It is seen that the terms $2\kappa\Omega_r/r + \Lambda$ in (A2) and $\kappa(\Omega_{rr} + \Omega_r/r) + \Lambda$ in (A3) are of magnitude $c^2$ times that of any other term in the equations. For a Newtonian approximation to be possible, these terms must vanish identically. Hence, it follows that

$$2\kappa\Omega_r/r + \Lambda = 0,$$
$$\kappa(\Omega_{rr} + \Omega_r/r) + \Lambda = 0.$$

Both equations are satisfied by the solution

$$\Omega = -\tfrac{1}{4}\Lambda r^2/\kappa, \qquad \text{(A6)}$$

which determines the function $\Omega$ to be of a form in agreement with the requirements for the metric.

A Newtonian approximation is now carried out in which $c$ is identified with an infinite velocity. The coordinate time $t$ then becomes the Newtonian absolute time $T$ and the coordinate $r$ becomes the radial distance from the origin in absolute Euclidean space. The fluid velocity $q$ is now defined by $q = u^1/u^4$.

The approximation of (A1) gives $u^4 = 1$. It is assumed that velocities are small, $q^2 \ll c^2$. The Newtonian forms of (A2) to (A5) then become

$$\rho q^2 + p = -\psi_{TT} + \Lambda\psi - \tfrac{1}{2}\Lambda r\psi_r - 2\pi G\psi_r^2, \qquad \text{(A7)}$$

$$p = -\psi_{TT} + \Lambda\psi + \tfrac{1}{2}\Lambda r\psi_r + 2\pi G\psi_r^2 + 2\pi G r^2\left(\frac{\Lambda}{8\pi G}\right)^2, \qquad \text{(A8)}$$

$$\rho = -\psi_{rr} - \frac{2\psi_r}{r} - \frac{\Lambda}{8\pi G}, \qquad \text{(A9)}$$

$$\rho q = \psi_{rT}. \qquad \text{(A10)}$$

The constants $n_1$ and $n_2$ can also be determined by solving Eq. (A9). The solution is found to be

$$\psi = -\tfrac{1}{6}\rho r^2 - \frac{1}{12}\frac{L}{G}r^2,$$

in accordance with the previous results.

JOURNAL OF MATHEMATICAL PHYSICS VOLUME 10, NUMBER 4 APRIL 1969

# Algebraic Theory of Ray Representations of Finite Groups*

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(Received 5 March 1968)

A theory of characters of ray representations of finite groups, that does not use any reference to a covering group, is derived by defining two generalizations of the concept of a group's class. Orthogonality relations are obtained over one of these generalized classes. This theory is used to discuss subduction and induction of ray representations while the Frobenius reciprocity theorem and generalizations thereof are proved. The theory provides a more efficient method of deriving and treating ray representations of finite groups for a given factor system than has previously been made available.

## 1. INTRODUCTION

Since ray representations of point groups have become useful in the theory of nonsymmorphic space groups,[1] methods of obtaining them have been given by Döring.[2] Tables of them for particular factor systems have been tabulated for the 32 point groups. The theory of double groups is really a special case of ray representation theory and tables of these have long been available for the point groups.[3] Recently this theory was extended to space groups.[4] Most of these treatments build on those given by Schur in his three papers.[5]

In all these theories, one deals with a so-called covering group whose order is, in general, some multiple of the order of the group which is being ray-represented. (Usually it is twice as large.) The following theory will show that this is unnecessarily laborious.

* Research supported by Number 9-443984-21112-3, NSF-GP No. 5844.

[1] G. Lyabarski, *The Application of Group Theory in Physics* (Pergamon Press, Inc., New York, 1960), p. 95.

[2] Z. Döring, Z. Naturforsch. **14**, 343 (1959).

[3] W. Opechowski, Physica **7**, 552 (1940).

[4] M. Glück, Y. Gur, and J. Zak, J. Math. Phys. **8**, 787 (1967).

[5] I. Schur, J. Reine Angew. Math. **127**, 20 (1904); **132**, 85 (1907); **139**, 155 (1911).

It will be shown how one may quickly construct and use a concise square character table and a set of representations straightaway from any given factor system. In short, the following theory will make it possible for one to use already tabulated results more easily and quickly to obtain irreducible ray representations for other finite groups or factor systems which might be important in other areas of interest.[6] But what is most important, an understanding of the possible structures of these mathematical objects is obtained systematically; this in turn sheds more light on the structure of representations of groups themselves.

A systematic treatment of ray representations of groups was begun by Rudra in a series of three papers.[7] Unfortunately, many of his results are not valid for factor systems that are not equivalent to the trivial one. In order to include the nontrivial cases, it will be necessary to begin the following treatment by starting with the basic definitions of a ray representation.

## 2. BASIC DEFINITIONS

If one nonsingular $n$ by $n$ matrix $\mathcal{R}(R)$ is assigned to each element $R$ of a group $\mathcal{G} = \{\cdots R, S, T, \cdots\}$ such that

$$\mathcal{R}(R)\mathcal{R}(S) = \omega_{R,S}\mathcal{R}(RS)$$

(where $\omega_{R,S}$ = complex number) for all $R$ and $S$ in $\mathcal{G}$, then that set $\{, \mathcal{R}(R), \mathcal{R}(S), \mathcal{R}(T), \}$ of matrices is called an $n$th *degree ray representation* of $\mathcal{G}$, and the constants $\omega_{RS}$ make up what is called a *factor system* $\{\omega\}$ over $\mathcal{G}$.

Now if all the matrices of this ray representation are multiplied by different constants, the resulting set of matrices $\{\mathcal{R}'(R) \equiv C_R\mathcal{R}(R), \mathcal{R}'(S) \equiv C_S\mathcal{R}(S), \cdots\}$ is also a ray representation but with a new factor system $\{\omega'\}$ as is shown below:

$$\begin{aligned}\mathcal{R}'(R)\mathcal{R}'(S) &= C_RC_S\mathcal{R}(R)\mathcal{R}(S)\\ &= (C_RC_S/C_{RS})\omega_{R,S}\mathcal{R}'(RS)\\ &\equiv \omega'_{R,S}\mathcal{R}'(RS),\end{aligned} \tag{1}$$

where $\omega'_{R,S} = (C_RC_S/C_{RS})\omega_{R,S}$. If two factor systems $\{\omega\}$ and $\{\omega'\}$ are related in this manner, they are said to be *projective equivalent* or *p-equivalent*[8] or in the same *class*.[9]

[6] A. O. Barut, J. Math. Phys. 7, 1908 (1966).

[7] P. Rudra, J. Math. Phys. 6, 1273, 1278 (1965); 7, 935 (1966).

[8] J. S. Lomont, *Applications of Finite Groups* (Academic Press, London, 1959), p. 729.

[9] M. Hamermesh, *Group Theory and Its Application to Physical Problems* (Addison-Wesley Publ. Co., Inc., Reading, Mass., 1962), p. 462.

If the $\mathcal{R}$ are $n \times n$ matrices, one may let $C_R$ equal any $n$th root of det $\mathcal{R}(R)$ for all $R$ in $\mathcal{G}$ and obtain the following:

$$\begin{aligned}\det[\mathcal{R}'(R)\mathcal{R}'(S)] &= \det[\omega'_{R,S}\mathcal{R}'(RS)],\\ C_R^n[\det\mathcal{R}(R)]C_S^n[\det\mathcal{R}(S)] &= (\omega'_{R,S})^nC_{RS}^n[\det\mathcal{R}'(RS)],\\ 1 &= (\omega'_{R,S})^n.\end{aligned} \tag{2}$$

Hence every factor system associated with an $n$th degree ray representation is $p$-equivalent to one in which all the factors are $n$th roots of unity; i.e., $(\omega'_{R,S})^n = 1$ and $\omega^*_{R,S}\omega_{R,S} = 1$ for all $R$ and $S$ in $\mathcal{G}$. If $n = 1$, then $\omega'_{R,S} = 1$ and $\{\omega\}$ is $p$-equivalent to the trivial factor system, which is sometimes called the *vector* factor system.

## 3. ASSOCIATIVITY AND THE REGULAR REPRESENTATION

It is helpful to imagine an abstract set of $g$ elements $\{a_R, a_S, a_T\}$ (one for each element $R$, $S$, or $T$ of a group $\mathcal{G}$) that obey the relations

$$a_Ra_S = \omega_{R,S}a_{RS}, \tag{3}$$

where the $\omega_{R,S}$ belong to a given factor system $\{\omega\}$. Such a collection forms what is defined as a ring or associative algebra. (But it is not a group unless $\omega_{R,S} \equiv 1$.) I call the set of all linear combinations of the elements $a_R$ (using complex coefficient) the *ray algebra* $\mathcal{A}(\mathcal{G}, \omega)$. The elements $a_R$ are here called a system of *base elements*, since any element in $\mathcal{A}(\mathcal{G}, \omega)$ can be written as a unique linear combination of them. Obviously there are other bases for which the same is true.

This definition is motivated by more general theories of rings and algebras. There the most general algebra $\mathcal{A}$ with $n$-base elements $\{a_1, a_2, a_3, \cdots, a_n\}$ has the following multiplicative structure:

$$a_ia_j = \sum_{k=1}^{k=n} c_{ij}^k a_k. \tag{4}$$

Here the complex numbers $c_{ij}^k$ are called structure constants and can be thought of as components of $n \times n$ matrices that will represent the algebra $\mathcal{A}$. For if one lets $c_{ij}^k = \mathcal{R}_{ik}(a_j)$, then the relations

$$\sum_{B=1}^{B=n} \mathcal{R}_{\alpha\beta}(a_i)\mathcal{R}_{\beta\gamma}(a_j) = \sum_{k=1}^{k=n} c_{ij}^k\mathcal{R}_{\alpha\gamma}(a_k) \tag{5}$$

follow directly from an expansion of the associativity relations $(a_ia_j)a_k = a_i(a_ja_n)$ which one demands of any ring or any set of matrices.

These $n$ $n$-by-$n$ matrices $\mathcal{R}(a_j)$ form what is called the *regular representation* of $\mathcal{A}$ in basis $\{a_j\}$. Now the regular representation of $\mathcal{A}(\mathcal{G}, \omega)$ in the basis $\{\cdots a_R, a_S \cdots\}$ will be made of $g$-by-$g$ matrices that have a very simple form. In particular, $\mathcal{R}_{RT}(a_S) = \omega_{R,S}$ if and only if $RS = T$. Otherwise $\mathcal{R}_{RT}(a_S) = 0$.

An easy way to construct these matrices is first to arrange a multiplication table of the base elements with the elements $a_R$ written in some order across the top, and the "inverses" $a_{R^{-1}}$ appearing in the same order down the side:

$$\begin{array}{c|ccccc}
 & \cdots & a_R & \cdots & a_T & \cdots \\
\hline
\vdots & & \vdots & & \vdots & \\
a_{R^{-1}} & \cdots & \omega_{R^{-1},R}a_1 & \cdots & \omega_{R^{-1},T}a_{R^{-1}T} & \cdots \\
\vdots & & \vdots & & \vdots & \\
a_{T^{-1}} & \cdots & \omega_{T^{-1},R}a_{T^{-1}R} & \cdots & \omega_{T^{-1},T}a_1 & \cdots \\
\vdots & & \vdots & & \vdots &
\end{array} \tag{6}$$

Then, since $\mathcal{R}_{RT}(a_S) = \omega_{R,S}$ if and only if $R^{-1}T = S$, you obtain matrices of the form

$$\mathcal{R}(a_S) = \left|\begin{array}{l}
\cdots\cdots\cdots\ \omega_{1,S}\ \cdots\cdots\cdots \\
\cdots\ \omega_{2,S}\ \cdots\cdots\cdots\cdots \\
\cdots\cdots\cdots\cdots\cdots\ \omega_{3,S}\ \cdots \\
\omega_{4,S}\ \cdots\cdots\cdots\cdots\cdots \\
\cdots\cdots\cdots \\
\cdots\cdots
\end{array}\right| \tag{7}$$

by simply inspecting the table. (Note that there is one and only one entry in each column and row.) Incidentally, the associativity of multiplication implies that

$$(a_R\, a_S)a_T = a_R(a_S a_T),$$

$$\omega_{R,S}\omega_{RS,T}a_{RST} = \omega_{S,T}\omega_{R,ST}a_{\omega ST},$$

or

$$\omega_{R,S} = \frac{\omega_{R,ST}\omega_{S,T}}{\omega_{RS,T}} \tag{8}$$

for all $R$, $S$, and $T$ in $\mathcal{G}$.

The regular representation of the unit element 1 obtained by the preceding construction is

$$\mathcal{R}(a_1) = \left|\begin{array}{cccc}
\omega_{11} & 0 & 0 & \cdots \\
0 & \omega_{21} & 0 & \cdots \\
0 & 0 & \omega_{31} & \cdots
\end{array}\right|. \tag{9}$$

But the associativity relations demand that

$$\omega_{R,1} = \frac{\omega_{R,1}\omega_{1,1}}{\omega_{R1}} = \omega_{1,1} \tag{10}$$

for all $R$ in $\mathcal{G}$. (Similarly $\omega_{1,S} = \omega_{1,1}$.) Clearly no loss of generality will occur if we assume from now on that

$$\omega_{R,1} = \omega_{1,1} = \omega_{1,S} = 1, \tag{11}$$

so that $\mathcal{R}(a_1)$ is a unit matrix.

Now the trace of $\mathcal{R}(a_1)$ is $g$, the order of group $\mathcal{G}$. From this one sees that $\omega_{R^{-1},R} = \omega_{R,R^{-1}}$, since

$$\operatorname{Tr}\mathcal{R}(a_R)\mathcal{R}(a_{R^{-1}}) = \operatorname{Tr}\mathcal{R}(a_{R^{-1}})\mathcal{R}(a_R) \tag{12}$$

implies that

$$\omega_{R,R^{-1}}\operatorname{Tr}\mathcal{R}(a_1) = \omega_{R^{-1},R}\operatorname{Tr}\mathcal{R}(a_1). \tag{13}$$

Note, however, that the trace of $\mathcal{R}(a_S)$ (for $S \neq 1$) is zero.

Finally, it is possible to show that each matrix $\mathcal{R}(a_S)$ is unitary. Since $\mathcal{R}_{RT}(a_S) = \omega_{R,S}$ (assuming now that $RS = T$), one has that $\mathcal{R}_{TR}(a_{S^{-1}}) = \omega_{T,S^{-1}}$. Now the inverse of matrix $\mathcal{R}(a_S)$ is

$$[\mathcal{R}(a_S)]^{-1} = (\omega_{S^{-1},S})^{-1}\mathcal{R}(a_{S^{-1}}),$$

while the Hermitian conjugate of the matrix $\mathcal{R}(a_S)$ is $\mathcal{R}^{+}(a_S)$, where

$$\begin{aligned}
\mathcal{R}^{+}_{TR}(a_S) = \omega^{*}_{R,S} = \frac{1}{\omega_{R,S}} &= \omega_{RS,S^{-1}}/(\omega_{R,S}\omega_{RS,S^{-1}}) \\
&= \omega_{RS,S^{-1}}/(\omega_{R,SS^{-1}}\omega_{S,S^{-1}}) \\
&= \omega_{R,S^{-1}}/\omega_{S,S^{-1}} \\
&= \mathcal{R}_{RT}(a_{S^{-1}})/\omega_{S,S^{-1}} \\
&= [\mathcal{R}_{TR}(a_S)]^{-1}.
\end{aligned} \tag{14}$$

The associativity relations (8) have been used along with the conventions (11) $\omega_{R,1} = 1$ and (2)

$$\omega_{R,S}^{*}\omega_{R,S} = 1.$$

## 4. NILPOTENTS, IDEMPOTENTS, AND OTHER ALGEBRAIC CONCEPTS

In this section it is necessary to make some statements without proof, since such proofs are still quite lengthy. The references (10, 11, 12) which contain these proofs are concerned with a type of algebra or ring, of which the ray algebras are a special case.

[10] H. V. McIntosh, "Abelian Groups with Operators," RIAS Technical Report 57-2, 1958.

[11] C. L. Curtis and I. Reiner, *Representation Theory of Finite Groups and Associative Algebras* (Interscience Publishers, Inc., New York, 1962).

[12] W. G. Harter, Doctoral dissertation, University of California at Irvine, 1967.

It is convenient to define a *contagiously nilpotent element* $\eta$ of algebra $\mathcal{A} = \{\cdots a, b, c, \cdots\}$ to be a nonzero element that transforms every element $a$ of $\mathcal{A}$ into a nilpotent $n = \eta a$ or into zero $\eta a = 0$. (A nilpotent $n$ is a nonzero element that goes to zero if raised to some power, i.e., $n^k = nnn \cdots n = 0$, for some $k = 2, 3, 4, \cdots$.) If it can be shown that contagiously nilpotent elements do not exist in a given $\mathcal{A}$, then[9–11] any element $a$ of algebra $\mathcal{A}$ can be written as a linear combination of a certain convenient set of base elements

$$\{P^{(\alpha)}_{11} P^{(\alpha)}_{12} \cdots P^{(\alpha)}_{1l(\alpha)} P^{(\alpha)}_{21} P^{(\alpha)}_{22} \cdots P^{(\alpha)}_{l(\alpha)l(\alpha)} P^{(\beta)}_{11} P^{(\beta)}_{12} \cdots\},$$

which will be called *unit dyads.*

$$a = \sum_{(a)} \sum_{L} \sum_{m} \mathfrak{D}^{(\alpha)}_{Lm}(a) P^{(\alpha)}_{Lm}. \tag{15}$$

These unit dyads have a simple multiplicative structure something like their naming would imply:

$$P^{(\alpha)}_{ij} P^{(\beta)}_{nL} = \begin{cases} 0 & \text{if } (\alpha) \neq (\beta) \quad \text{or} \quad j \neq k, \\ P^{(\alpha)}_{iL} & \text{if } \alpha = \beta \quad \text{and} \quad j = k. \end{cases} \tag{16}$$

Note that $P^{(\alpha)}_{ij}$ is idempotent if $i = k$, i.e., $P^{(\alpha)}_{jj} P^{(\alpha)}_{jj} = P^{(\alpha)}_{jj}$. By construction[10–12] these idempotents all share the following property. There is an $a$ in $\mathcal{A}$ such that $P^{(\alpha)}_{ii} a P^{(\alpha)}_{jj}$ is nonzero and

$$P^{(\alpha)}_{ii} b P^{(\alpha)}_{jj} = \mu P^{(\alpha)}_{ii} a P^{(\alpha)}_{jj} = \nu P^{(\alpha)}_{ij} \tag{17}$$

for any $b$ in $\mathcal{A}$, where $\mu$ and $\nu$ are constants. The meaning and application of these elements is further explored shortly.

First it must be shown that this expansion (15) is valid for ray algebras, by showing that there are no contagiously nilpotent elements in $\mathcal{A}(\mathcal{G}, \omega)$. This we do by showing that no nonzero element

$$\eta = \sum_{R=1}^{R=g} \eta_R a_R \quad \text{of} \quad \mathcal{A}(\mathcal{G}, \omega)$$

makes *every* element $a$ of $\mathcal{A}(\mathcal{G}, \omega)$ into a nilpotent $\eta a = n$ or into zero $\eta a = 0$. If

$$a = \sum_S \frac{\eta^*_S a_{S^{-1}}}{\omega_{S^{-1},S}},$$

then one has the following:

$$\eta a = \sum_{R,S} \eta_R \eta^*_S a_R \frac{a_{S^{-1}}}{\omega_{S^{-1},S}}.$$

The regular representation of this is

$$\mathcal{R}(\eta a) = \sum_{R,S} \eta_R \eta^*_S \mathcal{R}(a_R) \mathcal{R}^+(a_S),$$

which is a Hermitian matrix since the $\mathcal{R}(a_R)$ are unitary. Also one sees that

$$\operatorname{Tr} \mathcal{R}(\eta a) = \sum_{R,S} \eta_R \eta^*_S \operatorname{Tr} \frac{\mathcal{R}(a_R a_{S^{-1}})}{\omega_{S^{-1}S}} = \sum_R |\eta_R|^2 g,$$

which shows $\mathcal{R}(\eta a)$ is a diagonalizable matrix with a nonzero trace. Clearly $\eta a$ is not nilpotent, and the basis of unit dyads is valid.

Now imagine the construction of the regular representation $\mathcal{R}^P$ of the unit dyads *using the unit dyads as a basis* and relations (16). For example, one has the following:

$$\mathcal{R}^p(P^{(\alpha)}_{11}) = \begin{vmatrix} \begin{matrix}10\\00\\ & \ddots\end{matrix} & & & & & \\ & \begin{matrix}10\\00\\ & \ddots\end{matrix} & & & & \\ & & \begin{matrix}10\\00\\ & \ddots\end{matrix} & & & \\ & & & \ddots & & \\ & & & & \begin{matrix}00\\00\\ & \ddots\end{matrix} & \\ & & & & & \begin{matrix}00\\00\\ & \ddots\end{matrix} \\ & & & & & & \ddots \end{vmatrix}, \tag{18}$$

$$\mathcal{R}^p(P^{(\alpha)}_{12}) = \begin{vmatrix} \begin{matrix}01\\00\\ & \ddots\end{matrix} & & & & & \\ & \begin{matrix}01\\00\\ & \ddots\end{matrix} & & & & \\ & & \begin{matrix}01\\00\\ & \ddots\end{matrix} & & & \\ & & & \ddots & & \\ & & & & \begin{matrix}00\\00\\ & \ddots\end{matrix} & \\ & & & & & \begin{matrix}00\\00\\ & \ddots\end{matrix} \\ & & & & & & \ddots \end{vmatrix}, \tag{19}$$

and

$$\mathcal{R}^P(P_{11}^{(\beta)}) = \begin{vmatrix} \boxed{\begin{matrix}0&0\\0&0\\&&\ddots\end{matrix}} &&&&&& \\ & \boxed{\begin{matrix}0&0\\0&0\\&&\ddots\end{matrix}} &&&&& \\ && \boxed{\begin{matrix}0&0\\0&0\\&&\ddots\end{matrix}} &&&& \\ &&& \ddots &&& \\ &&&& \boxed{\begin{matrix}1&0\\0&0\\&&\ddots\end{matrix}} && \\ &&&&& \boxed{\begin{matrix}1&0\\0&0\\&&\ddots\end{matrix}} & \\ &&&&&& \ddots \end{vmatrix}. \tag{20}$$

This regular representation can represent any element $a$ of $\mathcal{A}(\mathcal{G}, \omega)$ according to (15). It is equivalent to the regular representation (7) over the basis $\{\cdot\, a_R \,\cdot\}$ constructed in Sec. 3, but it has the following form:

$$\mathcal{R}^P(a) = \begin{vmatrix} \boxed{\mathfrak{D}^{(\alpha)}(a)} &&&&&& \\ & \boxed{\mathfrak{D}^{(\alpha)}(a)} &&&&& \\ && \boxed{\mathfrak{D}^{(\alpha)}(a)} &&&& \\ &&& \ddots &&& \\ &&&& \boxed{\mathfrak{D}^{(\beta)}(a)} && \\ &&&&& \boxed{\mathfrak{D}^{(\beta)}(a)} & \\ &&&&&& \ddots \end{vmatrix}. \tag{21}$$

Here the $\mathfrak{D}^{(\alpha)}(a)$ are matrices of the coefficients $\mathfrak{D}^{(\alpha)}_{Lm}(a)$ in (15). These $\mathfrak{D}^{(\alpha)}$ are in fact the matrices of the irreducible representations of $\mathcal{A}(\mathcal{G}, \omega)$. Furthermore, the frequency of $\mathfrak{D}^{(\alpha)}$ on the diagonal of $\mathcal{R}^P$ is equal to the dimension $l^{(\alpha)}$. Also it is true that

$$g = l^{(\alpha)^2} + l^{(\beta)^2} + \cdots, \tag{22}$$

which will be called the *diophantine solution* of $\mathcal{A}(\mathcal{G}, \omega)$.

By definition the unit dyads are some linear combination of the old base elements $\{a_1 a_2 \cdots a_R \cdots a_g\}$:

$$P_{ij}^{(\beta)} = \sum_T P_{ij}^{(\beta)}(a_T) a_T. \tag{23}$$

We solve for the constants $P_{ij}^{(\beta)}(a_T)$. We find that

$$P_{ij}^{(\beta)}(a_T) = (g\omega_{T^{-1},T}) \operatorname{Tr} \mathcal{R}(P_{ij}^{(\beta)} a_{T^{-1}}),$$

since $\operatorname{Tr} \mathcal{R}(a_T)$ is zero if $T$ is not 1, but is $g$ if $T = 1$. Next, we obtain the result that

$$\begin{aligned} P_{ij}^{(\beta)}(a_T) &= (g\omega_{T^{-1},T})^{-1} \operatorname{Tr} \mathcal{R}\left(P_{ij}^{(\beta)}\left\{\sum_k \sum_m \sum_n \mathfrak{D}^{(\alpha)}_{mn}(a_{T^{-1}}) P^{(\alpha)}_{mn}\right\}\right) \\ &= \frac{l^{(\beta)}}{g\omega_{T^{-1},T}} \mathfrak{D}^{(\beta)}_{ji}(a_{T^{-1}}), \end{aligned}$$

which gives finally

$$P_{ij}^{(\beta)} = \frac{l^{(\beta)}}{g} \sum_T \frac{\mathfrak{D}^{(\beta)}_{ji}(a_{T^{-1}})}{\omega_{T^{-1},T}} a_T. \tag{24}$$

These $\mathfrak{D}^{(\alpha)}$, like the regular representation, can be chosen unitary,[13] where

$$[\mathfrak{D}^{(\beta)}(a_{T^{-1}})/\omega_{T^{-1},T}] = [\mathfrak{D}^{(\beta)}(a_T)]^{+}. \tag{25}$$

We shall assume that the $\mathfrak{D}^{(\alpha)}$ are unitary so that

$$P_{ij}^{(\beta)} = \frac{l^{(\beta)}}{g} \sum_L \mathfrak{D}^{(\beta)*}_{ij}(a_T) a_T. \tag{26}$$

It is now a simple matter to derive representation orthogonality relations and the Schur's Lemmas.

*Theorem 1*[14]*:*

$$\frac{l^{(\gamma)}}{g} \sum_T \frac{1}{\omega_{T^{-1},T}} \mathfrak{D}^{(\gamma)}_{mn}(a_{T^{-1}}) \mathfrak{D}^{(\beta)}_{Lj}(a_T) = \delta^{\gamma\beta} \delta_{Ln} \delta_{jm}.$$

*Proof:* By inspecting (18), (19), and (20), one may write

$$\mathfrak{D}^{(\beta)}_{Lj}[P^{(\gamma)}_{nm}] = \delta^{(\gamma)\beta} \delta_{Lm} \delta_{jm}. \tag{27}$$

Now expanding $P^{(\gamma)}_{nm}$ according to (24) yields the desired relation.

*Theorem 2*[14]*:* The only (square) $l^{(\alpha)} \times l^{(\alpha)}$ matrix $m$ that commutes with all the matrices of an irreducible ray representation $\{\cdots \mathfrak{D}^{(\alpha)}(a_T) \cdots\}$ of group $\mathcal{G}$ is a multiple of the unit matrix.

*Proof:* Let $m$ be an $l^{(\alpha)} \times l^{(\alpha)}$ matrix that satisfies $m\mathfrak{D}^{(\alpha)}(a_T) = \mathfrak{D}^{(\alpha)}(a_T)m$ for all $T$. This implies that $m\mathfrak{D}^{(\alpha)}[P^{(\alpha)}_{ij}] = \mathfrak{D}^{(\alpha)}[P^{(\alpha)}_{ij}]m$ for all $i$ and $j$. From (27) one sees that $m$ is a multiple of the unit matrix.

*Theorem 3*[14]*:* The only (in general rectangular) $l^{(\alpha)} \times l^{(\beta)}$ matrix $m$ that satisfies the relations

$$m\mathfrak{D}^{(\alpha)}(a_T) = \mathfrak{D}^{(\beta)}(a_T)m$$

for all $a_T$, when $\alpha \neq \beta$, is the zero matrix.

[13] P. Rudra, J. Math. Phys. **6**, 1273 (1965), p. 1274.
[14] Reference 13, p. 1275.

*Proof:* Define the elements

$$P^{(\alpha)} = \sum_{j=1}^{j=l^{(\alpha)}} P_{jj}^{(\alpha)} \tag{28}$$

and note that $\mathfrak{D}^{(\alpha)}[P^{(\alpha)}]$ is the unit matrix, while $\mathfrak{D}^{(\beta)}[P^{(\alpha)}]$ is the zero matrix if $\alpha \neq \beta$ using (27). Clearly

$$m\mathfrak{D}^{(\alpha)}[P^{(\alpha)}] = \mathfrak{D}^{(\beta)}[P^{(\alpha)}]m$$

or

$$m = 0.$$

## 5. THE CENTER OF A RAY ALGEBRA

The *center* $C$ of $\mathcal{A}(\mathcal{G}, \omega)$ is the maximal set of elements in $\mathcal{A}(\mathcal{G}, \omega)$ that commute with every element of $\mathcal{A}(\mathcal{G}, \omega)$. $C$ is a subalgebra of $\mathcal{A}(\mathcal{G}, \omega)$, and it would contain the idempotents $P^{(\alpha)}$ defined in (28). We now set out to determine how many $P^{(\alpha)}$ there may be for a general $\mathcal{A}(g, \omega)$ and learn how to obtain them for a particular ray algebra.

Suppose that $c = \sum_R \gamma_R a_R$ is in $C$. Then

$$ca_S = a_S c \tag{29}$$

for all $a_S$ in $\mathcal{A}(\mathcal{G}, \omega)$. Also one has that

$$(a_{S^{-1}} c a_S)/\omega_{S^{-1},S} = c. \tag{30}$$

From this one sees that

$$\sum_{S=1}^{S=g} \frac{a_{S^{-1}} c a_S}{\omega_{S^{-1},S}} = \sum_S c = gc, \tag{31}$$

where $g$ is the order of group $\mathcal{G}$. One is assured that any element in the center must have the form

$$c = \frac{1}{g}\sum_{2=1}^{s=1} a_{S^{-1}}\left[\left(\sum_R \gamma_R a_R\right)\Big/\omega_{S^{-1},S}\right]a_S$$

$$= \sum_R \frac{\gamma_R}{g}\left\{\sum_S \frac{a_{S^{-1}} a_R a_S}{\omega_{S^{-1},S}}\right\}.$$

So every element of the center must be some linear combination of some of the $g$ elements $c_R$ defined by

$$c_R = \sum_{s=1}^{s=s} \frac{a_{S^{-1}} a_R s_S}{\omega_{S^{-1}S}} = \sum_{s=1}^{s=s} \frac{\omega_{S^{-1},R}\,\omega_{S^{-1}R,S}}{\omega_{S^{-1}S}}\, a_{S^{-1}RS}. \tag{32}$$

We must now find which $c_R$, if any, are linearly dependent. For the number of linearly independent $c_R$ is exactly equal to the number of inequivalent IR of $\mathcal{A}(\mathcal{G}, \omega)$.

Now $c_R$ is the linear combination of $a_R, a_{R'}, \cdots$, where $R, R', \cdots$, are in the class of $R$ in $\mathcal{G}$. Suppose

$$c_R = (\alpha a_R + \beta a_{R'} \cdots), \tag{33}$$

where all terms have been collected. If in group $\mathcal{G}$ it happens that an element $T$ transforms $R$ into $R'(T^{-1}RT = R')$ so that

$$a_{T^{-1}} a_R a_T = \omega_{T^{-1},RT}\,\omega_{R,T}\, a_{R'}, \tag{34}$$

then

$$a_{T^{-1}} c_R a_T / \omega_{T^{-1},T}$$

$$= (\omega_{T^{-1},T})^{-1}(\alpha a_{T^{-1}} a_R a_T + \cdots)$$

$$= (\omega_{T^{-1},T})^{-1}(\alpha\omega_{T^{-1},RT}\,\omega_{R,T}\, a_{R'} + \cdots), \tag{35}$$

where $(a_{T^{-1}} c_R a_T)/\omega_{T^{-1}T} = c_R$ by (30). Equating the coefficients of the first terms of (33) and (35) (which is correct since $a_{R'}$ appears only once in either series), one has

$$\beta = \alpha[(\omega_{T^{-1},RT}\,\omega_{R,T})/\omega_{T^{-1},T}].$$

It will be conventional to let $\alpha = \nu_R$ and

$$\epsilon_{R'} \equiv \beta/\alpha = (\omega_{T^{-1},RT}\,\omega_{R,T})/\omega_{T^{-1},T} \tag{36}$$

so that

$$c_R = \nu_R(a_R + \epsilon_{R'} a_{R'} + \epsilon_{R''} a_{R''} + \cdots), \tag{37}$$

where the $\epsilon_{R''}$ is a root of unity,

$$\epsilon^*_{R''}\epsilon_{R''} = 1,$$

and $\nu_R$ is some number whose possible values will be discovered shortly.

Now define the character of an irreducible representation $\mathfrak{D}^{(\alpha)}$ of $\mathcal{A}(\mathcal{G}, \omega)$ to be $\chi^{(\alpha)}(a_R) \equiv \operatorname{Tr}\mathfrak{D}^{(\alpha)}(a_R)$. Now note that $\chi^{(\alpha)}(a_{R'})$ is not necessarily equal to $\chi^{(\alpha)}(a_R)$ even if $R'$ is in the class of $R$ in $\mathcal{G}$. Instead one may take the trace of the equation [where it is assumed that $R' = T^{-1}RT$ as in (34)]

$$[\mathfrak{D}^{(\alpha)}(a_T)]^{-1}\mathfrak{D}^{(\alpha)}(a_R)\mathfrak{D}^{(\alpha)}(a_T) = \frac{\omega_{T^{-1},RT}\,\omega_{R,T}}{\omega_{T^{-1},T}}\mathfrak{D}^{(\alpha)}(a_{R'})$$

to obtain

$$\chi^{(\alpha)}(a_R) = \frac{\omega_{T^{-1},RT}\,\omega_{R,T}}{\omega_{T^{-1},T}}\chi^{(\alpha)}(a_{R'}). \tag{38}$$

If one substitutes $a = c_R$ in Eq. (15), there results

$$c_R = \sum_\beta \sum_L \sum_\kappa \mathfrak{D}^{(\beta)}_{L\kappa}(c_R) P^{(\beta)}_{L\kappa}.$$

But since $c_R$ is in the center, Schur's Lemmas demand that

$$\mathfrak{D}^{(\beta)}_{L\kappa}(c_R) = \begin{cases} 0 & \text{if } L \neq \kappa, \\ \dfrac{1}{l^{(\beta)}}\chi^{(\beta)}(c_R) & \text{if } L = \kappa, \end{cases}$$

whence

$$c_R = \sum_\beta \sum_L \frac{1}{l^{(\beta)}}\chi^{(\beta)}(c_R)P^{(\beta)}_{LL} = \sum_\beta \frac{1}{l^{(\beta)}}\chi^{(\beta)}(c_R)P^{(\beta)}. \tag{39}$$

Of course one must compute $\chi^{(\beta)}(c_R)$. From (37) one has

$$\chi^{(\beta)}(c_R) = \nu_R[\chi^{(\beta)}(a_R) + \epsilon_{R'}\chi^{(\beta)}(a_{R'}) + \cdots].$$

Using (36) and (38), one obtains

$$\chi^{(\beta)}(c_R) = O(C_R)\nu_R\chi^{(\beta)}(a_R), \qquad (40)$$

where $O(C_R)$ is the order of the class of $R$ in $\mathcal{G}$. Hence we obtain

$$c_R = O(C_R)\nu_R \sum_\beta \frac{\chi^{(\beta)}(a_R)}{l^{(\beta)}} P^{(\beta)}. \qquad (41)$$

Similarly, if $R' = T^{-1}RT$, then

$$c_{R'} = O(C_R)\nu_{R'} \sum_\beta \frac{\chi^{(\beta)}(a_{R'})}{l^{(\beta)}} P^{(\beta)},$$

since $O(C_{R'}) \equiv O(C_R)$. Now, using (38) and assuming that $c_R$ is not zero, one obtains

$$c_{R'} = \left(\frac{\omega_{T^{-1},T}}{\omega_{T^{-1},RT}}\, \omega_{R,T}\, \frac{\nu_{R'}}{\nu_R}\right) c_R \equiv \gamma(T) c_R. \quad (42)$$

So, if $c_R$ is not zero, all the other $c_{R'}$ (corresponding to elements $R'$ of class $C_R$) are simply proportional to $c_R$. We now evaluate $\gamma(T)$ in (42).

Substituting $c_R$ for $c$ in (31) and using (37), we obtain

$$gc_R = \nu_R\left(\sum_S \frac{a_{S^{-1}}a_R a_S}{\omega_{S^{-1}S}} + \epsilon_{R'} \sum_S \frac{a_{S^{-1}}a_{R'}a_S}{\omega_{S^{-1},S}} + \cdots\right)$$
$$= \nu_R(c_R + \epsilon_{R'}c_{R'} + \cdots),$$

which by (42) becomes

$$gc_R = \nu_R\left(c_R + \epsilon_{R'}\frac{\omega_{T^{-1},T}}{\omega_{T^{-1},RT}}\,\omega_{R,T}\,\frac{\nu_{R'}}{\nu_R}\, c_R + \cdots\right),$$

or

$$gc_R = (\nu_R + \nu_{R'} + \cdots)c_R, \qquad (43)$$

where (36) gave the last result. The series on the right has $o(C_R)$ terms $\nu_{R'}$. The highest value that any $\nu_R$ or $\nu_{R'}$ can have is $(g/o(C_R))$, since this is the number of elements $T, U, V, \cdots$ in $\mathcal{G}$ that transform $R$ into $R' = T^{-1}RT = U^{-1}RU = V^{-1}RV = \cdots$, where we assume all factors are modulus unity [see (2)]. Clearly, by (43), either each of the $\nu_R$, $\nu_{R'}$, etc., is exactly $(g/o(C_R))$, or else $c_R = c_{R'} = \cdots = 0$. In the latter case $\nu_R = \nu_{R'} = \cdots = 0$.

*Theorem 1:* Given

$$c_R = \sum_{S=1}^{S=S} \frac{\omega_{S^{-1},R}\,\omega_{S^{-1}R,S}}{\omega_{S^{-1},S}}\, a_{S^{-1}RS}$$

for all $R$ in group $\mathcal{G}$, there are no more linearly independent $c_R$ than there were classes in $\mathcal{G}$. Furthermore, $c_R$ is zero (nonzero) if and only if all $c_{R'}$ are zero (nonzero) for all $R'$ in the class $C_R$. Finally, if $c_R$ is nonzero, then

$$c_R = g/o(C_R)(a_R + \epsilon_{R'}a_{R'} + \cdots) = \epsilon_{R'}c_{R'}, \quad (44)$$

where $\epsilon_{R'}$ is given by (36).

*Definition:* If the elements $\{a_R, a_{R'}, \cdots\}$ correspond to a class $C_R$ in $\mathcal{G}$ that gives nonzero $c_R$, they will be said to belong to the *ray class* of $a_R$ in $\mathcal{A}(G, \omega)$. If $c_R = 0$, they will be said to belong to the *zeroing class* of $a_R$.

The term "zeroing class" will hereafter be abbreviated to *zass*, and a "ray class" will be called a *rass.*

*Definition:* If $c_R$ is nonzero, then the element

$$\kappa_R = (a_R + \epsilon_{R'}a_{R'} + \cdots) = \frac{1}{\nu_R} c_R \qquad (45)$$

[where $\epsilon_{R'}$ is given by (36)] is called the *rass sum* of $a_R$.

From (28), (26), and (38) one obtains

$$P^{(\alpha)} = \sum_{j=1}^{j=l^{(\alpha)}} P_{jj}^{(\alpha)} = \frac{l^{(\alpha)}}{g} \sum_R \chi^{(\alpha)*}(a_R)a_R. \qquad (46)$$

This can be written as a linear combination of independent rass sums, since it is in the center:

$$\sum_s \frac{a_{s^{-1}}P^{(\alpha)}a_s}{\omega_{s^{-1},s}} = gP^{(\alpha)} = \frac{l^{(\alpha)}}{g} \sum_{R=1}^{R=g} \chi^{(\alpha)}(a_R)c_R,$$

$$P^{(\alpha)} = \frac{l^{(\alpha)}}{g^2} \sum_{\substack{\text{independent}\\ \text{rasses } j}} (\chi^{(\alpha)*}(a_{R_j})c_{R_j} + \chi^{(\alpha)*}(a_{R_j'})c_{R_j'} + \cdots). \quad (47)$$

In the series above, there is one term for each element of the $j$th rass of $\mathcal{A}(\mathcal{G}, \omega)$. Assuming, as before, that $R_j' = T^{-1}R_jT$ and using (44) and (38), there results

$$P^{(\alpha)} = \frac{l^{(\alpha)}}{g^2} \sum_{\substack{\text{rasses}\\ j}} \left(\chi^{(\alpha)}(a_{R_j})c_{R_j} + \frac{1}{\epsilon_{R_j}^*\epsilon_{R_j}}\chi^{(\alpha)}(a_{R_j}),c_R + \cdots\right),$$
$$= \frac{l^{(\alpha)}}{g^2} \sum_{\substack{\text{rasses}\\ j}} O(C_{R_j})\chi^{(\alpha)*}(a_{R_j})c_{R_j},$$
$$= \frac{l^{(\alpha)}}{g} \sum_{\substack{\text{rasses}\\ j}} \chi^{(\alpha)*}(a_{R_j})\kappa_{R_j}, \qquad (48)$$

where $\kappa_{R_j}$ is given by (45).

Finally, (41) and (45) give

$$\kappa_{R_j} = O(C_{R_j}) \sum_{\substack{\text{irreducible}\\ \text{representative}\\ (\beta)}} \frac{\chi^{(\beta)}(a_{R_j})}{l^{(\beta)}} P^{(\beta)}. \qquad (49)$$

## 6. CHARACTER THEORY INVOLVING RASSES

The following two theorems show that only the rasses need be considered in manipulations of characters, while the rest of the ray algebra (the zasses) can be ignored.

*Theorem 2:* If $\chi^{(\alpha)}(a_R) = 0$ for all $(\alpha)$, then $c_R = 0$.

*Proof:* This follows immediately from (41).

*Theorem 3:* If $c_{R_k} = 0$, then $\chi^{(\beta)}(a_{R_k}) = 0$ for all $(\beta)$.

*Proof:* The expressions (47) for $P^{(\alpha)}$ do not include any elements from a zass. The expressions (46) for $P^{(\alpha)}$ show then that $\chi^{(\alpha)}(a_R)$ is identically zero for an element $a_R$ in a zass.

Now suppose $\{\mathcal{R}(a_1)\ \mathcal{R}(a_2) \cdots \mathcal{R}(a_g)\}$ is a representation of $\mathcal{A}(\mathcal{G}, \omega)$, i.e., a ray representation of $\mathcal{G}$ with factor system $\{\omega\}$. An immediate consequence of the preceding formalism is that if $\mathcal{R}$ is not equivalent to one of the $\mathfrak{D}^{(\alpha)}$, it must be equivalent to a direct sum of them. Let

$$\mathcal{R} \sim \oplus \sum_{(\alpha)} f^{(\alpha)}(\mathcal{R}) \mathfrak{D}^{(\alpha)},$$

where the integer $f^{(\alpha)}(\mathcal{R})$ is the *frequency* of $\mathfrak{D}^{(\alpha)}$ in $\mathcal{R}$, or the $(\alpha)$ *contents* of $\mathcal{R}$. Now one has

$$f^{(\alpha)}(\mathcal{R}) = \operatorname{Tr} \mathcal{R}(P_{ii}^{(\alpha)}) = \frac{1}{l^{(\alpha)}} \operatorname{Tr} \mathcal{R}(P^{(\alpha)}),$$

which by (48) expands to

$$f^{(\alpha)}(R) = \frac{1}{g} \sum_{\substack{\text{rasses} \\ j}} o(C_{R_j}) \chi^{(\alpha)*}(a_{R_j}) \operatorname{Tr} \mathcal{R}(a_{R_j}). \quad (50)$$

It is an easy matter to obtain rass–IR orthogonality relations. Substituting (49) into (48) and comparing coefficients yields

$$\frac{1}{g} \sum_{\text{rasses}} o(C_{R_j}) \chi^{(\alpha)}(a_{R_j}) \chi^{(\beta)}(a_{R_j}) = \delta^{\alpha\beta}, \quad (51)$$

while substitution of (48) in (49) yields

$$\frac{o(C_{R_L})}{g} \sum_{(\beta)} \chi^{(\beta)}(a_{R_L}) \chi^{(\beta)}(a_{R_j}) = \delta_{Lj}. \quad (52)$$

## 7. CALCULATING CHARACTERS: AN EXAMPLE

The following is a table [see (6)] of $\mathcal{A}(D_6\omega)$ derived from a spinor representation $\mathfrak{D}^{(\frac{1}{2})}$ of $O(3)$:

| | $1$ | $r^2$ | $r^4$ | $\rho_1$ | $\rho_2$ | $\rho_3$ | $r^3$ | $r^5$ | $r$ | $\rho_1'$ | $\rho_2'$ | $\rho_3'$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $1$ | $1$ | $r^2$ | $r^4$ | $1$ | $2$ | $3$ | $r^3$ | $r^5$ | $r$ | $1'$ | $2'$ | $3'$ |
| $r^4$ | $r^4$ | $-1$ | $-r^2$ | $3$ | $-1$ | $-2$ | $-r$ | $-r^3$ | $r^5$ | $-3'$ | $-1'$ | $2'$ |
| $r^2$ | $r^2$ | $r^4$ | $-1$ | $2$ | $3$ | $-1$ | $r^5$ | $-r$ | $r^3$ | $2'$ | $-3'$ | $1'$ |
| $\rho_1$ | $1$ | $-3$ | $-2$ | $-1$ | $r^4$ | $r^2$ | $-1'$ | $-3'$ | $-2'$ | $r^3$ | $r$ | $r^5$ |
| $\rho_2$ | $2$ | $1$ | $-3$ | $-r^2$ | $-1$ | $r^4$ | $-2'$ | $-1'$ | $3'$ | $r^5$ | $r^3$ | $-r$ |
| $\rho_3$ | $3$ | $2$ | $1$ | $-r^4$ | $-r^2$ | $-1$ | $3'$ | $-2'$ | $1'$ | $-r$ | $r^5$ | $-r^3$ |
| $r^3$ | $r^3$ | $r^5$ | $r$ | $1'$ | $2'$ | $3'$ | $-1$ | $-r^2$ | $r_2^4$ | $-1$ | $-2$ | $3$ |
| $r$ | $r$ | $r^3$ | $r^5$ | $3'$ | $1'$ | $2'$ | $r^4$ | $-1$ | $r^2$ | $3$ | $-1$ | $2$ |
| $r^5$ | $r^5$ | $-r$ | $-r^3$ | $2'$ | $-3'$ | $-1'$ | $-r^2$ | $-r^4$ | $-1$ | $-2$ | $-3$ | $-1$ |
| $\rho_1'$ | $1'$ | $3'$ | $-2'$ | $-r^3$ | $-r$ | $r^5$ | $1$ | $-3$ | $2$ | $-1$ | $r^4$ | $-r^2$ |
| $\rho_2'$ | $2'$ | $1'$ | $3'$ | $-r^5$ | $-r^3$ | $-r$ | $2$ | $1$ | $3$ | $-r^2$ | $-1$ | $-r^4$ |
| $\rho_3'$ | $3'$ | $-2'$ | $-1'$ | $-r$ | $r^5$ | $r^3$ | $-3$ | $-2$ | $1$ | $r^4$ | $r^2$ | $-1$ |

(The notation $a_R$ has been replaced by $R$ for each element of $D_6$.) The element $r$ corresponds to a rotation by 60° around the six-fold axis, while $\rho_1\rho_2$, $\rho_3$, $\rho_1'$, $\rho_2'$, and $\rho_3'$ correspond to various 180° rotations around the axis shown in Fig. 1. (These are indicated by the numbers 1, 2, 3, 1′, 2′, and 3′ in the table.)

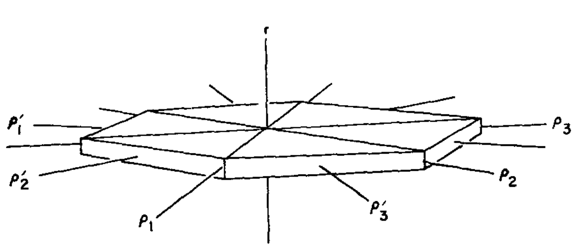



FIG. 1. The group of rotations in $D_6 = D_3 \times C_2$.

There are three rasses and three zasses. The rass sums are $k_1 = 1$, $k_2 = r^2 - r^4$ and $k_3 = r - r^5$. These three form the following algebra:

| | $k_1$ | $k_2$ | $k_3$ |
|---|---|---|---|
| $k_1$ | $k_1$ | $k_2$ | $k_3$ |
| $k_2$ | $k_2$ | $2k_1 - k_2$ | $k_3$ |
| $k_3$ | $k_3$ | $k_3$ | $2k_1 + k_2$ |

The minimal equations satisfied by $k_2$ and $k_3$ are the following:

$$(k_2)^2 + k_2 - 2k_1 = 0, \quad (k_3)^3 - 3k_1 = 0,$$

with the roots $(1, -2)$ for $k_2$, and $(\sqrt{3}, -\sqrt{3}, 0)$ for

$k_3$. These roots will serve (sometimes repeatedly[15]) as coefficients $O(C_R)\chi^{(\beta)}(a_R)/l^{(\beta)} = a_R^{(\beta)}$ in (49):

$$\kappa_R = O(C_R)\sum_\beta \frac{\chi^{(\beta)}(a_R)}{l^{(\beta)}} P^{(\beta)} = \sum_\beta a_R^{(\beta)} P^{(\beta)}. \quad (49')$$

The coefficients are tabulated below:

| | $k_1$ | $k_2$ | $k_3$ |
|---|---|---|---|
| $a^{(\alpha)}$ | 1 | 1 | $\sqrt{3}$ |
| $a^{(\beta)}$ | 1 | 1 | $-\sqrt{3}$ |
| $a^{(\gamma)}$ | 1 | $-2$ | 0 |

The character table is completed once the values of $l^{(\alpha)}$, $l^{(\beta)}$, and $l^{(\gamma)}$ are obtained. The relation (51) gives the following simple formula for these:

$$\left(\frac{1}{l^{(\alpha)}}\right)^2 = \frac{1}{g}\sum_{\substack{\text{rasses}\\ j}} \frac{1}{oC_{R_j}}\left(\frac{oC_{R_j}\chi^{(\alpha)}(a_{R_j})}{l^{(\alpha)}}\right)\left(\frac{oC_{R_j}\chi^{(\alpha)}(a_{R_j})}{l^{(\alpha)}}\right)$$

$$= \frac{1}{g}\sum_j \frac{1}{oC_{R_j}} |a_{R_j}^{(\alpha)}|^2.$$

The resulting ray character table of $D_6$ is shown below:

| | 1 | $r^2$ | $r$ |
|---|---|---|---|
| $\chi^{(\alpha)}$ | 2 | 1 | $\sqrt{3}$ |
| $\chi^{(\beta)}$ | 2 | 1 | $-\sqrt{3}$ |
| $\chi^{(\gamma)}$ | 2 | $-2$ | 0 |

This $\mathcal{A}(D_6, \omega)$ has diophantine solution (22) $2^2 + 2^2 + 2^2 = 12$. The characters of other rass elements $r^4$ and $r^5$ follow from (38). The characters of zass elements are zero.

Other "double value" representations of $O(3)$ subduce ray representations of $D_6$. The contents of these are easily obtained by writing "down" the traces of them for 1, $r^2$, and $r$, using the formula

$$\chi^{(J/2)}(\omega) = [\sin (J + 1)\omega/2]/(\sin \omega/2)$$

(where $\omega = 2\pi/3$ for $r^2$ and $\omega = \pi/3$ for $r$) and then applying (50).

## 8. CANONICAL RASSES

So far it has been assumed that one treated a given factor system without transformation, provided it was in a "unimodular" form corresponding to (2) and (11).

However, the theory of Sec. 5 inclines one to transform to a (generally) different factor system $\{\omega'\}$ that is $p$-equivalent to $\{\omega\}$ but more convenient to deal with. For if

$$\kappa_R = a_R + \epsilon_{R'} a_{R'} + \cdots \quad (45)$$

is a rass sum, one may write a new element

$$b_{R'} \equiv \epsilon_{R'} a_{R'} \quad (53)$$

for each element $a_{R'}$ in the sum where $\epsilon_{R'}$ is given by (36) as

$$\epsilon_{R'} = \frac{\omega_{T^{-1},RT}\,\omega_{R,T}}{\omega_{T^{-1},T}}$$

if $R' = T^{-1}RT$. Now if all these new elements $b_{R'}$ for all the rass sums in $\mathcal{A}(\mathcal{G}, \omega)$ are collected along with the $a_R$ (which we label $b_R \equiv a_R$ for notational convenience) and the members of zasses $a_z$ (which we label $b_z \equiv a_z$), the resulting set $\{b_R \cdots b_S \cdots b_T \cdots\}$ is a basis of a new ray algebra $\mathcal{A}(\mathcal{G}, \omega')$, which is said to have *canonical rasses.*

One advantage of $\mathcal{A}(\mathcal{G}, \omega')$ is that if $\{\cdots M(b_R) \cdots M(b_s) \cdots M(b_T) \cdots\}$ is a representation of it, then one has

$$\mathrm{Tr}\, M(b_{R'}) = \epsilon_{R'}\, \mathrm{Tr}\, M(a_{R'}) = \epsilon_{R'} \frac{\omega_{T^{-1},T}}{\omega_{T^{-1},RT}\,\omega_{R,T}}\, \mathrm{Tr}\, M(a_R),$$

or

$$\mathrm{Tr}\, M(b_{R'}) = \mathrm{Tr}\, M(a_R) = \mathrm{Tr}\, M(b_R), \quad (54)$$

provided $b_{R'}$ and $b_R$ are in the same rass. If they are both in a *zass*, then the identity

$$\mathrm{Tr}\, M(b_{R'}) = 0 = \mathrm{Tr}\, M(b_R) \quad (55)$$

holds in any case.

Furthermore, in this $\mathcal{A}(\mathcal{G}, \omega')$ it is true that

$$b_{v^{-1}} b_{R'} b_v = \omega'_{v^{-1},v}\, b_{v^{-1}Rv'} \quad (56)$$

for any base element $b_v$, provided $b_{R'}$ is in a rass.

## 9. INDUCTION INVOLVING RAY ALGEBRAS

The process of *induction* in groups is well described by Coleman[16] and Bradley.[17] A slight modification of their derivation allows us to perform the same operation with ray algebras without involving a covering group.

The idea is to obtain a representation [labeled $\mathcal{R} = M \uparrow \mathcal{A}(\mathcal{G}, \omega)$] of ray algebra $\mathcal{A}(\mathcal{G}, \omega)$ if you know a representation $\{\cdots M(a_H) \cdots\}$ of a subalgebra $\mathcal{A}(\mathcal{H}, \omega)$ of $\mathcal{A}(\mathcal{G}, \omega)$ corresponding to a subgroup $\mathcal{H}$ of $\mathcal{G}$.

[15] But the roots themselves (of the minimal equation) will never be repeated. If they were, a nilpotent $\eta$ in the center could be constructed, and this would be a contagious nilpotent in $\mathcal{A}(\mathcal{G}, \omega)$, since $\eta^n a^n = 0 = (\eta a)(\eta a) \cdots (\eta a) = (\eta a)^n$, contrary to Theorem 1.

[16] A. J. Coleman, *Induced Representations with Applications to $S_n$ and GL(n), Queens Papers No. 4* (Queen's University, Kingston, Ontario, Canada, 1966).

[17] C. J. Bradley, J. Math. Phys. 7, 1146 (1966).

Proceeding in the manner of Bradley,[17] one constructs left cosets of $\mathcal{H}$ in $\mathcal{G}$,

$$\mathcal{G} = \sum_R R\mathcal{H}, \tag{57}$$

while singling out one element $R$ from each coset to be the $R$th *coset leader*.

Now, returning to the corresponding ray algebras, suppose $m$ vectors $\{{}^1\Psi_1\, {}^1\Psi_2 \cdots {}^1\psi_m\}$ form a basis of representation $M$,

$$a_H\, {}^1\Psi_i = \sum_{i=1}^{i=m} {}^1\Psi_i M_{ij}(a_H). \tag{58}$$

Let $m$ new vectors $\{{}^R\Psi_1\, {}^R\Psi_2 \cdots {}^R\Psi_m\}$ be defined for each coset leader in (57):

$$a_R\, {}^1\Psi_j \equiv {}^R\Psi_j. \tag{59}$$

Then, if $a_g$ is any element of $\mathcal{A}(\mathcal{G}, \omega)$, one has that

$$a_g\, {}^S\Psi_j = a_g a_S\, {}^1\Psi_j = \left(\frac{a_R a_{R^{-1}}}{\omega_{R^{-1},R}}\right) a_g g_S\, {}^1\Psi_j. \tag{60}$$

If you pick a leader $R$ such that $a_{R^{-1}} a_g a_S$ is an element of $\mathcal{A}(\mathcal{H}, \omega)$, (60) becomes

$$a_g\, {}^S\Psi_j = \frac{a_R}{\omega_{R^{-1},R}} \sum_{i=1}^{i=m} {}^1\Psi_i M_{ij}(a_{R^{-1}} a_g a_S) \tag{61}$$

by (58), which in turn becomes

$$a_g\, {}^S\Psi_j = \sum_{i=1}^{i=m} {}^R\Psi_i M_{ij}(a_{R^{-1}} a_g a_S)/\omega_{R^{-1},R}$$

by (59). The vectors $\{{}^1\psi_1 \cdots {}^R\psi_j \cdots\}$ form the basis of the *induced ray representation* $M \uparrow \mathcal{A}(\mathcal{G}, \omega) \equiv \mathcal{R}$. One matrix $\mathcal{R}(a_g)$ of this is depicted in Fig. 2.

We now assume that $\mathcal{A}(\mathcal{G}, \omega)$ has canonical rasses and proceed to compute the traces of the induced representation. One has

$$\operatorname{Tr}(M \uparrow \mathcal{A}(\mathcal{G}, \omega))(a_g) = \sum_{\substack{\text{leaders} \\ R}} \operatorname{Tr} M(a_{R^{-1}} a_g a_R)/\omega_{R^{-1},R}\{\delta(\mathcal{H}, R^{-1}gR)\},$$

where $\delta$ is defined by

$$\delta(\mathcal{H}, R^{-1}gS) \equiv \begin{cases} 1, & \text{if } R^{-1}gR \in \mathcal{H}, \\ 0, & \text{if } R^{-1}gR \notin \mathcal{H}. \end{cases}$$

Using (56), one obtains

$$\operatorname{Tr}(M \uparrow \mathcal{A})(a_g) = \sum_{\substack{\text{leaders} \\ R}} \operatorname{Tr} M(a_{R^{-1}gR})\{\delta(\mathcal{H}, R^{-1}gR\}, \tag{62}$$

which can be converted to a sum over all $g$ using the following lemmas with (54) and (55).

*Lemma 1:* If a leader $R$ transforms $g$ into $h \in \mathcal{H}$ ($R^{-1}sR = L \in \mathcal{H}$), then all the members of coset $R\mathcal{H}$ will transform $g$ into the class $C_h$ of $h$ in $\mathcal{H}$.

*Proof:* If $R^{-1}gR = h$ is in $\mathcal{H}$, and if $h'$ is in $\mathcal{H}$, then $h'^{-1}(R^{-1}gR)h' = h'^{-1}hh'$. This becomes

$$(Rh')^{-1}g(Rh') = h'^{-1}hh'.$$

*Lemma 2:* If a leader $R$ fails to transform $g$ into an $h \in \mathcal{H}$, then no member of coset $R\mathcal{H}$ can transform $g$ into $\mathcal{H}$.

*Proof:* Assume $R^{-1}gR = G$ is not in $\mathcal{H}$. Then $(Rh')^{-1}g(Rh') = h'^{-1}Gh'$, and clearly this is not in $\mathcal{H}$. The result is

$$\operatorname{Tr}(M \uparrow \mathcal{A})(a_g) = \frac{1}{h} \sum_{\substack{\text{all} \\ \text{elements} \\ G \in \mathcal{G}}} \operatorname{Tr} M(a_{G^{-1}} a_g a_G)\delta(\mathcal{H}, G^{-1}gG). \tag{63}$$

Now assume that $g$ is in class $C_j$ of $\mathcal{G}$ and that $a_g$ is in the $j$th rass of $\mathcal{A}(\mathcal{G}, \omega)$. Then it follows that

$$\operatorname{Tr}(M \uparrow \mathcal{A})(a_g) = \operatorname{Tr}(M \uparrow \mathcal{A})_j = \frac{g}{ho(C_j)} \sum_{\substack{\text{all } h \\ \text{in } C_j \cap \mathcal{H}}} \operatorname{Tr} M(a_h). \tag{64}$$

Assuming that this $j$th class $C_j$ of $\mathcal{G}$ contains classes $C_1^j, C_2^j, C_3^j, \cdots, C_{n_j}^j$ of $\mathcal{H}$, one has

$$\operatorname{Tr}(M \uparrow \mathcal{A})_j = \frac{g}{ho(C_j)} \sum_{n=1}^{n=n_j} o(C_n^j) \operatorname{Tr}(M)_n^j, \tag{65}$$

using the notation $\operatorname{Tr}(M)_n^j = \operatorname{Tr}(M(a_n))$ where $a_n$ is any member of rass[18] $C_n^j$ of $\mathcal{A}(\mathcal{H}, \omega)$.

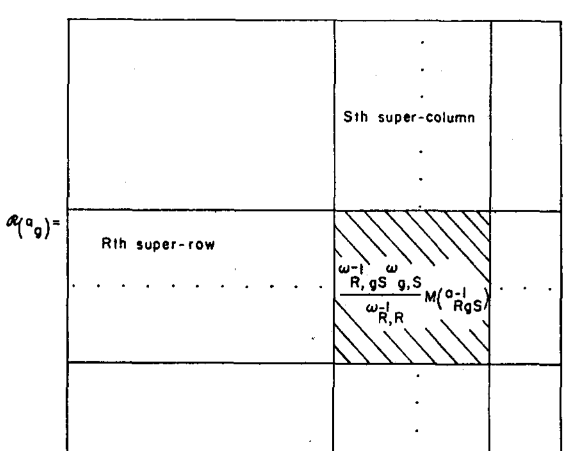



FIG. 2. The matrix $M \uparrow \mathcal{A}(\mathcal{G}, \omega) \equiv \mathcal{R}(a_g)$ induced by representation $M$ of ray-subalgebra $\mathcal{A}(\mathcal{H}, \omega)$.

[18] One should note that there cannot be a zass of $\mathcal{A}(\mathcal{H}, \omega)$ within a rass of $\mathcal{A}(\mathcal{G}, \omega)$.

The formalism is now set up to prove a Frobenius reciprocity theorem for ray algebras. Using Bradley's notation (↓) for subduction, letting $D^{(A)}$ denote an irreducible representation of $\mathcal{A}(\mathcal{H})$ and $\mathfrak{D}^{(\alpha)}$, an irreducible representation of $\mathcal{A}(\mathcal{G})$, one obtains the following theorem:

*Theorem 5:*

$$f^{(\alpha)}(D^{(A)} \uparrow \mathcal{A}(\mathcal{G})) = f^{(A)}(\mathfrak{D}^{(\alpha)} \downarrow \mathcal{A}(\mathcal{H})).$$

*Proof:* Using (50) and (65), one has

$$f^{(\alpha)}(D^{(A)} \uparrow \mathcal{A}(\mathcal{G})) = \frac{1}{g} \sum_{\substack{\text{rasses } j \\ \text{of } \mathcal{A}(\mathcal{G})}} o(C_j)\chi_j^{(\alpha)*}\left\{\frac{g}{h_0(C_j)} \sum_{\substack{\text{rasses } n \\ \text{of } \mathcal{A}(\mathcal{H}) \\ \text{in } C_j}}^{n=n_j} o(c_n^j) X^{(A)}\binom{j}{n}\right\},$$

where $X^{(A)}\binom{j}{n}$ is a character for rass $c_n^j$ of $\mathcal{A}(\mathcal{H})$. Rearranging this equation, one obtains

$$f^{(\alpha)}(D^{(\alpha)} \uparrow (\mathcal{G})) = \frac{1}{h} \sum_{\substack{\text{all rasses} \\ \binom{j}{n} \text{ of } \mathcal{A}(\mathcal{H})}} o(c_n^j)\chi_j^{(\alpha)*} X^{(A)}\binom{j}{n},$$

which, by (50) and the fact that $f^{(\alpha)*} = f^{(\alpha)}$, proves the theorem.

With this established, it is an easy matter to prove the following two generalizations [Eqs. (66) and (67) below]:

*Theorem 6:* If $\mathcal{A}(\mathcal{H}')$ and $\mathcal{A}(\mathcal{H}'')$ are two ray subalgebras of $\mathcal{A}(\mathcal{G})$, $D^{(A)'}$ is an IR of $\mathcal{A}(\mathcal{H}')$, $D^{(B)''}$ is an IR of $\mathcal{A}(\mathcal{H}'')$, and $\mathfrak{D}^{(\alpha)}$ is an IR of $\mathcal{A}(\mathcal{G})$, then one has the following relations:

$$f^{(A)'}(D^{(B)''} \uparrow \mathcal{A}(\mathcal{G})) \downarrow \mathcal{A}(\mathcal{H}')) = f^{(B)''}((D^{(A)'} \uparrow \mathcal{A}(\mathcal{G})) \downarrow \mathcal{A}(\mathcal{H}'') \quad (66)$$

and

$$f^{(A)'}(D^{(B)''} \downarrow \mathcal{A}(\mathcal{H}' \cap \mathcal{H}'')) \uparrow \mathcal{A}(\mathcal{H}') = f^{(B)''}(D^{(A)'} \downarrow \mathcal{A}(\mathcal{H}' \cap \mathcal{H}'')) \uparrow \mathcal{A}(\mathcal{H}''). \quad (67)$$

*Proof:* The proof of (67) is virtually identical to the proof of (66), which is now given. Assuming, as in Sec. 6, that

$$(D^{(A)'} \uparrow \mathcal{A}(\mathcal{G})) \sim \oplus \sum_{(\alpha)} f^{(\alpha)}(D^{(A)'} \uparrow \mathcal{A}(\mathcal{G}))\mathfrak{D}^{(\alpha)}$$

and

$$\mathfrak{D}^{(\alpha)} \downarrow \mathcal{A}(\mathcal{H}'') \sim \oplus \sum_{(B)''} f^{(B)''}(\mathfrak{D}^{(\alpha)} \downarrow \mathcal{A}(\mathcal{H}''))D^{(B)''},$$

one obtains the following expression for the right-hand side of (66):

$$f^{(B)''}(D^{(A)'} \uparrow \mathcal{A}(\mathcal{G})) \downarrow \mathcal{A}(\mathcal{H}'')) = \sum_{(\alpha)} f^{(\alpha)}(D^{(A)'} \uparrow \mathcal{A}(\mathcal{G}))f^{(B)''}(\mathfrak{D}^{(\alpha)} \downarrow \mathcal{A}(\mathcal{H}'')).$$

Applying Theorem 5, one has

$$f^{(B)''}(D^{(A)'} \uparrow \mathcal{A}(\mathcal{G}) \downarrow \mathcal{A}(\mathcal{H}'')) = \sum_{(\alpha)} f^{(\alpha)}(D^{(A)'} \uparrow \mathcal{A}(\mathcal{G}))f^{(\alpha)}(D^{(B)''} \uparrow \mathcal{A}(\mathcal{G})),$$

which is clearly equal to the left side of (66).

## 10. OBTAINING RAY REPRESENTATION EXAMPLES

The reciprocity theorems allow the extension of the recursive techniques of Seitz[19] and Boerner[20] (for finding the IR of solvable groups) to the corresponding ray algebras. This will be described here without detailed proof, and examples will be treated.

Suppose $\mathcal{A}(\mathcal{H})$ is a ray subalgebra of $\mathcal{A}(\mathcal{G})$ corresponding to normal subgroup $\mathcal{H}$ of $\mathcal{G}$ with prime index $p$ $[O(\mathcal{G}/\mathcal{H}) = p]$. Then the IR $\{\mathfrak{D}^{(\alpha)}, \mathfrak{D}^{(\beta)}, \cdots\}$ of $\mathcal{A}(\mathcal{G})$ will be obtained from the IR $\{D^{(A)}, D^{(B)}, \cdots\}$ of $\mathcal{A}(\mathcal{H})$ by the process of induction $\mathfrak{D}^{(\alpha)} = D^{(B)} \uparrow \mathcal{A}(\mathcal{G})$, as described in Sec. 9, and by a process called *extension* $\mathfrak{D}^{(\beta)} = D^{(B)} \to \mathcal{A}(\mathcal{G})$, which is described now.

Extension is used when $D^{(B)} \uparrow \mathcal{A}(\mathcal{G})$ is not an irreducible representation of $\mathcal{A}(\mathcal{G})$. This can be checked by using Eqs. (65) and (51). In this case, and under the conditions listed above, there is an IR $\mathfrak{D}^{(\beta)}$ of $\mathcal{A}(\mathcal{G})$ such that $\mathfrak{D}^{(\beta)} \downarrow \mathcal{A}(\mathcal{H}) = D^{(B)}$. Furthermore, the representation $D^{(B)}G$, defined by

$$D^{(B)}G(a_H) \equiv \frac{D^{(B)}(a_{G^{-1}}a_H a_G)}{\omega_{G^{-1},G}} \quad (68)$$

for a given $G$ in $\mathcal{G}$ and all $a_H$ in $\mathcal{A}(\mathcal{H})$, is equivalent to $D^{(B)}$; i.e.,

$$D^{(B)}G = \mathcal{C}^{-1}D^{(B)}\mathcal{C} \quad (69)$$

for some matrix $\mathcal{C}$. One may obtain all the solutions $\mathcal{C}$ to (67) by using the unit dyads of $\mathcal{A}(\mathcal{H})$. Clearly then,

$$\mathfrak{D}^{(\alpha)}(a_G) = \mu_G^{(\alpha)}\mathcal{C}, \quad (70)$$

where the allowed constants $\mu^{(\alpha)}, \mu^{(\beta)}, \cdots$, are obtained by inspecting $\mathcal{A}(\mathcal{G})$. A number of inequivalent IR $\{\mathfrak{D}^{(\alpha)}, \mathfrak{D}^{(\beta)}, \cdots\}$ of $\mathcal{A}(\mathcal{G})$ will result, corresponding to the number of distinct allowed constants $(\mu^{(\alpha)}, \mu^{(\beta)}, \cdots)$.

### A. Double-Valued Representations of $D_n$

The group $D_n$ has a cyclic subgroup $C_n$ of index 2, which is generated by an element $R$ which satisfies the relation

$$R^n = 1. \quad (71)$$

The ray algebra $\mathcal{A}(D_n)$ derived from spinors has a ray subalgebra $\mathcal{A}(C_n)$, which is generated by an element

[19] C. M. Seitz, Ann. Math. 37, 17 (1936).
[20] H. Boerner, *Representations of Groups* (Interscience Publishers, Inc., New York, 1963), p. 95.

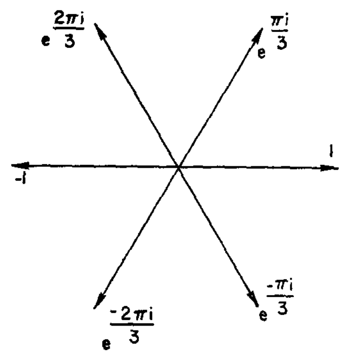



FIG. 3. Irreducible representations of the generator of $C_6$.

$a_R$ which satisfies the relation below:

$$(a_R)^n = -a_1. \tag{72}$$

Now there are as many IR (labeled as $\{D_{(R)}^{(0)} D_{(R)}^{(1)} \cdots D_{(R)}^{(A)} \cdots\}$) of $C_n$ as there are $n$th roots of unity. Similarly, there are as many IR $\{\cdots D^{(A)}(a_R) \cdots\}$ of $\mathcal{A}(C_n)$ as there are roots of minus one. In fact, for $C_n$ one has $D^{(A)}(R)$ = [an $n$th root of (1)], while for $\mathcal{A}(C_n)$ one has $D^{(A)}(R)$ = [an $n$th root of $(-1)$].

Now the IR $\mathfrak{D}^{(\alpha)}$ of $D_n(\mathcal{A}(D_n))$ are obtained from the IR $D^{(A)}$ of $C_n(\mathcal{A}(C_n))$ by induction or extension, depending on whether case (a) or case (b), shown below, is relevant.

*Case (a):* $D^{(A)}$ not equal to $D^{(A)*}$. In this case

$$D^{(A)} \uparrow D_n \sim D^{(A)*} \uparrow D_n[D^{(A)} \uparrow \mathcal{A}(D_n) \sim D^{(A)*} \uparrow \mathcal{A}(D_n)]$$

and induction yields one two-dimensional IR of $D_n$ [of $\mathcal{A}(D_n)$] for each conjugate pair $D^{(A)}$ and $D^{(A)*}$.

*Case (b):* $D^{(A)}$ equals $D^{(A)*}$. In this case extension yields two inequivalent one-dimensional IR of $D_n$ [of $\mathcal{A}(D_n)$].

Examination of the following tables for $n = 2, 3, 4$, and 6 should make this clear, and the results for arbitrary $n$ should thereby be transparent.

$n = 6$. The IR of $C_6$ and $\mathcal{A}(C_6)$ are indicated by vectors in complex plane in Figs. 3 and 4, respectively.

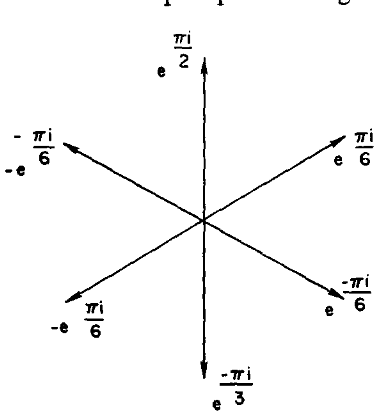



FIG. 4. Irreducible representations of the generator of $\mathcal{A}(C_6, \omega)$.

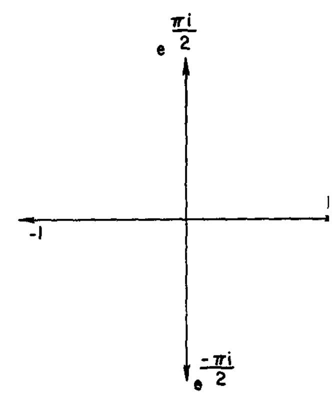
These yield IR of $D_6$ and $\mathcal{A}(D_6)$, whose character tables are drawn below:

| | 1 | $R^2$ | $\rho$ | $R^3$ | $R^5$ | $\rho'$ |
|---|---|---|---|---|---|---|
| (extended) | 1 | 1 | 1 | 1 | 1 | 1 |
| (extended) | 1 | 1 | $-1$ | 1 | 1 | $-1$ |
| (induced) | 2 | $-1$ | 0 | 2 | $-1$ | 0 |
| (extended) | 1 | 1 | 1 | $-1$ | $-1$ | $-1$ |
| (extended) | 1 | 1 | $-1$ | $-1$ | $-1$ | 1 |
| (induced) | 2 | $-1$ | 0 | $-2$ | 1 | 0 |

| | 1 | $R^2$ | $R^5$ |
|---|---|---|---|
| (induced) | 2 | 1 | $\sqrt{3}$ |
| (induced) | 2 | 1 | $-\sqrt{3}$ |
| (induced) | 2 | $-2$ | 0 |

$n = 4$. The IR of $C_4$ and $\mathcal{A}(C_4)$ are indicated in Figs. 5 and 6, respectively. These yield IR of $D_4$ and $\mathcal{A}(D_4)$

FIG. 5. Irreducible representations of the generator of $C_4$.




with the following character tables:

| | 1 | $R^2$ | $R^3$ | $C_1$ | $\dot{C}_2$ |
|---|---|---|---|---|---|
| (extended) | 1 | 1 | 1 | 1 | 1 |
| (extended) | 1 | 1 | $-1$ | $-1$ | 1 |
| (extended) | 1 | 1 | $-1$ | 1 | $-1$ |
| (extended) | 1 | 1 | 1 | $-1$ | $-1$ |
| (induced) | 2 | $-2$ | 0 | 0 | 0 |

| | 1 | $R$ |
|---|---|---|
| (induced) | 2 | $\sqrt{2}$ |
| (induced) | 2 | $-\sqrt{2}$ |

$n = 3$. The IR of $C_3$ and $\mathcal{A}(C_3)$ are indicated in Figs. 7 and 8, respectively. These yield the following

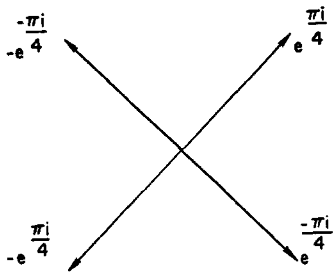



FIG. 6. Irreducible representations of the generator of $\mathcal{A}(C_4, \omega)$.

character tables for the IR of $D_3$ and $\mathcal{A}(D_3)$:

| | $a_R$ | | |
|---|---|---|---|
| (extended) | 1 | 1 | 1 |
| (extended) | 1 | 1 | −1 |
| (induced) | 2 | −1 | 0 |

| | | | |
|---|---|---|---|
| (extended) | 1 | −1 | $i$ |
| (extended) | 1 | −1 | $-i$ |
| (induced) | 2 | 1 | 0 |

.

Note that $\mathcal{A}(D_3)$ is $p$-equivalent to $D_3$.

$n = 2$. The IR of $C_2$ and $\mathcal{A}(C_2)$ are indicated in Figs. 9 and 10.

The resulting IR of $D_2$ and $\mathcal{A}(D_2)$ are characterized as follows:

| | 1 | $R$ | $R'$ | $R''$ |
|---|---|---|---|---|
| (extended) | 1 | 1 | 1 | 1 |
| (extended) | 1 | −1 | 1 | −1 |
| (extended) | 1 | −1 | 1 | −1 |
| (extended) | 1 | −1 | −1 | 1 |

| | 1 |
|---|---|
| (induced) | 2 |

### B. The Groups $T$ and $D$

The tetrahedral group $T$ is of order 12 and contains the group $D_2$ discussed in the preceding section. The representations of a nontrivial ray algebra $\mathcal{A}(T)$, derived from spinors, are characterized below along

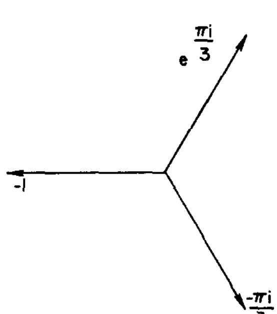



FIG. 7. Irreducible representations of the generator of $C_3$.

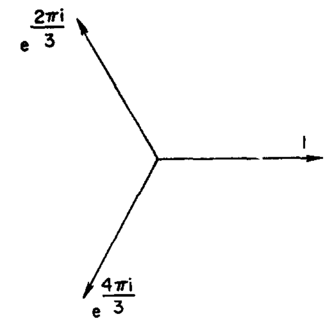



FIG. 8. Irreducible representations of the generator of $\mathcal{A}(C_3, \omega)$.

with the representations of the group $T$ itself:

| $T$ | 1 | $r$ | $r^2$ | $R(2)$ |
|---|---|---|---|---|
| | 1 | 1 | 1 | 1 |
| | 1 | $\exp\left(\frac{2\pi i}{3}\right)$ | $\exp\left(\frac{-2\pi i}{3}\right)$ | 1 |
| | 1 | $\exp\left(\frac{-2\pi i}{3}\right)$ | $\exp\left(\frac{2\pi i}{3}\right)$ | 1 |
| | 3 | 0 | 0 | −1 |

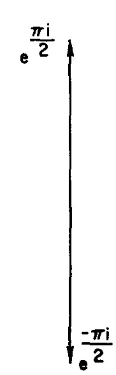
| $\mathcal{A}(T)$ | $a_1$ | $a_r$ | $a_{r_2}$ |
|---|---|---|---|
| | 2 | 1 | 1 |
| | 2 | $\exp\left(\frac{2\pi i}{3}\right)$ | $\exp\left(\frac{-2\pi i}{3}\right)$ |
| | 2 | $\exp\left(\frac{-2\pi i}{3}\right)$ | $\exp\left(\frac{2\pi i}{3}\right)$ |

Here $r$ and $r^2$ indicate the classes (rasses) of 120° rotations around the points and faces of a tetrahedron,

FIG. 9. Irreducible representations of the generator of $C_2$.


while the $R(2)$ indicates the class $\{R, R', R''\}$ of 180° rotations about the tetrahedron edges. [Clearly, $a_{R(2)}$ is in a zass of $\mathcal{A}(T)$.] Note that all the representations of $\mathcal{A}$(T) are extensions of the one IR of $\mathcal{A}(D_2)$.

FIG. 10. Irreducible representations of the generator of $\mathcal{A}(C_2, \omega)$.



The octahedral (cubic) group $O$ is of order 24 and has $T$ as a subgroup. The representations of $O$ and a nontrivial ray algebra $\mathcal{A}(O)$ are characterized below. Here class $r$ of $O$ contains classes $r$ and $r^2$ of $T$. $R(4)$ and $R(2)$ are classes of 90° and 180° rotations, respectively, around cube faces. $i(2)$ is the class of 180° rotations around cube edges.

| 0 | 1 | $r$ | $R_{(4)}$ | $R_{(2)}$ | $i_{(2)}$ |
|---|---|---|---|---|---|
| (extended) | 1 | 1 | 1 | 1 | 1 |
| (extended) | 1 | 1 | −1 | 1 | −1 |
| (induced) | 2 | −1 | 0 | 2 | 0 |
| (extended) | 3 | 0 | −1 | −1 | 1 |
| (extended) | 3 | 0 | 1 | −1 | −1 |

| $\mathcal{A}(0)$ | $a_1$ | $a_r$ | $a_{R(4)}$ |
|---|---|---|---|
| (extended) | 2 | 1 | $\sqrt{2}$ |
| (extended) | 2 | 1 | $-\sqrt{2}$ |
| (induced) | 4 | −1 | 0 |

The first two IR of $\mathcal{A}(0)$ are simply extensions of the one real IR of $\mathcal{A}(\mathrm{T})$. The third one can be induced by either of the complex IR of $\mathcal{A}(\mathrm{T})$.

The *diophantine solution* of $\mathcal{A}(0)$ is $2^2 + 2^2 + 4^2 = 24$. One sees from Theorem 1 that the solution $2^2 + 2^2 + 2^2 + 2^2 + 2^2 + 2^2 = 24$ could never exist for any ray algebra of 0.

## 11. OUTER PRODUCTS OF RAY REPRESENTATIONS

Suppose one has a number $n$ of different non-$p$-equivalent factor systems defined over a group $\mathcal{G}$, such that

$$\{\omega^{(1)}_{R,S} = 1\}\{\omega^{(2)}_{R,S}\cdots\}\{\cdots\omega^{(3)}_{R,S}\cdots\}\cdots\{\cdots\omega^{(n)}_{R,S}\cdots\}.$$

(Here $\omega^{(1)}$ is the trivial factor system.) Suppose also that no other factor system can be defined over $\mathcal{G}$ that is not $p$-equivalent to one of these.

Denote by $\{\mathfrak{D}^{(\alpha)}, \mathfrak{D}^{(\alpha)'}, \cdots\}$ the set of all IR of $\mathcal{A}(\mathcal{G}, \omega^{(a)})$ and by $\{\mathfrak{D}^{(\beta)}, \mathfrak{D}^{(\beta)}, \cdots\}$ the IR of $\mathcal{A}(\mathcal{G}, \omega^{(b)})$. Thus,

$$\mathfrak{D}^{(\alpha)}(a_R)\mathfrak{D}^{(\alpha)}(a_S) = \omega^{(a)}_{R,S}\mathfrak{D}(a_{RS}),$$
$$\mathfrak{D}^{(\beta)}(a_R)\mathfrak{D}^{(\beta)}(a_S) = \omega^{(b)}_{R,S}\mathfrak{D}(a_{RS}). \quad (73)$$

Consider the outer (tensor) products of two of these representations:

$$\{\cdots\mathfrak{D}^{(\alpha)}(a_R)\otimes\mathfrak{D}^{(\beta)}(a_R),\cdots,\mathfrak{D}^{(\alpha)}(a_S)\otimes\mathfrak{D}^{(\beta)}(a_S),\cdots\}.$$

These form a ray representation of $\mathcal{G}$:

$$[\mathfrak{D}^{(\alpha)}(a_R)\otimes\mathfrak{D}^{(\beta)}(a_R)][\mathfrak{D}^{(\alpha)}(a_S)\otimes\mathfrak{D}^{(\beta)}(a_S)] = \omega^{(a)}_{R,S}\omega^{(b)}_{R,S}[\mathfrak{D}^{(\alpha)}(a_{RS})\otimes\mathfrak{D}^{(\beta)}(a_{RS})]. \quad (74)$$

But by (74) the factor system of this representation is (in general) $p$-equivalent to some other factor system $\omega^{(c)}$. Therefore $(\mathfrak{D}^{(\alpha)} \otimes \mathfrak{D}^{(\beta)})$ can be reduced by similarity transformation $\mathcal{C}$, and projective transformations $\{\cdots c_R\cdots c_S\cdots\}$, to a direct sum of the IR $\{\mathfrak{D}^{(\gamma)}, \mathfrak{D}^{(\gamma)'}\}$ of $\mathcal{A}(\mathcal{G}, \omega^{(c)})$:

$$\mathcal{C}^{-1}[c_R\mathfrak{D}^{(\alpha)}(a_R)\otimes\mathfrak{D}^{(\beta)}(a_R)]\mathcal{C} = \begin{vmatrix} \boxed{\mathfrak{D}^{(\gamma)}(a_R)} & & \\ & \boxed{\mathfrak{D}^{(\gamma)'}(a_R)} & \\ & & \ddots \end{vmatrix}. \quad (75)$$

One may use the unit dyads to compute the constants $c_R$ and the matrices $\mathcal{C}$ indicated in (75). The components of the $\mathcal{C}$ matrix are *Clebsch–Gordan coefficients.*

One should note the difference between the preceding theory and that of Rudra. For example, in Eq. (21) of his first paper[7] he defines the direct product of two representations $\Gamma_\mu$ and $\Gamma_\nu$ belonging to a particular factor system $\{\omega_{R,S}\}$ by

$$\Gamma_{\mu\otimes\nu}(R)_{ik,jL} \equiv (\Gamma_\mu(R)\otimes\Gamma_\mu(R))_{ik,jL} = \Big[\prod_{P\in\mathcal{G}}\omega_{R,P}\omega_{P,R}\Big]^{-\frac{1}{2}g}\Gamma_\mu(R)_{ij}\Gamma_\nu(R)_{kL}$$

and supposedly obtains a representation $\Gamma_{\mu\otimes\nu}$ that belongs to that *same* factor system. But we have shown that this is impossible if $\{\omega_{R,S}\}$ is not $p$-equivalent to the trivial factor system.