since otherwise $\mathscr{A}(C) \subset \Phi^+(C')$, then since $\mathscr{A}(C)$ is a set of simple roots, $\mathscr{A}(C') = \mathscr{A}(C)$, which is impossible. So, pick $\alpha \in \mathscr{A}(C) - \Phi^+(C')$ and set $C'' = \sigma_\alpha(C)$. Noting that $C - \{\alpha\}$ is σ_α-stable by Lemma 7.3.3 and $\sigma_\alpha(\alpha) = -\alpha$, then $\Phi^+(C'') = (\Phi^+(C) - \{\alpha\}) \cup \{-\alpha\}$ and so $\ell(C'', C') = \ell(C, C') - 1$. $\square$

From which have the beautiful

Theorem 7.3.1. *Let C, C' be Weyl chambers, then there exists a sequence of simple reflections, of length $\ell(C, C')$, belonging to $\mathscr{A}(C)$, that map C to C'.*

Proof. We do induction on $s = \ell(C, C') = |\Phi^+(C) \cap \Phi^-(C')|$. If $s = 0$, then $\Phi^+(C) \cap \Phi^-(C') = \emptyset$, this is clearly true if and only if $C = C'$. Now, assume the result true for $s = k$ and so it suffices to prove the result true for $s = k + 1$. By applying Lemma 7.3.3, then there is a Weyl chamber, C'', adjacent to C such that $\ell(C'', C') = \ell(C, C') - 1$. Applying the inductive hypothesis to the pair C'', C', then we have a sequence $w := \sigma_{\alpha_{i_1}} \cdots \sigma_{\alpha_{i_k}}$, where $\alpha_{i_j} \in \mathscr{A}(C'')$ and $C' = w(C'')$. Also, since C'' is adjacent to C, then Proposition 7.3.3 implies $C'' = \sigma_\alpha(C)$ for some unique $\alpha \in \mathscr{A}(C)$. Note that this implies $\beta_j := \sigma_\alpha(\alpha_{i_j}) \in \mathscr{A}(C)$ and since $\sigma_{\beta_j} = \sigma_\alpha \sigma_{\alpha_{i_j}} \sigma_\alpha$, then $w' := \sigma_\alpha \sigma_{\beta_1} \cdots \sigma_{\beta_k} = w\sigma_\alpha$ is such that $w'(C) = C'$ and has length $\ell(C'', C') + 1 = \ell(C, C')$ and is a product of simple roots belonging to $\mathscr{A}(C)$; as required. $\square$

Notation 7.3.2. Let C be a Weyl chamber, define $W_C = \langle \sigma_\alpha : \alpha \in \mathscr{A}(C) \rangle$.

Corollary 7.3.5. *W_C acts simply transitively on the set of Weyl chambers.*

Corollary 7.3.6. *Let C, C' be Weyl chambers, then $W_C \cong W_{C'}$. So, we let $W_\Phi := W_C$.*

These last two results are really quite incredible, they connect the geometry of conjugation in G to the symmetries and combinatorics of a finite group of isometries of $\mathfrak{t}$.

7.4 Geometric Characterisation of the Weyl Group

The aim of this section is to prove the following theorem, giving us the geometric characterisation of the Weyl group that we desire.

Theorem 7.4.1.

$$\boxed{W_\Phi \cong W.}$$

From which we have the gracious

Corollary 7.4.1. *The geometric structure of the Stiefel diagram is independent of T.*

With all the work that we have done so far, it is not surprising that this result is now not so difficult to prove. The method of proof is to show that W acts simply transitively on the set of Weyl chambers, by the previous section this would imply that there exists a reduced word $s_1 \ldots s_r$, where the s_i are all simple reflections, which corresponds to each element of W. The problem is now reduced to a a proposition and a corollary. The following proof is related to [6] pp. 193-194.

Proposition 7.4.1. *If $w \in W$ and $w(C) = C$, then $w = 1$.*

Proof. Note that since W is finite, then the order of $w \in W$ is finite, call the order n. Fix $t \in C$ and define

$$\hat{t} = \sum_{k=1}^{n} w^k \cdot t.$$

Observe that since $w(C)$ then $w \cdot \mathscr{A}(C) = \mathscr{A}(C)$ and so if $\alpha \in \mathscr{A}(C)$ then as $\alpha(t) > 0$ we must have $\alpha(\hat{t}) > 0$. Hence, $\hat{t} \in \mathscr{C}(\mathscr{A}(C)) = C$.

Write $w = nT$ where $n \in N(T)$, and so n and $H := \exp(\hat{t})$ commute. By Corollary 5.3.4, this is is contained in some maximal torus T'. However, since $h \notin U^\alpha$ for any $\alpha \in \Phi$, then $T = T'$ and so $n \in T$. In particular, we must have $w = 1$. $\square$

Corollary 7.4.2. *W acts simply transitively on the set of Weyl chambers.*

Proof of Theorem 7.4.1. Let $w \in W \setminus W_\Phi$ then consider the action of w on a Weyl chamber C. Note that $w(C)$ is another Weyl chamber, by Lemma 7.3.2 and so since W_Φ acts simply transitively on the set of Weyl chambers, by Corollary 7.4.2, then the only possibility is $w(C) = C$. By Proposition 7.4.1 we must have $w = 1$, but since $1 \in W_\Phi$ then $1 \notin W \setminus W_\Phi$. Hence, we conclude that $W \setminus W_\Phi = \emptyset$. $\square$

We now have a natural definition of the determinant of an element of W.

Corollary 7.4.3. *Any $w \in W$ can be written as a product of $\ell(C, w(C))$ simple reflections in the walls of its Weyl chamber. Consequently,* $\det w = (-1)^{\ell(C,w(C))}$.

Chapter 8

Weyl's Formulae

In this chapter we examine four formulae, due to Weyl. The first is the Weyl integration formula, this relates integration across the group to integration across G-orbits and a maximal tori, an immediate corollary is an integration formula for class functions. This leads into the celebrated Weyl character formula, which provides a method of calculating irreducible characters for a compact Lie group G in terms of its highest weight; immediate corollaries of this marvellous result are the denominator and dimension formulae. The author adds that this chapter is by far the most computational, but this is by no means surprising in consideration of the work that is being undertaken.

There are a number of different proofs of the Weyl character formula, with each displaying another aspect of Lie theory. The path that is taken by the author is that of an analytic nature, using the theory of Fourier series. It is worth remarking that this proof is quite different from Weyl's (and [6] ch. VI §1), predominantly because our characters are computed as opposed to deduced. For a collection of results equivalent to the Weyl character formula, with different proofs I would certainly recommend [10] p. 440-444.

8.1 The Integration Formula

Consider the process of diagonalising a matrix in $U(n)$, if the eigenvalues are distinct then there are precisely $|\mathfrak{S}_n| = n!$ ways of doing this. This generalises to the following

Lemma 8.1.1. *The map $\Psi : G/T \times T \to G : (gT, t) \mapsto gtg^{-1}$ is a well-defined, W-fibred, smooth mapping of manifolds.*

Proof. Let gT, $hT \in G/T$ be such that $gT = hT$ and pick $t' \in T_{\text{reg}}$. Note that clearly $h^{-1}g = t \in T$ and so

$$\Psi(gT, t') = gt'g^{-1} = (ht)t'(ht)^{-1} = h(tt't^{-1})h^{-1} = ht'h^{-1} = \Psi(hT, t'),$$

since T is abelian. Hence, Ψ is well-defined.

Note that T is a submanifold of G, and since G/T is a manifold it is now clear that $G/T \times T$ is a manifold too. Indeed, since conjugation is smooth it is immediate that Ψ is a smooth mapping of manifolds and by Sard's Theorem ([20] p. 460), the set of singular points of this transformation has Haar measure zero.

Now consider $\Psi_{\text{reg}} := \Psi|_{G/T \times T_{\text{reg}}}$ and let $g \in G_{\text{reg}} = \operatorname{im} \Psi_{\text{reg}}$. So, g belongs to precisely one maximal torus T' and by Theorem 5.3.1 $T' = hTh^{-1}$ for some $h \in G$. Hence, $g = hth^{-1}$, for some $t \in T$. However, let $nT \in W$, then since $g = (hn)(n^{-1}tn)(hn)^{-1}$ and $n^{-1}tn \in T$ and since W acts transitively on regular elements of G then $\Psi_{\text{reg}}^{-1}(g) \cong W$.

Once noting that G_{reg} is open and dense in G (c.f. Proposition 6.3.2) the result is now proven in its entirety. $\square$

It is not surprising that this map has a pleasant Jacobian; this is computed in the

Lemma 8.1.2.

$$|\det(\Psi)_\star(gT, t)| = |\det(\operatorname{Ad}(t^{-1}) - I)_{\mathfrak{g}/\mathfrak{t}}| \quad \mu - a.e.$$

Proof. Since G_{reg} is open and dense in G we need only consider Ψ_{reg}. Note that since $(\Psi_{\text{reg}})_\star(gT, t)$ is a linear map from $T_{gT}(G/T) \times T_t(T_{\text{reg}})$ to $T_{gtg^{-1}}(G) \cong \mathfrak{g}/\mathfrak{t} \oplus \mathfrak{t}$ we may as well consider the actions on each of the two tangent spaces in the above product.

Following [24] pp. 6-7, $s \mapsto (g\exp(sX)T, t)$ is a smooth map in a neighbourhood of (gT, t) with tangent vector $[X, 0] \in T_{gT}(G/T) \times T_t(T_{\mathrm{reg}})$ when $s = 0$. Hence,

$$\begin{aligned}
(\Psi_{\mathrm{reg}})_\star(gT,t)[X,0]_{(gT,t)} &= \frac{\mathrm{d}}{\mathrm{d}s}\left[\Psi_{\mathrm{reg}}(g\exp(sX)T,t)\right]_{s=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}s}\left[g(tg^{-1}gt^{-1})\exp(sX)t(g^{-1}g)\exp(-sX)g^{-1}\right]_{s=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}s}\left[(gtg^{-1})g(t^{-1}\exp(sX)t)g^{-1}g\exp(-sX)g^{-1}\right]_{s=0} \\
&= \mathrm{Ad}(g)(\mathrm{Ad}(t^{-1}) - I)[X,0]_{gtg^{-1}}.
\end{aligned}$$

Similarly, $s \mapsto (gT, t\exp(sY))$ is a smooth map in a neighbourhood of (gT, t) with tangent vector $[0, Y] \in T_{gT}(G/T) \times T_t(T_{\mathrm{reg}})$ when $s = 0$. Thus,

$$\begin{aligned}
(\Psi_{\mathrm{reg}})_\star(gT,t)[X,0]_{(gT,t)} &= \frac{\mathrm{d}}{\mathrm{d}s}\left[\Psi_{\mathrm{reg}}(gT,t\exp(sY))\right]_{s=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}s}\left[gt\exp(sY)g^{-1}\right]_{s=0} \\
&= \mathrm{Ad}(g)[0,Y]_{gtg^{-1}}.
\end{aligned}$$

Indeed, it now follows that

$$(\Psi_{\mathrm{reg}})_\star(gT,t)[X,Y]_{(gT,t)} = \mathrm{Ad}(g)\left((\mathrm{Ad}(t^{-1}) - I)[X,0]_{gtg^{-1}} + [0,Y]_{gtg^{-1}}\right).$$

Note that $\mathrm{Ad}(g)$ is a unitary representation, and so $|\det \mathrm{Ad}(g)| = 1$. Hence, $|\det(\Psi_{\mathrm{reg}})_*(gT,t)| = |\det(\mathrm{Ad}(t^{-1}) - I)_{\mathfrak{g}/\mathfrak{t}}|$. □

We now introduce the following

Notation 8.1.1. For each Weyl chamber C, let

$$\delta_C(t) = \prod_{\alpha \in \Phi^+(C)} (1 - t^{-\alpha}).$$

When the choice of Weyl chamber is unimportant, then it will not be specified.

For $SU(2)$ note that there is only one positive root and so:

$$\delta\begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} = 1 - t^{-2}.$$

On $U(3)$, we have

$$\delta\begin{pmatrix} t_1 & & \\ & t_2 & \\ & & t_3 \end{pmatrix} = \left(1 - \frac{t_2}{t_1}\right)\left(1 - \frac{t_3}{t_2}\right)\left(1 - \frac{t_3}{t_1}\right) = \frac{\Delta(t_1, t_2, t_3)}{t_1^2 t_2},$$

where Δ is the Vandermonde determinant. For $U(n)$, it is clear the computation generalises to:

$$\delta\begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix} = \frac{\Delta(t_1, \ldots, t_n)}{t_1^{n-1} t_2^{n-2} \cdots t_{n-1}}.$$

The next result was something that impressed Schur ([5] p. 35). Analysing the W-symmetric functions functions on T in order to understand the character theory is almost intractible, due to its directness. The next result alludes that studying W-antisymmetric functions may be worthwhile, something that is completely unobvious, as they are not even class functions.

Lemma 8.1.3.

$$|\det(\mathrm{Ad}(t^{-1}) - I)_{\mathfrak{g}/\mathfrak{t}}| = \delta_C(t)\overline{\delta_C(t)}.$$

Proof. Notice that

$$\begin{aligned} \det\begin{pmatrix} \cos\alpha(t)-1 & -\sin\alpha(t) \\ \sin\alpha(t) & \cos\alpha(t)-1 \end{pmatrix} &= 2 - 2\cos\alpha(t) \\ &= 1 - (e^{i\alpha(t)} + e^{-i\alpha(t)}) + e^{i\alpha(t)}e^{-i\alpha(t)} \\ &= (1 - e^{i\alpha(t)})(1 - e^{-i\alpha(t)}) \end{aligned}$$

So, since $\mathfrak{g}/\mathfrak{t} = \bigoplus_{\alpha \in \Phi^+(C)} T_\alpha^{\mathbb{R}}$ as T-modules computation above implies the result. $\square$

For $U(n)$, it is clear:

$$\left|\delta\begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix}\right| = \frac{|\Delta(t_1,\ldots,t_n)|}{|t_1|^{2n-2}|t_2|^{2n-4}\cdots|t_{n-1}|^2}.$$

We now arrive at the first of Weyl's many contributions to the theory of compact Lie groups. It is also worth remarking that a theorem of Harish-Chandra generalises this result to an arbitrary real reductive Lie group ([20] pp. 483-485).

Theorem 8.1.1 (Weyl Integration Formula). *Let $f \in L^1(G)$ then*

$$\boxed{\int_G f(g)\,\mathrm{d}g = \frac{1}{|W|}\iint_{G/T\times T} f(gtg^{-1})\delta(t)\overline{\delta(t)}\,\mathrm{d}t\,\mathrm{d}(gT),}$$

where $\mathrm{d}(gT)$ is the Haar measure on the homogeneous space G/T induced from G with $\mu(G/T)=1$, and $\mathrm{d}t$ is Haar measure on T.[1]

Proof. Now, using Theorem 5.19 in [6] p. 51, for a volume form $\omega \in \Omega^{\dim G}(G)$

$$\int_{G_{\mathrm{reg}}} \omega = \frac{1}{|W|}\iint_{G/T\times T_{\mathrm{reg}}} \Phi^\star\omega.$$

Hence, applying the definitions and results already obtained

$$\int_{G_{\mathrm{reg}}} f(g)\,\mathrm{d}g = \frac{1}{|W|}\iint_{G/T\times T_{\mathrm{reg}}} f(gtg^{-1})\delta(t)\overline{\delta(t)}\,\mathrm{d}t\,\mathrm{d}(gT).$$

Noting that $\mu(G\setminus G_{\mathrm{reg}})$, $\mu(T\setminus T_{\mathrm{reg}})=0$, the result follows. $\square$

This formula is quite unpleasant computationally for a general $f \in L^1(G)$. This is due to the fact that we would be required to explicitly understand the measure on G/T, which is normally constructed out of partitions of unity. However in the case that f is an L^1 class function, then we have the all important

[1] For the existence and uniqueness of such a measure see [20] pp. 470-471.

Corollary 8.1.1. *If f is a (measurable) class function then*

$$\int_G f(g)\,\mathrm{d}g = \frac{1}{|W|}\int_T f(t)\delta(t)\overline{\delta(t)}\,\mathrm{d}t.$$

Corollary 8.1.2. *The correspondence*

$$\{L^2(G) \text{ class functions}\} \ni f \to \frac{f|_T\delta}{\sqrt{|W|}} \in L^2(T),$$

is an isometry.

This last corollary is informing us how the $L^2(G)$ class functions are mangled (the loss of conjugation) by the group when they are restricted to a maximal torus. The mangling is captured by the **Weyl factor** δ, which summarises conjugation in the group; agreeing with intuition.

8.2 The Weight Space Decomposition

Consider the natural action of $U(n)$ on $\mathbb{C}^n$. This restricts to a homomorphism of the diagonal subgroup T on $\mathbb{C}^n$. If for $i = 1, \ldots, b$, $\beta_i \in (\mathbb{R}^n)^*$ is the map $\beta_i(\boldsymbol{\theta}) = \theta_i$, then

$$\mathbb{C}^n = \bigoplus_{i=1}^{n} T^{\mathbb{C}}_{\beta_i}.$$

In general, let $\pi_V : G \to GL(V)$ be a representation of G on the finite dimensional, complex vector space V, then $\pi_V|_T : T \to GL(V)$ is a representation of T. It follows from Theorem 2.2.1 that V is an orthogonal direct sum of irreducible T-modules.

Definition 8.2.1 (Weights, Weight Space). For $\alpha \in \hat{\Lambda}$ let $n_\alpha(V)$ denote the degree of $T^{\mathbb{C}}_\alpha$ in V. Define $\Phi(V,T) := \{\alpha \in \widehat{\Lambda} : n_\alpha(V) \neq \{0\}\}$, called the **weights** of V.
Also, let $V_\alpha := n_\alpha T^{\mathbb{C}}_\alpha \subset V$. We refer to V_α as the **weight space** corresponding to the weight α.

Note that since W acts on T and $\widehat{\Lambda}$, and so $w\cdot : \pi_V(w) \in \mathrm{Aut}\,(V)$. Hence,

$$V = w^{-1}\cdot V \cong \bigoplus_{\alpha\in\widehat{\Lambda}} n_\alpha(V) w^{-1}\cdot T^{\Bbbk}_\alpha = \bigoplus_{\alpha} n_\alpha(V) T^{\Bbbk}_{(w^{-1})^*\alpha} = \bigoplus_{\alpha'} n_{w\cdot\alpha'}(V) T^{\Bbbk}_{\alpha}.$$

Noting that each $T^{\mathbb{C}}_\alpha$ is irreducible, then we have proven the following

Proposition 8.2.1.

$$\boxed{n_{w\cdot\alpha}(V) = n_\alpha(V) \quad \forall w\in W \quad \forall \alpha\in\widehat{\Lambda}}$$

In particular, W permutes the weights of V.

8.3 The Character Formula

Before we can state this result we require another

Definition 8.3.1 (Highest Weight). Let V be a G-module, then a weight $\lambda \in \hat{\Lambda}$ is said to be of **highest weight for** V if $\lambda+\alpha$ is not a weight for any nonzero $\alpha \in \operatorname{span}_{\mathbb{N}} \Phi^+(C)$.

The existence of a highest weight is obvious by finite dimensionality of V. We are now in a position to state the

Theorem 8.3.1 (Weyl's Character Formula). *Let G be a compact, connected Lie group and V an irreducible, complex G-module. Then, for a choice of maximal torus, and a Weyl Chamber C there is unique, highest weight λ with degree equal to one. Moreover, for any $t \in T_{\mathrm{reg}}$:*

$$\boxed{\chi_V(t) = \frac{\sum_{w\in W} (\det w)\, t^{w^*\lambda + (w^*\rho_C - \rho_C)}}{\delta_C(t)},}$$

where $\rho_C = \frac{1}{2}\sum_{\alpha\in\Phi^+(C)} \alpha$.

What is so remarkable about this result is it's explicitness. It was the first such formula for the characters of a compact Lie group, and is still regarded as being the most beautiful. The proof contained in this section follows Duistermaat's exposition in [9] pp. 252-256 quite closely.

As in Section 8.2 we write $V = \bigoplus_{\lambda \in \hat{\Lambda}} n_\lambda(V) T_\lambda^{\mathbb{C}}$, then

$$\chi_V(t) = \sum_{\lambda \in \hat{\Lambda}} n_\lambda(V) t^\lambda.$$

Also, since $\psi_V := (\chi_V|_T)\delta_C$ belongs to $L^2(T)$ (by Corollary 8.1.2), then it also has ([30] pp. 233-236) a Fourier expansion

$$\psi_V(t) = \sum_{\lambda \in \hat{\Lambda}} p_\lambda(V) t^\lambda,$$

with convergence in $L^2(T)$.

Notation 8.3.1. When there is no cause for confusion, $n_\lambda(V) = n_\lambda$ and $p_\lambda(V) = p_\lambda$.

The way in which we prove the formula is to examine how the W-action effects the Fourier coefficients of ψ_V. This is a very elegant approach to the problem, as it really captures the geometry of the situation. Namely, the interaction between the torus T and the non-uniqueness of conjugating into the torus, the precise thing we need to understand to prove a result related to class functions.

What the author really likes about this proof is that the highest weight theory is very accessible, many expositions are hard to follow because the motivation is unclear; yet in this proof it arises naturally. Let us now move into the proof, which is broken into a sequence of lemmata.

Our first result is a bland expansion of the δ function.

Lemma 8.3.1.

$$\boxed{\delta_C(t) = \sum_{P \subset \Phi^+(C)} (-1)^{|P|} t^{-\sum_{\alpha \in P} \alpha}}$$

Proof (By Example). I provide a computation for $U(3)$, the general computation is identical.

$$\delta\begin{pmatrix} t_1 & & \\ & t_2 & \\ & & t_3 \end{pmatrix} = \left(1 - \frac{t_2}{t_1}\right)\left(1 - \frac{t_3}{t_1}\right)\left(1 - \frac{t_3}{t_2}\right)$$
$$= 1 - \left(\frac{t_2}{t_1} + \frac{t_3}{t_1} + \frac{t_3}{t_2}\right) + \left(\frac{t_2}{t_1}\frac{t_3}{t_1} + \frac{t_2}{t_1}\frac{t_3}{t_2} + \frac{t_3}{t_2}\frac{t_3}{t_1}\right) - \frac{t_2}{t_1}\frac{t_3}{t_2}\frac{t_3}{t_1}.$$

□

This formula is quite cumbersome, but it does turn out to be useful for the following

Lemma 8.3.2. [2]

$$\boxed{p_\lambda = \sum_{P \subset \Phi^+(C)} (-1)^{|P|}\, n_{\lambda + \sum_{\alpha \in P} \alpha} \in \mathbb{Z} \qquad \forall \lambda \in \hat{\Lambda}.}$$

In particular, the Fourier expansion of $(\chi_V|_T)\delta$ on T has at most $|W|$ non-zero coefficients.

Proof. By direct expansion we have

$$\sum_{\lambda \in \hat{\Lambda}} p_\lambda t^\lambda = \sum_{\lambda \in \hat{\Lambda}} \sum_{P \subset \Phi^+(C)} n_\lambda (-1)^{|P|}\, t^{\lambda - \sum_{\alpha \in P} \alpha}.$$

So equating Fourier coefficients yields

$$p_\lambda = \sum_{P \subset \Phi^+(C)} (-1)^{|P|}\, n_{\lambda + \sum_{\alpha \in P} \alpha}.$$

In fact, since $\dim V < \infty$, then at most finitely many of the $n_\mu \neq 0$, and since $n_\mu \in \mathbb{N}$ then $m_\lambda \in \mathbb{Z}$ too.

[2] A better form of the above expansion is the Weyl denominator formula, which is proven later.

Also, since V is irreducible Corollary 2.3.1 implies $\int_G |\chi_V(g)|^2 \, \mathrm{d}\mu(g) = 1$ and so Corollary 8.1.2 informs us the norm of ψ_V in $L^2(T)$ is $|W|$. By Parseval's Theorem ([15] p. 226) we obtain

$$\sum_{\lambda \in \hat{\Lambda}} p_\lambda^2 = |W|.$$

But the $p_\lambda \in \mathbb{Z}$, so at most $|W|$ of the Fourier coeffiecients of ψ_V can be non-zero. □

From these two (trivial) lemmata, we have the somewhat surprising

Corollary 8.3.1. *If λ is a highest weight:*

$$\boxed{n_\lambda = p_\lambda.}$$

Proof. Let $P \subset \Phi^+(C)$, then clearly $\sum_{\alpha \in P} \alpha = 0$ if and only if $P = \emptyset$ since $P \subset \operatorname{span}_{\mathbb{N}} \mathscr{A}(C)$ and $\mathscr{A}(C)$ is linearly independent. Also, since λ is highest weight, then $\kappa := \lambda + \sum_{\alpha \in P} \alpha$ is not a weight for any $P \neq \emptyset$ and so $n_\kappa = 0$ for any $P \neq \emptyset$. Applying the formula obtained in Lemma 8.3.2 now implies the result. □

This is wonderful, we have just connected the two Fourier expansions in a concrete fashion. Let us remove ourselves from the highest weight theory for a while and proceed with the task of computing the Fourier coefficients of ψ_V combinatorially.

Since χ_V is W-invariant it would seem fair to imply that the W-symmetries of the m_λ's are only dependent on δ_C. So, let us now investigate the symmetries of δ_C. Note that W's action on δ_C is given by $(w^*\delta_C)(t) = \delta_C(w^{-1} \cdot t)$ and so we have:

$$\begin{aligned}
(w^*\delta_C)(t) &= \prod_{\alpha \in \Phi^+(C)} (1 - (w^{-1} \cdot t)^{-\alpha}) \\
&= \prod_{\alpha \in \Phi^+(C)} (1 - t^{-w^*\alpha}) \\
&= \prod_{\alpha \in \Phi^+(w(C))} (1 - t^{-\alpha}) \\
\therefore \quad (w^*\delta_C)(t) &= \delta_{w(C)}(t).
\end{aligned}$$

From which we infer that the W-action on δ correponds to a W-action on the Stiefel diagram.

Example 8.3.1 ($U(3)$).

$$[(12)^*\delta]\begin{pmatrix} t_1 & & \\ & t_2 & \\ & & t_3 \end{pmatrix} = \delta\begin{pmatrix} t_2 & & \\ & t_1 & \\ & & t_3 \end{pmatrix} = \frac{\Delta(t_2, t_1, t_3)}{t_2^2 t_1} = -\left(\frac{t_1}{t_2}\right)\delta\begin{pmatrix} t_1 & & \\ & t_2 & \\ & & t_3 \end{pmatrix}.$$

A further observation is that $\Phi^+ \cap (12) \cdot \Phi^- = \{\alpha_{12}\}$.

Since $\mathfrak{S}_3$ is generated by transpositions it would appear that for $U(3)$, δ is almost $\mathfrak{S}_3$-antisymmetric; with the factor out the front deserving of some special attention. We now generalise this computation to an arbitrary compact Lie group. The following notation would seem appropriate in light of our example.

Notation 8.3.2. For $w \in W$, let

$$\eta(w) := \sum_{\alpha \in \Phi^+(C) \setminus \Phi^+(w(C))} \alpha.$$

The symmetry result we desire is now displayed in the

Proposition 8.3.1. *Let $w \in W$, then*

$$(w^*\delta_C)(t) = (\det w)\delta_C(t)t^{\eta(w)}$$

Proof. The main observation is that since

$$\Phi^+(C) = (\Phi^+(C) \cap \Phi^+(w(C))) \cup (\Phi^+(C) \setminus \Phi^+(w(C))),$$

and

$$\begin{aligned} \Phi^+(w(C)) \setminus \Phi^+(C) &= \Phi^+(w(C)) \cap \Phi^-(C) \\ &= -(\Phi^+(C) \cap \Phi^-(w(C))) \\ &= -(\Phi^+(C) \setminus \Phi^+(w(C))), \end{aligned}$$

then we have:

$$\Phi^+(w(C)) = (\Phi^+(w(C)) \cap \Phi^+(C)) \cup -(\Phi^+(C) \setminus \Phi^+(w(C))).$$

Hence,

$$\begin{aligned} \delta_{w(C)}(t) &= \left(\prod_{\alpha \in \Phi^+(w(C)) \cap \Phi^+(C)} (1 - t^{-\alpha}) \right) \times \left(\prod_{\alpha \in -(\Phi^+(C) \setminus \Phi^+(w(C)))} (1 - t^{-\alpha}) \right) \\ &= \left(\frac{\prod_{\alpha \in \Phi^+(C)} (1 - t^{-\alpha})}{\prod_{\alpha \in \Phi^+(C) \setminus \Phi^+(w(C))} (1 - t^{-\alpha})} \right) \times \left(\prod_{\alpha \in \Phi^+(C) \setminus \Phi^+(w(C))} -t^{\alpha}(1 - t^{-\alpha}) \right) \\ &= \delta_C(t) \times \left(\prod_{\alpha \in \Phi^+(w(C)) \cap \Phi^+(C)} -t^{\alpha} \right). \\ &= (\det w)\delta_C(t)t^{\eta(w)}, \end{aligned}$$

where the last equality makes use of Corollary 7.4.3. ∎

The η function not particularly unpleasant computationall, it would be nice to obtain a more useful expression. So, we introduce the

Notation 8.3.3. Let $\rho_C := \frac{1}{2}\sum_{\alpha\in\Phi^+(C)} \alpha$.

Let us compute an

Example 8.3.2 ($U(n)$). For $n = 3$ we clearly have

$$\rho\begin{pmatrix} i\theta_1 & & \\ & i\theta_2 & \\ & & i\theta_3 \end{pmatrix} = \frac{1}{2}(\theta_1 - \theta_2 + \theta_2 - \theta_3 + \theta_1 - \theta_3) = \theta_1 - \theta_3 = 2\theta_1 + \theta_2 - (\theta_1 + \theta_2 + \theta_3).$$

More generally,

$$\rho\begin{pmatrix} i\theta_1 & & \\ & \ddots & \\ & & i\theta_n \end{pmatrix} = \frac{1}{2}\sum_{1\le r<s\le n}(\theta_r - \theta_s) = \sum_{r=1}^{n-1}(n-r)\theta_r + \beta\left(\sum_{r=1}^{n}\theta_r\right),$$

where $\beta \in \mathbb{Q}$. This decomposition will be important when we compute the characters of $U(n)$.

One should be careful with this definition as the following example demonstrates.

Example 8.3.3. ρ is *not* a weight of $SO(3)$ as the only positive root is $\alpha = \mathrm{id}$.

What this means is that we are not supposed to treat ρ like a weight, if we wanted to we would have to pass to some bigger Lie group (i.e. the universal covering group), analogous to passing to a field extension when trying to solve polynomials.[3] Consider the following

Example 8.3.4 ($U(3)$). Let $(321) \in \mathfrak{S}_3$, then

$$[(321)^*\rho]\begin{pmatrix} i\theta_1 & & \\ & i\theta_2 & \\ & & i\theta_3 \end{pmatrix} = \rho\begin{pmatrix} i\theta_3 & & \\ & i\theta_1 & \\ & & i\theta_2 \end{pmatrix} = \theta_3 - \theta_2.$$

So,

$$\rho\begin{pmatrix} i\theta_1 & & \\ & i\theta_2 & \\ & & i\theta_3 \end{pmatrix} - [(321)^*\rho]\begin{pmatrix} i\theta_1 & & \\ & i\theta_2 & \\ & & i\theta_3 \end{pmatrix} = \alpha_{13} + \alpha_{23} = \omega_1 + 2\omega_2.$$

[3]Technically speaking, the computation would generally need to be carried out in the representation ring $\mathbb{Z}\left[\frac{1}{2}\Phi\right]$. See [2] pp. 134-141 for the details.

However, $(321)^*\Phi^+ = \{\alpha_{32}, \alpha_{31}, \alpha_{12}\}$ and so $\Phi^+ \setminus (321)^*\Phi^+ = \{\alpha_{13}, \alpha_{23}\}$. Hence,

$$\eta(321) = \alpha_{13} + \alpha_{23} = \omega_1 + 2\omega_2.$$

Inspired by the previous example it would appear that a computational formula for η, involving ρ_C may be possible. The proof is lifted from [9] p. 255 and merely provided for completeness.

Lemma 8.3.3.

$$\boxed{\eta(w) = \rho - w^*\rho \in \widehat{\Lambda} \qquad \forall w \in W.}$$

Proof.

$$\begin{aligned}
w^*\rho &= \frac{1}{2} \sum_{\alpha \in \Phi^+(w(C))} \alpha \\
&= \frac{1}{2} \left(\sum_{\alpha \in \Phi^+(C) \cap \Phi^+(w(C))} \alpha + \sum_{\alpha \in -(\Phi^+(C) \setminus \Phi^+(w(C)))} \alpha \right) \\
&= \frac{1}{2} \left(\sum_{\alpha \in \Phi^+(C)} \alpha - 2 \sum_{\alpha \in \Phi^+(C) \setminus \Phi^+(w(C))} \alpha \right) \\
&= \rho - \eta(w).
\end{aligned}$$

□

So we arrive at the following symmetry relation for δ:

$$\boxed{(w^*\delta)(t) = (\det w)\delta(t)t^{\rho - w^*\rho} \qquad \forall w \in W.}$$

Example 8.3.5 ($U(3)$).

$$[(321)^*\delta](t) = \delta(t)t^{\omega_1 + 2\omega_2}.$$

Although δ is not quite W-antisymmetric we are now in a much better position to explicitly describe how the Fourier coefficients of ψ_V behave under the W-action. This is contained in the somewhat miraculous

Proposition 8.3.2.

$$\boxed{p_{w^*\lambda+(w^*\rho-\rho)} = (\det w)p_\lambda \qquad \forall\lambda \in \hat{\Lambda}, \quad \forall w \in W.}$$

Proof. Since the Fourier expansion has only finitely many non-zero terms we have no difficulties with convergence. Now,

$$\psi_V(t) = \sum_{\lambda\in\hat{\Lambda}} p_\lambda t^\lambda \implies (w^*\psi_V)(t) = \sum_{\lambda\in\hat{\Lambda}} p_{w^*\lambda} t^\lambda.$$

However,

$$\begin{aligned}(w^*\psi_V)(t) &= \chi_V|_T(t)(w^*\delta)(t)\\ &= (\det w)t^{\rho-w^*(\rho)}\chi_V|_T(t)\delta(t)\\ &= \sum_{\lambda\in\hat{\Lambda}}(\det w)p_\lambda t^{\lambda+\rho-w^*(\rho)}.\end{aligned}$$

The result is clear once we equate the Fourier coefficients for the two expressions of $w^*\psi_V$.

□

So, the Fourier coefficients of ψ_V are W-antisymmetric with respect to the *shifted* W action:

$$\boxed{w \star \lambda := w^*\lambda + (w^*\rho - \rho).}$$

This is very unexpected, there is really no reason for this symmetry to emerge, yet it is the natural symmetry of the Fourier coefficients. From this shifted W action it now becomes clear where the highest weight theory fits in. These are formally digested in the subsequent results.

Proposition 8.3.3. *The shifted W-action is free and transitive on highest weights of V.*

Proof. Let λ be a highest weight and suppose that $w^*\lambda + w^*\rho - \rho = \lambda$ for some $w \in W$, then $w^*\lambda = \lambda + \rho - w^*\rho = \lambda + \eta(w)$, by Lemma 8.3.3. Proposition 8.2.1 informs us that $n_{w^*\lambda} = n_\lambda > 0$ and so that $\eta(w) = 0$, making use of Proposition 7.4.1 yields $w = 1$. Moreover, the shifted W-orbit has at most $|W|$ elements, by virtue of Lemma 8.3.2. Putting these two facts together implies that the shifted W-orbit of a highest weight λ has precisely $|W|$ elements, so the action is free and transitive. □

After all our work it would appear that we are very close to the end of this spectacular theorem.

Corollary 8.3.2. *If λ is a highest weight, then $n_\lambda = p_\lambda = 1$.*

Proof. Let $w \in W$, then since the shifted W-orbit has $|W|$ elements, we must have $p_\lambda^2 = 1$ and so $p_\lambda = \pm 1$, by Lemma 8.3.2 and the transtivity of the action, by Proposition 8.3.3. But Corollary 8.3.1 implies that $p_\lambda = n_\lambda \in \mathbb{Z}^+$, so $p_\lambda = 1$. □

We now have the Fourier coefficients in all their glory.

Corollary 8.3.3. *For a highest weight λ and $w \in W$:*

$$\boxed{p_{w\star\lambda} = \det w.}$$

In fact, the proof of the character formula is complete except for the uniqueness of the highest weight, which we prove now.

Proposition 8.3.4. *V has a unique highest weight.*

Proof. Existence of a highest weight is obvious by finite dimensionality. Suppose that λ' were another highest weight, then the transitivity condition implies that $w^*\lambda + w^*\rho - \rho = \lambda'$

for some $w \in W$ and so $w^*\lambda = \rho - w^*\rho + \lambda' = \eta(w) + \lambda'$, again by Lemma 8.3.3. Since λ' is a highest weight and $n_{w^*\lambda} = n_\lambda > 0$ by Proposition 6.1.1 and so $n_{\eta(w)+\lambda'} > 0$. Again, we have $\eta(w) = 0$ and so $w = 1$, hence $\lambda = \lambda'$. $\square$

So, to recapitulate we have proven that there is a unique highest weight λ such that

$$\psi_V(t) = \sum_{\lambda \in \widehat{\Lambda}} p_\lambda t^\lambda = \sum_{w \in W} \det w \, t^{w\star\lambda}$$

Let us look at a trivial example to tell us what the Weyl character formula says for $SU(2)$. For an irreducible representation V, there is a highest weight l. Note that $l \in \mathbb{Z}$ and since $W(SU(2)) = \mathbb{Z}_2$ we have for $\theta \notin \pi\mathbb{Z}$

$$\chi_V \begin{pmatrix} e^{i\theta} & \\ & e^{-i\theta} \end{pmatrix} = \frac{-e^{(-k-2)i\theta} + e^{ik\theta}}{1 - e^{-2i\theta}} = \frac{\sin(k+1)\theta}{\sin\theta}.$$

In other words, the Weyl character formula trivialises the pages of computations for $SU(2)$.

There is a converse to the Weyl character formula, namely Cartan's theorem. It will turn out that the weights that come out of the Weyl character formula are *dominant*, a concept that will be defined when we prove Cartan's theorem.

8.4 Miscellaneous Formulae

Let us get a nice formula for that δ function that has been so helpful.

Corollary 8.4.1 (Weyl Denominator).

$$\boxed{\delta_C(t) = \sum_{w \in W} (\det w) t^{w^*(\rho) - \rho}, \quad t \in T_{\text{reg}}.}$$

Proof. We note that if V is the trivial representation, then $\chi_V(t) = 1$ and so since the trivial representation has highest 0, applying the Weyl character formula from Theorem 8.3.1 implies

$$1 = \frac{\sum_{w \in W} (\det w) t^{w^*(\rho) - \rho}}{\delta_C(t)} \quad t \in T_{\text{reg}},$$

which is obviously equivalent to the result. □

Observe for the $SU(2)$ case, $1 \notin T_{\text{reg}}$ and so to calculate $\chi_V(1) = \dim V$ we would have to use L'Hopital's rule. In fact, this is how the procedure is carried out in general.

Corollary 8.4.2 (Weyl's Dimension Formula). *If V is an irreducible represention of highest weight λ then*

$$\boxed{\dim V = \prod_{\alpha \in \Phi^+(C)} \frac{(\alpha, \lambda + \rho)}{(\alpha, \rho)}.}$$

Proof. This proof is standard, one just carefully applies L'Hopital's rule, see [9] p. 257 for the details. □

Let us make use of this formula in the following

Example 8.4.1 ($U(3)$). For a highest weight λ the dimension of the highest weight module λ (if it exists) would be

$$\frac{(1,0,0)\cdot(\lambda_1+1,\lambda_2+1,\lambda_3)}{(1,0,0)\cdot(1,1,0)} \times \frac{(0,1,0)\cdot(\lambda_1+1,\lambda_2+1,\lambda_3)}{(0,1,0)\cdot(1,1,0)} \times \frac{(1,1,0)\cdot(\lambda_1+1,\lambda_2+1,\lambda_3)}{(1,1,0)\cdot(1,1,0)} = \frac{1}{2}(\lambda_1+1)(\lambda_2+1)(\lambda_1+\lambda_2+2).$$

8.5 Cartan's Theorem

As mentioned before, Cartan's theorem is a converse to the Weyl character formula. The proof that is provided is non-constructive and uses the Peter-Weyl theorem. Constructing representation spaces is very difficult in general, as you are actually classifying all closed subgroups of matrices isomorphic to a given Lie group. There are a number of models, the standard ones being the Borel-Weil theorem (see [31] ch. 3) and the construction using Verma modules ([20] ch. V). The former model is essentially a statement in algebraic geometry, a very eloquent result. The latter is an algebraic construction involving quotients of the universal enveloping algebra. Another construction, worthy of mention is the Gel'fand-Cetlin method; this model is combinatorial, the application to $GL(n, \mathbb{C})$ can be found in [34] ch. X.

From this slight digression let us return to looking at Cartan's theorem.

Definition 8.5.1 (Dominant Weight). A weight $\lambda \in \widehat{\Lambda}$ is said to be **dominant** if $\lambda(C) \geq 0$. That is, $H_\lambda \in \overline{C}$.

Example 8.5.1 ($SU(2)$). Since $\overline{C} = \{\boldsymbol{\theta} \in \mathbb{R} : \theta \geq 0\}$ we conclude that a dominant weight $\lambda \in \mathbb{N}$.

Cartan's theorem proves that for each dominant weight there is an irreducible representation. I will follow [9] pp. 258-262 for this section. We are required to make the following

Definition 8.5.2 (Ladder). Given $\lambda \in \widehat{\Lambda}$ and $\alpha \in \Phi$ define the α-**ladder** from λ to $\sigma_\alpha(\lambda)$ to be

$$\{\lambda - k\alpha \,:\, k \in [0, \lambda(\alpha^\vee)] \cap \mathbb{Z}\},$$

if $\lambda(\alpha^\vee) \geq 0$ and the same set if $\lambda(\alpha^\vee) \leq 0$ with α replaced by $-\alpha$.

Lemma 8.5.1. *Let $\lambda \in \Phi(V, T)$, then for any $\alpha \in \Phi$, the α-ladder of λ to $\sigma_\alpha(\lambda)$ consists of weights of V.*

Proof. Omitted, see [9] p. 259. ☐

The next proposition proves that the highest weights that are constructed in the Weyl character formula are all dominant.

Proposition 8.5.1. *Suppose that $V \in \widehat{G}$, then the highest weight of V is dominant.*

Proof. Let $\alpha \in \Phi^+(C)$, then it suffices to prove that $(\lambda, \alpha) \geq 0$. For if not, then $(\lambda, \alpha) < 0$ and so by Lemma 7.2.2, $\lambda + \alpha \in \Phi$. If $\lambda + \alpha \in \Phi(V, T)$, then since λ is a highest weight,

$\alpha \in \Phi^-(C)$, which is a contradiction. If $\lambda + \alpha \notin \Phi(V, T)$, then this contradicts Lemma 8.5.1. □

Here is the result we have been waiting for.

Theorem 8.5.1 (Cartan). *Let λ be a dominant weight, then there is an irreducible G-module V with highest weight vector λ and the character χ_V as in Theorem 8.3.1.*

Proof. Let $\chi_\lambda : T_{\mathrm{reg}} \to \mathbb{C}$ be the function

$$\chi_\lambda(t) = \frac{\sum_{w\in W} (\det w)\, t^{w^*\lambda + (w^*\rho_C - \rho_C)}}{\delta_C(t)}.$$

A straightforward computation proves that χ_λ is a W-invariant, continuous function on T_{reg} (see [9] p. 261 for the details). Using the map Ψ_{reg} constructed earlier in this chapter we can uniquely extend χ_λ to a continuous class function on G_{reg} and so G (c.f. Proposition 6.3.2). By Corollary 8.1.1 we also have (we are using the extension now)

$$\int_G \chi_\lambda(g)\overline{\chi_\lambda(g)}\, \mathrm{d}\mu(g) = \frac{1}{|W|}\int_T \left| \sum_{w\in W} (\det w)\, t^{w^*\lambda + (w^*\rho_C - \rho_C)} \right|^2 \mathrm{d}\mu(t) = 1,$$

by the orthonormality of the t^μ's.

Moreover, if V is some other irreducible G-module, by Proposition 8.3.4 it has a unique highest weight ξ. If $\lambda = \xi$, then we are done, otherwise we note that if $w^*(\xi + \rho) - \rho = \lambda$ or the same with λ swapped with ξ, the calculation provided in Proposition 8.3.4 would demonstrate that $w = 1$ and $\xi = \lambda$, a contradiction. By Proposition 8.3.3, the shifted W-action is free and transitive on $\Phi(V, T)$ and by definition of χ_λ the shifted W-action is free on λ and so the shifted W-orbits are disjoint. So,

$$\int_G \chi_\lambda(g)\, \overline{\chi_V(g)}\, \mathrm{d}\mu(g) = 0$$

By Theorem 4.3.1, this implies that $\chi_\lambda = 0 \quad \mu - a.e.$ which is impossible. So χ_λ is the character of some representation of highest weight λ. $\square$

Corollary 8.5.1. *There is a one-to-one correspondence:*

$$\boxed{\{Dominant\ Weights\} \longleftrightarrow \widehat{G}}$$

Part IV

The Cayley Transform

Chapter 9

The Cayley map $z \mapsto \frac{1+z}{1-z}$ is well known to be a conformal mapping of the right half plane into the unit disc. For this reason, it is really a perfect mapping of an infinite line about a circle. Weyl was fascinated by this map and its appropriate generalisations to Lie theory, making use of it a number of times in [33]. This is a good enough reason to want to understand this map better, but it is a map that makes reappearances too.

Recent work by Kostant[22] introduced the notion of a generalised Cayley map for an algebraic group, proving a raft of results analogous to properties that the classical map possesses, and a great deal more. The work that we will be looking at is theoretically related to [25], the derivations are very different and arose out of correspondences with the author's supervisor, Norman Wildberger.

9.1 Preliminaries

Following [33] p. 56 we make the following

Definition 9.1.1. The **exceptional matrices** is the set of matrices V_n

$$V = \{A \in M_n(\mathbb{C}) \,:\, \det(A + I) \neq 0\}.$$

The computations for this chapter will be concerned with $U(n)$ and $SU(2)$. Consequently, the following result is fundamental.

Lemma 9.1.1.

$$\mathfrak{u}(n) \subset V.$$

Proof. We recall the fact from linear algebra: if A is skew-hermitian then A is diagonalizable and has all imaginary eigenvalues. It is now a matter of remarking that since A has all imaginary eigenvalues, then $\det(A+I) = \det(iD+I)$ (where $A \sim iD$ and D is real diagonal) and so will be non-zero. $\square$

So, we adopt some more

Notation 9.1.1. Let $\mathfrak{g} = \mathfrak{u}(n)$ and $G' = U(n) \cap V$.

We are now in position to define the Cayley map for our purposes. The former results were just to make sure that the map works properly.

Definition 9.1.2 (Cayley Map). The **Cayley Map** is the map $\Phi_n : \mathfrak{g} \to G' : A \mapsto (A+I)(-A+I)^{-1}$.

We need to prove that the map is well-defined and invertible.

Proposition 9.1.1. *Φ_n is a bijection from $\mathfrak{g}$ to G'.*

Proof. Suppose that $A \in \mathfrak{g}$, then

$$\begin{aligned}
\Phi_n(A)^* &= [(A+I)(A^*+I)^{-1}]^* \\
&= (A+I)^{-1}(A^*+I) \\
&= [(A^*+I)^{-1}(A+I)]^{-1} \\
&= \Phi_n(A)^{-1}.
\end{aligned}$$

Hence, $\Phi_n(A) \in U(n)$. Moreover, notice that

$$\begin{aligned}
\det(\Phi_n(A)+I) &= \det([A+I][A^*+I]^{-1}+I) \\
&= \det(A+I+A^*+I)\det(A^*+I)^{-1} \\
&= \det(2I)\det(A^*+I)^{-1}.
\end{aligned}$$

So, $\Phi_n(A) \in V$ too.

To show that Φ_n is a bijection we simply prove that the inverse map $\Phi_n^{-1} : G' \to \mathfrak{g}' : Z \mapsto (Z - I)(Z + I)^{-1}$ has a well-defined image.

$$\begin{aligned}
\Phi_n^{-1}(Z)^* &= ([Z - I][Z + I]^{-1})^* \\
&= [Z^* + I]^{-1}[Z^* - I] \\
&= [Z^* + I]^{-1}(Z^* Z)[Z^* - I] \\
&= [Z^* + I]^{-1} Z^{-1}[ZZ^* - Z] \\
&= [Z(Z^* + I)]^{-1}[I - Z] \\
&= [Z + I]^{-1}[I - Z] \\
&= -\Phi_n^{-1}(Z).
\end{aligned}$$

So, Φ_n is a bijection. □

Corollary 9.1.1. *Let $A \in \mathfrak{g}'$ then $A\Phi_n(A) = \Phi_n(A)A$.*

Proposition 9.1.2. *Φ_n maps $Ad(G)$-orbits to G-orbits.*

Proof. It suffices to prove that $\Phi_n(gAg^{-1}) = g\Phi_n(A)g^{-1}$, for all $g \in U(n)$ and $A \in \mathfrak{g}$.

$$\begin{aligned}
\Phi_n(gAg^{-1}) &= (gAg^{-1} + I)(gA^* g^{-1} + I)^{-1} \\
&= g(A + I)g^{-1} g(A^* + I)^{-1} g^{-1} \\
&= g\Phi_n(A)g^{-1}.
\end{aligned}$$

□

Notation 9.1.2. For $a_1, \ldots, a_n \in \mathbb{R}$ we let $[a_j] = \bigoplus_{i=1}^n (a_i)$.

Lemma 9.1.2. *Let $d_1, \ldots, d_n \in \mathbb{R}$ then $\Phi_n([id_j]) = [\Phi_1(id_j)]$. In particular, Φ_n maps the Lie algebra of the maximal torus to the maximal torus of G.*

Proof.

$$\begin{aligned}\Phi_n([id_j]) &= ([id_j] + [1])\,(-[id_j] + [1])^{-1} \\ &= [1 + id_j][(1 - id_j)^{-1}] \\ &= [\Phi_1(id_j)].\end{aligned}$$

□

The next proposition is very cute. An equivalent result appears in [33] p. 62.

Proposition 9.1.3 (Weyl). *Let A, $B \in \mathfrak{g}$ commute, then*

$$\Phi_n(A)\Phi_n(B) = \Phi_n\left(\frac{A+B}{I+AB}\right),$$

provided AB is exceptional.

Proof. Everything commutes, so we can deal with this like the scalar case.

$$\begin{aligned}\Phi_n(A)\Phi_n(B) &= \frac{A+I}{-A+I}\frac{B+I}{-B+I} \\ &= \frac{AB+I+A+B}{AB+I-A-B} \\ &= \frac{\left(\frac{A+B}{I+AB}\right)+I}{-\left(\frac{A+B}{I+AB}\right)+I} \\ &= \Phi_n\left(\frac{A+B}{I+AB}\right).\end{aligned}$$

□

Remark 9.1.1. The proposition looks like the addition of velocity law in special relativity.

We now examine the restriction of the Cayley map to $SU(2)$ in detail.

9.2 Restriction to $SU(2)$

Note that, since $\mathfrak{su}(2) \subset \mathfrak{u}(2)$ then all elements of $\mathfrak{su}(2)$ are exceptional. Moreover, since $\det \Phi_2(X) = \det(X+I)\overline{\det(X+I)}^{-1}$ then the above calculation proves that $\det \Phi_2(X) = 1$ for all $X \in \mathfrak{su}(2)$. Hence, the image of Φ_2 restricted to $\mathfrak{su}(2)$ is contained in $SU(2)$.

It suffices to prove that the inverse of Φ_2, Φ_2^{-1}, restricted to $SU(2)' := SU(2) \cap V$ all have trace 0. Note that (clearly) $SU(2)' = SU(2) \setminus \{-I\}$. Also, for $u \in SU(2)'$ we can write

$$u = \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix},$$

where $|\alpha|^2 + |\beta|^2 = 1$. Hence,

$$\begin{aligned}
(u-I)(u+I)^{-1} &= \begin{pmatrix} \alpha - 1 & \beta \\ -\overline{\beta} & \overline{\alpha} - 1 \end{pmatrix} \begin{pmatrix} \alpha + 1 & \beta \\ -\overline{\beta} & \overline{\alpha} + 1 \end{pmatrix}^{-1} \\
&= \frac{1}{2(\Re\alpha + 1)} \begin{pmatrix} \alpha - 1 & \beta \\ -\overline{\beta} & \overline{\alpha} - 1 \end{pmatrix} \begin{pmatrix} \overline{\alpha} + 1 & -\beta \\ \overline{\beta} & \alpha + 1 \end{pmatrix}^{-1} \\
&= \frac{1}{2(\Re\alpha + 1)} \begin{pmatrix} \alpha - \overline{\alpha} & 2\beta \\ -2\overline{\beta} & \overline{\alpha} - \alpha \end{pmatrix}.
\end{aligned}$$

From which it is clear that $\operatorname{tr} \Phi_2^{-1}(u) = 0$.

For completeness, we now compute the image of $X \in \mathfrak{su}(2)$ under Φ_2. Some notation, we let $\|X\| = t^2 + v^2 + w^2$.

$$\begin{aligned}
\Phi(X) &= \begin{pmatrix} 1+it & x+iy \\ -x+iy & 1-it \end{pmatrix} \begin{pmatrix} 1-it & -x-iy \\ x-iy & 1+it \end{pmatrix}^{-1} \\
&= \frac{1}{1+\|X\|^2} \begin{pmatrix} 1+it & x+iy \\ -x+iy & 1-it \end{pmatrix} \begin{pmatrix} 1+it & x+iy \\ -x+iy & 1-it \end{pmatrix} \\
&= \begin{pmatrix} \frac{1-\|X\|^2}{1+\|X\|^2} + \frac{2it}{1+\|X\|^2} & \frac{2(x+iy)}{1+\|X\|^2} \\ \frac{2(-x+iy)}{1+\|X\|^2} & \frac{1-\|X\|^2}{1+\|X\|^2} - \frac{2it}{1+\|X\|^2} \end{pmatrix}
\end{aligned}$$

9.3 Stereographic Projection

Let us try and realise the Cayley map for $SU(2)$ in a more geometric sense. Observe that since Φ_2 takes Ad orbits to conjugation orbits then it suffices to work with diagonal elements of $\mathfrak{su}(2)$. So, let $T = \left(\begin{smallmatrix} it & \\ & -it \end{smallmatrix}\right)$ and then notice that if $\lambda = \frac{1+t^2}{2}$ then

$$\begin{aligned}
\lambda(\Phi(T) + I) &= \frac{1}{2} \begin{pmatrix} (1+it)^2 + 1 + t^2 & \\ & (1-it)^2 + 1 + t^2 \end{pmatrix} \\
&= \frac{1}{2} \begin{pmatrix} 2+2it & \\ & 2-2it \end{pmatrix} \\
&= T + I.
\end{aligned}$$

So $\Phi(T)$, T and $-I$ are collinear. Since $SU(2)$ naturally identified as S^3 and $\mathfrak{g}$ is a vector space, then the Cayley map is stereographic projection from the "south pole".

Remark 9.3.1. Unfortunately, this result does not seem to generalise that well. In fact, for $U(n)$ the result seems to be that $\Phi(X)$, X and $-I$ are collinear if and only if X^2 is a diagonal matrix.

9.4 An Integration formula

The aim of this section is to relate Haar measure on $SU(2)$ with ordinary Lebesgue measure on $\mathfrak{su}(2) \cong \mathbb{R}^3$; via a rational transformation. This is a special case of a general result, that appears in [25] (Lemma 3.11). The result is obvious once we observe that $|\det(1-\boldsymbol{X})|^{-3} = (1+\|\boldsymbol{X}\|^2)^3$.

Proposition 9.4.1. *Let $f \in L^1(SU(2))$ then*

$$\int_{SU(2)} f(g)\,\mathrm{d}g = \frac{8}{\pi}\int_{\mathbb{R}^3} \frac{f \circ \Phi_2(\boldsymbol{X})}{(1+\|\boldsymbol{X}\|^2)^3}\,\mathrm{d}\boldsymbol{X}$$

9.5 Lifted Characters

By remarking that the Cayley transform commutes with conjugation, and the characters of $SU(2)$ are class functions, then it is natural to ask what the characters look like under a lift onto $\mathfrak{su}(2)$

This is quite interesting, since we would then have a complete orthonormal sequence on the space of L^2 functions, which only depend on the value that they take in $\mathfrak{t}/W$. Note that for the one-dimensional case we have

$$\Phi(it) = \frac{1+it}{1-it} = e^{i\theta(t)}.$$

So, if $X \in \mathfrak{su}(2)$ (with the parametrisation given earlier) then a trivial calculation shows that the eigenvalues of X are $\pm i\|X\|$. This has a two fold implication, the conjugacy classes of $\mathfrak{su}(2)$ (and so $SU(2)$) are parametrised by the spheres centred at the origin; the other is that the lifted character $\tilde{\chi}_\ell : \mathfrak{su}(2) \ni X \mapsto \chi \circ \Phi_2(X) \in \mathbb{R}$ is a radial function.

Hence,

$$
\begin{aligned}
\tilde{\chi}_\ell(X) &= \tilde{\chi}_\ell \begin{pmatrix} i\|X\| & 0 \\ 0 & -i\|X\| \end{pmatrix} \\
&= \frac{\Phi(i\|X\|)^{2\ell+1} - \Phi(-i\|X\|)^{2\ell+1}}{\Phi(i\|X\|) - \Phi(-i\|X\|)} \\
&= \frac{\left(\frac{1+i\|X\|}{1-i\|X\|}\right)^{2\ell+1} - \left(\frac{1-i\|X\|}{1+i\|X\|}\right)^{2\ell+1}}{\frac{1+i\|X\|}{1-i\|X\|} - \frac{1-i\|X\|}{1+i\|X\|}} \\
&= \frac{\frac{(1+i\|X\|)^{4\ell+2}-(1-i\|X\|)^{4\ell+2}}{(1+\|X\|^2)^{2\ell+1}}}{\frac{(1+i\|X\|)^2-(1-i\|X\|)^2}{1+\|X\|^2}} \\
&= \frac{2i\Im(1+i\|X\|)^{4\ell+2}}{4i\|X\|(1+\|X\|^2)^{2\ell}} \\
&= \frac{\sum_{k=0}^{2\ell+1}(-1)^k\binom{4\ell+2}{2k+1}\|X\|^{2k}}{2(1+\|X\|^2)^{2\ell}}.
\end{aligned}
$$

Note that we now have a complete orthonormal system of radial, rational functions on $\mathbb{R}^3$ with respect to the measure

$$
\frac{8\,\mathrm{d}\boldsymbol{X}}{\pi(1+\|\boldsymbol{X}\|^2)^3}.
$$

9.6 Orthogonal Rational Functions

For $j \in \mathbb{N}$ let

$$
\kappa_j : \mathbb{R}^+ \to \mathbb{R} : x \mapsto \chi_{j/2}(x, 0, 0).
$$

Then,

$$
\kappa_j(x) = \frac{\sum_{k=0}^{j+1}(-1)^k\binom{2j+2}{2k+1}x^{2k}}{2(1+x^2)^j}.
$$

In particular, $\{\kappa_j\}_{j\in\mathbb{N}}$ is a complete orthonormal system on $L^2_\nu(\mathbb{R}^+)$, where

$$
\mathrm{d}\nu = \frac{16x^2\,\mathrm{d}x}{(1+x^2)^3},
$$

and $\mathrm{d}x$ is Lebesgue measure on $\mathbb{R}$. These functions are clearly related to the Jacobi polynomials, as the χ_ℓ are; but this sequence of rational functions has not been studied before. A graph is provided in Figure 9.6 for the first few functions.

9.6.1 A Recurrence Relation

Note that Theorem 3.3.1 informs us that for $\ell \geq 1/2$

$$\chi_\ell \chi_{1/2} = \chi_{\ell-1/2} + \chi_{\ell+1/2},$$

so we infer that

$$2\kappa_j(x)(1-x^2) = (1+x^2)(\kappa_{j-1}(x) + \kappa_{j+1}(x)),$$

with the initial condition $\kappa_0(x) = 1$.

9.6.2 A Differential Equation

Using the theory of Jacobi polynomials (see [27]), one can show

$$4\kappa_j''(x) + j(j+2)(1+x^2)^2\kappa_j(x) = 0.$$

This is just an exercise in patience using the chain rule.

9.7 Lifted Matrix Coefficients of $SU(2)$

This calculation is straightforward, but the simplification is brutal; it is omitted for the reader's pleasure. Using the matrix coefficient formulae developed earlier, one eventually gets the **lifted matrix coefficients** $\tilde{\pi}_\ell^{ij} : \mathfrak{su}(2) \to \mathbb{C}$ where $i,\, j \in \{-\ell, \ldots, \ell\}$.

$$\tilde{\pi}_\ell^{ij}(X) = \frac{\sum_{s=-\ell}^{i}(-1)^{j-s}2^{i+j-2s}\binom{\ell+j}{\ell+s}\binom{\ell-j}{i-s}([1-\|X\|^2]^2+4t^2)^{\ell+i+j}(\|X\|^2-t^2)^{j-s}(1-\|X\|^2+2it)^{i+/j}(x+iy)^{i-j}}{(1+\|X\|^2)^{2\ell}}$$

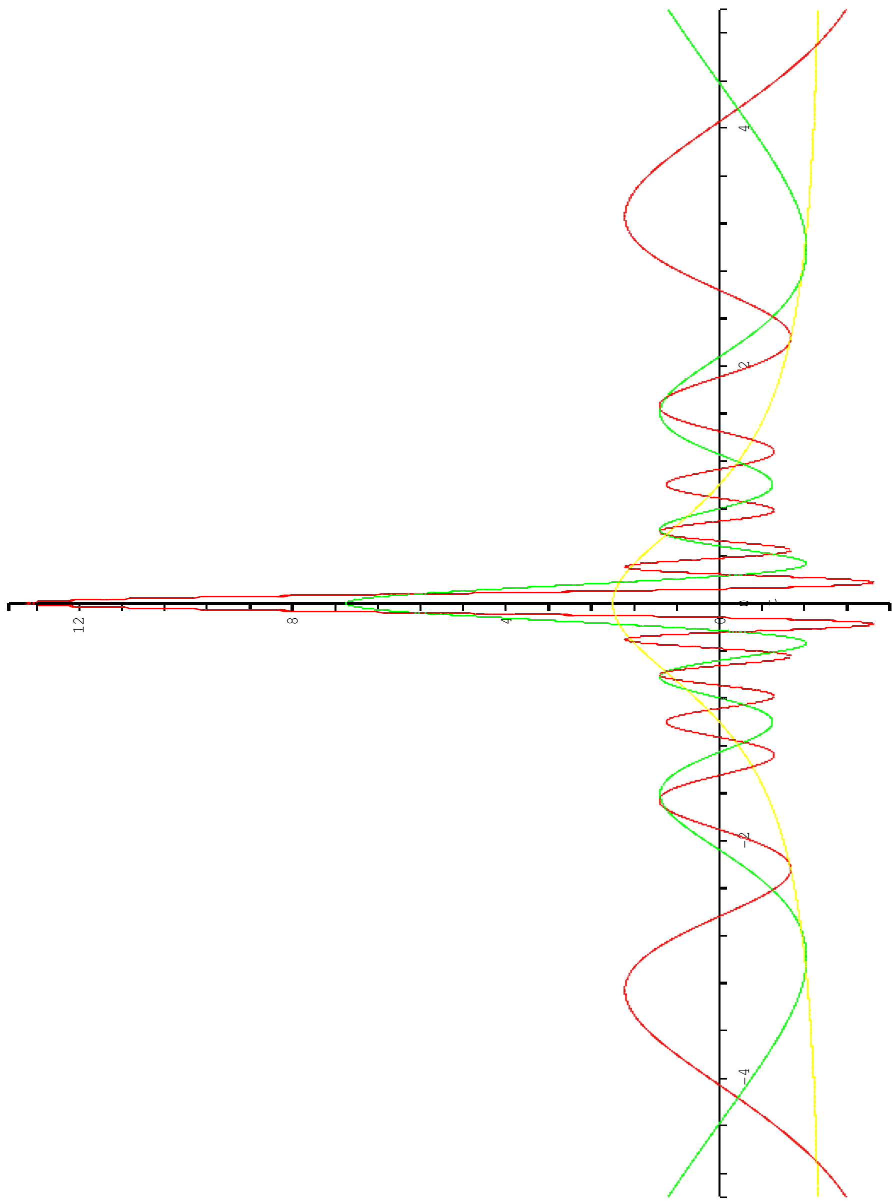

Figure 9.1: κ_j

Chapter 10

Conclusion

The character theory of compact Lie groups is one of the most beautiful in mathematics, as it is completely described by the Weyl character formula. The aim of this thesis was to show that this theory could be completed within the realms of real analysis; something that is not often done anymore.[1]

A Lie group is a very beautiful structure in its own right and that beauty truly captured in the geometry of the Stiefel diagram. It is not surprising that this features prominently in the Weyl character formula.

By far and way the most beautiful thing about Lie groups is that with all the structure that they have, character theory is reduced to a purely combinatorial analysis of the roots.

Thank you for taking the time to read my thesis.

[1] Adams [2] does his study using algebraic topology, in a somewhat similar flavour to real analytic methods.

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