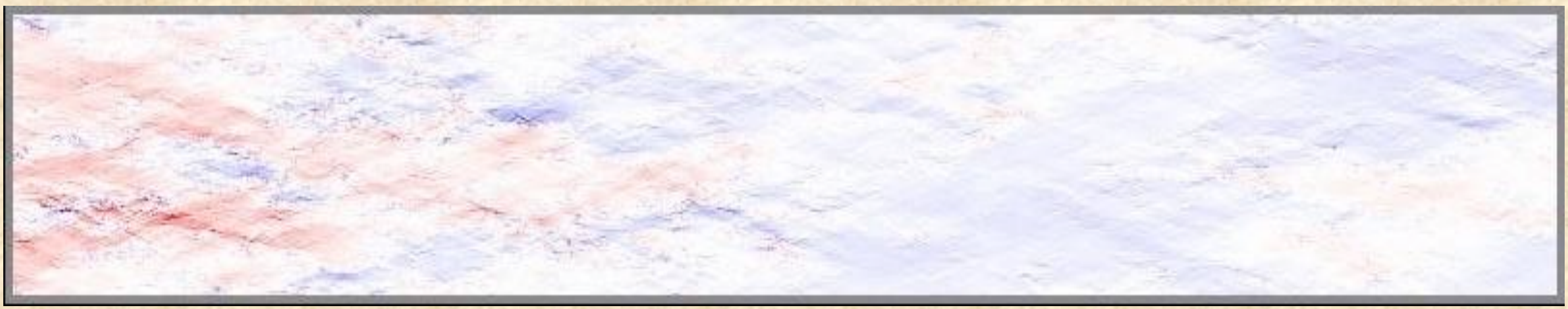
Characterizing dynamics with covariant Lyapunov vectors

Francesco Ginelli

Institut des Systèmes Complexes Paris Ile de France & CEA/Saclay

Joint work with:

A Politi, H. Chaté, R. Livi, P. Poggi, A. Turchi



Lyapunov Exponents

- Chaotic dynamics is characterized by exponential sensitivity to initial conditions:

$$\vec{x}_{t+1} = \vec{\mathbf{F}}(\vec{x}_t)$$

$$\|\delta\,\vec{x}_t\| \approx \|\delta\,\vec{x}_0\| \cdot \exp[\lambda_1\ t] \qquad t >> 1$$

$$\frac{d}{dt}\vec{x}_t = \vec{\mathbf{F}}(\vec{x}_t)$$

- Tangent evolution of linearized perturbations is ruled by the *Jacobian:*

$$\mathbf{J}_t\ : \qquad [\mathbf{J}_t]_{\mu\nu} = \frac{\partial F_\mu(\vec{x}_t)}{\partial x_\nu}$$

$$\delta\,\vec{x}_t = \mathbf{M}(\vec{x}_0,t)\ \delta\,\vec{x}_0$$

$$\mathbf{M}(\vec{x}_0,t) = \left(\mathbf{J}_{t-1}\mathbf{J}_{t-2}\cdots\mathbf{J}_{t_0+1}\mathbf{J}_{t_0}\right)$$

$$\frac{d}{dt}\mathbf{M}(\vec{x}_0,t) = \mathbf{J}_t\,\mathbf{M}(\vec{x}_0,t) \qquad \mathbf{M}(\vec{x}_0,0) = \mathbf{I}$$

Lyapunov Exponents

- The existence of a complete set of N LEs is granted by the Oseledec theorem:

$$\Lambda_+(\vec{x}_0) = \lim_{t\to\infty}\left[\mathbf{M}^{\mathrm{T}}(\vec{x}_0,t)\mathbf{M}(\vec{x}_0,t)\right]^{1/(2t)}$$

$$\Lambda_+(\vec{x}_0)\vec{e}_+^{\,j}(\vec{x}_0) = \gamma_j\,\vec{e}_+^{\,j}(\vec{x}_0)$$

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N \qquad \lambda_j = \ln\gamma_j$$

- There exist a sequence of nested subspaces connected with these growth rates:

$$\mathbf{R}^N = \Gamma_{\vec{x}_0}^{(1)} \supset \Gamma_{\vec{x}_0}^{(2)} \supset \cdots \supset \Gamma_{\vec{x}_0}^{(N)} \qquad \lambda_j \quad \text{exp. growth rate of } \vec{u} \in \Gamma_{\vec{x}_0}^{(j)} \backslash \Gamma_{\vec{x}_0}^{(j+1)}$$

$$\dim\left(\Gamma_{\vec{x}_0}^{(j)}\right) = N - j + 1$$

- LEs quantify the growth of volumes in tangent space

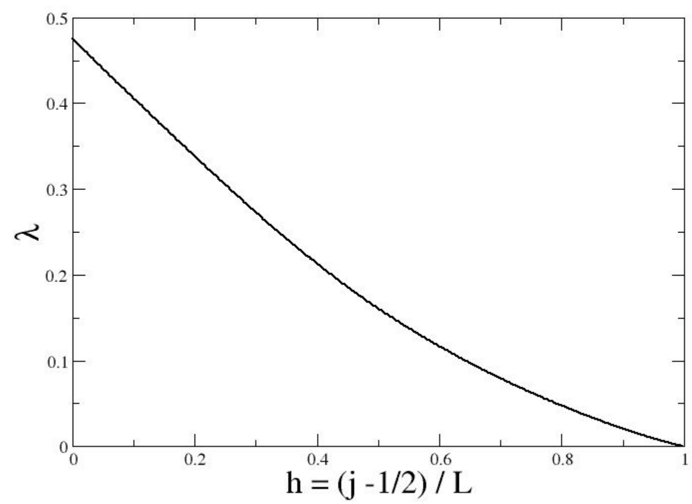
- Entropy production (Kolmogorov-Sinai entropy):

$$H_{KS} = \sum_{\lambda_i > 0} \lambda_i$$

- Attractor dimension (Kaplan Yorke Formula)

$$D_{KY} = k + \frac{\sum_{i=1}^{k} \lambda_i}{|\lambda_{k+1}|}$$

- There exist a thermodynamic limit for Lyapunov spectra in spatially ext. systems:



Lyapunov Vectors ?

- After exponents (i.e. eigenvalues), people got interested in vectors (i.e. eigenvectors ?) to quantify stable and unstable directions in tangent space.

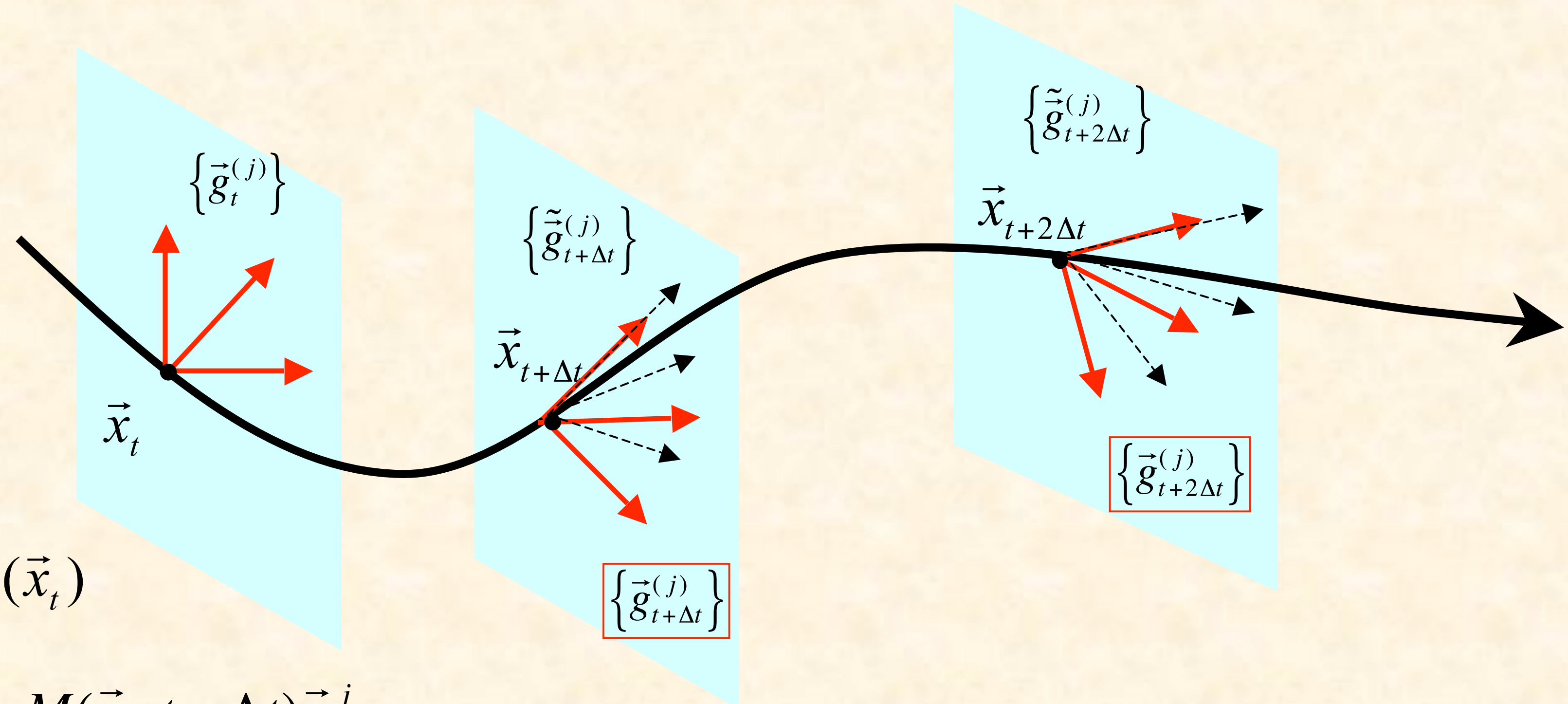
- Hierarchical decomposition of spatiotemporal chaos

- Optimal forecast in nonlinear models (e.g. in geophysics)

- Study of "hydrodynamical modes" in near-zero exponents and vectors (access to transport properties ?)

But… which vectors ?

Gram Schmidt vectors



$$\frac{d\,\vec{x}_t}{dt} = \vec{\mathbf{F}}(\vec{x}_t)$$

$$\tilde{\vec{g}}^{\,j}{}_{t+\Delta t} = M(\vec{x}_t, t+\Delta t)\vec{g}^{\,j}{}_t$$

Gram Schmidt vectors are obtained by GS orthogonalization (Benettin *et al.* 1980)

$$\tilde{\mathbf{G}}_t = \left(\tilde{g}_t^{\,1}\middle|\cdots\middle|\tilde{g}_t^{\,N}\right)$$

$$\mathbf{Q}_t = \left(g_t^{\,1}\middle|\cdots\middle|g_t^{\,N}\right)$$

Upper triangular

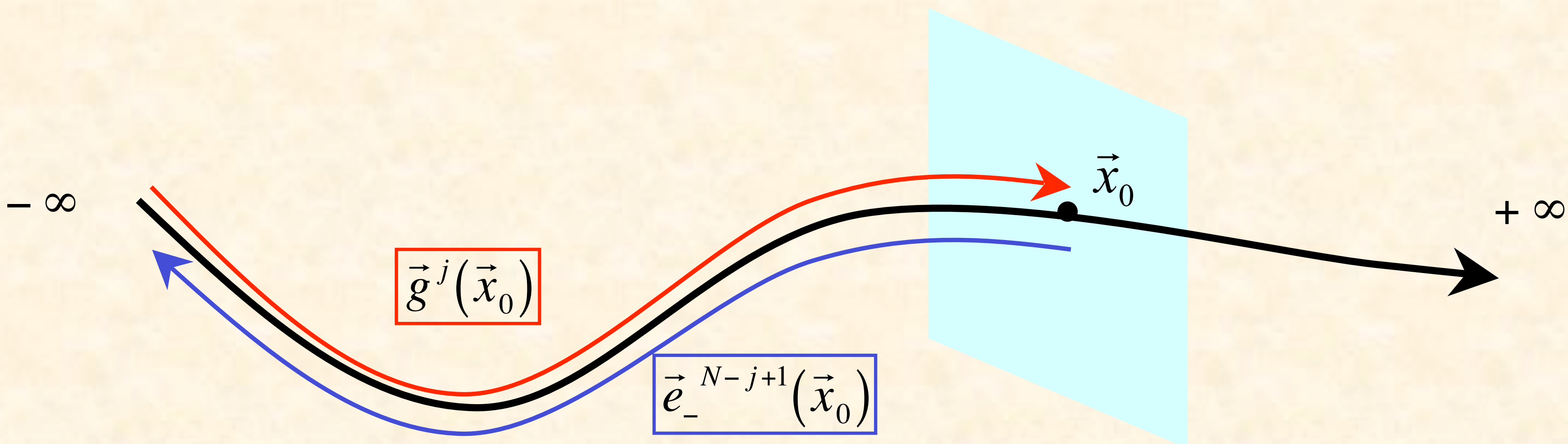
$$\tilde{\mathbf{G}}_{t+\Delta t} = \mathbf{Q}_{t+\Delta t}\mathbf{R}_{t,\Delta t}$$

- It can be shown that any orthonormal set of vectors eventually converge to a well defined basis *(Ershov and Potapov, 1998)*

- For time-invertible systems they coincide with the eigenvectors of the backward Oseledec matrix:

$$\vec{g}^{\,j} \longrightarrow \vec{e}_-^{\,N-j+1}$$

$$\Lambda_-(x_0) = \lim_{t \to -\infty} \left[\mathbf{M}^{-1}(x_0,t)^T \mathbf{M}^{-1}(x_0,t) \right]^{1/2t}$$



But…

- They are orthogonal, while *stable* and *unstable* manifolds are generally not.

- Dynamical properties are "washed away" by orthonormalization, which is norm dependent, while LEs are not (for a wide class of norms).

- They are not invariant under time reversal, while LEs are (sign-wise):

$$\vec{g}_{+}^{\ j} \neq \vec{g}_{-}^{\ N-j+1} \qquad \lambda^{+}_{j} = -\lambda^{-}_{N-j+1}$$

- They are not covariant with dynamics and do not yield correct growth factors:

$$\mathbf{M}(\vec{x}_t, t+\Delta t)\vec{g}_t^{\ j} \neq \gamma_j \vec{g}_{t+\Delta t}^{\ j} \qquad \left\langle \ln\left\| \mathbf{M}(\vec{x}_t, t+\Delta t)\vec{g}_t^{\ j} \right\| \right\rangle \neq \lambda_j$$

Covariant Lyapunov vectors v

- Ruelle (1979) – Oseledec splitting

$$\vec{v}^{\,j} \quad \text{spans} \quad \mathbf{E}_{\vec{x}_0}^{(j)} = \mathbf{\Gamma}_{\vec{x}_0}^{(j)} \cap \overline{\mathbf{\Gamma}}_{\vec{x}_0}^{(j)}$$

$$\mathbf{\Gamma}_{\vec{x}_0}^{(j)} = \mathbf{U}_+^{(J)}(\vec{x}_0) \oplus \cdots \oplus \mathbf{U}_+^{(N)}(\vec{x}_0)$$

$$\overline{\mathbf{\Gamma}}_{\vec{x}_0}^{(j)} = \mathbf{U}_-^{(N-j+1)}(\vec{x}_0) \oplus \cdots \oplus \mathbf{U}_-^{(N)}(\vec{x}_0)$$

$$\mathbf{U}_\pm^{(J)}(\vec{x}_0) \quad \text{eigenspaces of} \quad \Lambda_\pm(\vec{x}_0)$$

$$\dim\left[\mathbf{\Gamma}_{\vec{x}_0}^{(j)}\right] = N - j + 1 \qquad \dim\left[\overline{\mathbf{\Gamma}}_{\vec{x}_0}^{(j)}\right] = j$$

- They are covariant with dynamics and do yield correct growth factors (LEs):

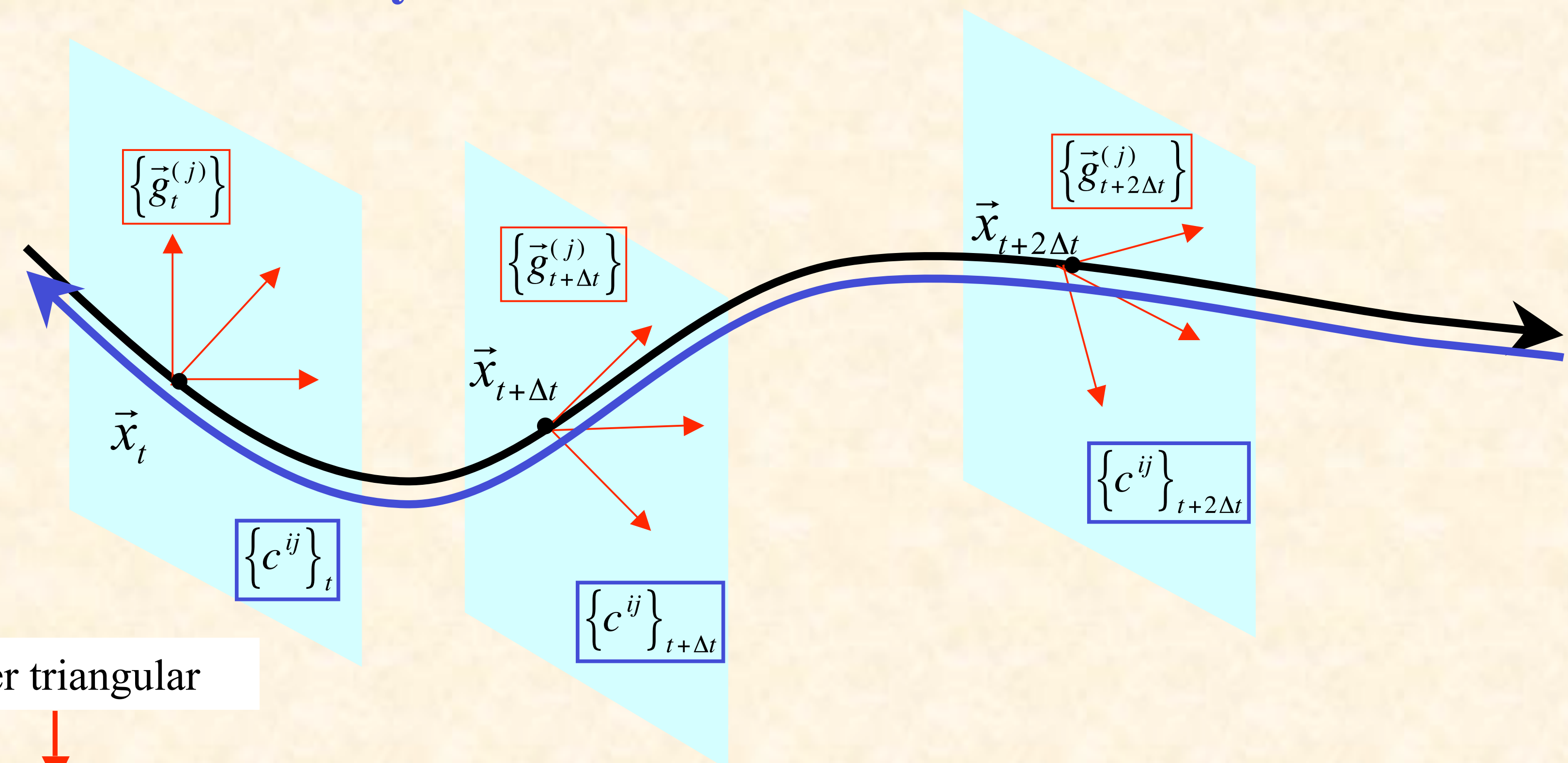
$$\mathbf{M}(\vec{x}_t, t + \Delta t)\vec{v}_t^{\,j} = \gamma_j \, \vec{v}_{t+\Delta t}^{\,j} \qquad \left\langle \ln\left\|\mathbf{M}(\vec{x}_t, t + \Delta t)\vec{v}_t^{\,j}\right\|\right\rangle = \lambda_j$$

After Ruelle

- Brown, Bryant & Abarbanel (1991) – Covariant vectors in time series data analysis
- Legras & Vautard; Trevisan & Pancotti (1996) – Covariant vectors in Lorenz 63
- Politi *et. al.* (1998) – Covariant vectors satisfy a node theorem for periodic orbits
- Wolfe & Samelson (2007) – Intersection algorithm, more efficient for $j \ll N$

Lack of a practical algorithm to compute them
No studies of ensemble properties in large systems

Computing covariant Lyapunov Vectors *v* by forward-backward iterations



Upper triangular

$$[\mathbf{C}_t]_{ij} = c_t^{ij} = \left(\vec{g}_t^{\,i} \cdot \vec{v}_t^{\,j}\right)$$

Consider vectors which are linear combinations of the first *j* Gram-Schmidt vectors *g*

$$\vec{v}_t^{\,j} = \sum_{i=1}^{j} c_t^{ij} \vec{g}_t^{\,i}$$

$$\sum_{i=1}^{j} \left[c_t^{ij}\right]^2 = 1$$

1. R evolves the coefficients C according to tangent dynamics

Covariant evolution means:

$$\mathbf{V}_{t+\Delta t}\,\Delta_{t,\Delta t} = \mathbf{M}_{t,\Delta t}\,\mathbf{V}_t \qquad \left(\mathbf{M}_{t,\Delta t} \equiv \mathbf{M}(\vec{x}_t, t+\Delta t)\right)$$

(Expand CLV on GS basis)

$$\left(v_t^1 \middle| v_t^2 \middle| \cdots \middle| v_t^N\right) \equiv \mathbf{V}_t = \mathbf{Q}_t \mathbf{C}_t$$

$$\mathbf{Q}_{t+\Delta t}\mathbf{C}_{t+\Delta t}\,\Delta_{t,\Delta t} = \mathbf{M}_{t,\Delta t}\,\mathbf{Q}_t\mathbf{C}_t$$

(use QR decomposition)

$$\mathbf{M}_{t,\Delta t}\mathbf{Q}_t = \tilde{\mathbf{G}}_{t+\Delta t} = \mathbf{Q}_{t+\Delta t}\mathbf{R}_{t,\Delta t}$$

$$\mathbf{Q}_{t+\Delta t}\mathbf{C}_{t+\Delta t}\,\Delta_{t,\Delta t} = \mathbf{Q}_{t+\Delta t}\mathbf{R}_{t,\Delta t}\mathbf{C}_t$$

one gets the evolution rule

$$\mathbf{C}_{t+\Delta t}\,\Delta_{t,\Delta t} = \mathbf{R}_{t,\Delta t}\mathbf{C}_t$$

2. Moving backwards insures convergence to the “right” covariant vectors

$$\mathbf{R}_{t,\Delta t}{}^{-1}\tilde{\mathbf{C}}_{t+\Delta t}\,\tilde{\Delta}_{t,\Delta t} \to \mathbf{C}_t$$

(*consider two different random initial conditions*)

$$\tilde{\mathbf{C}}_{t+\Delta t}\,\Delta_{t,\Delta t} = \mathbf{R}_{t,\Delta t}\tilde{\mathbf{C}}_t \qquad \tilde{\tilde{\mathbf{C}}}_{t+\Delta t}\,\tilde{\tilde{\Delta}}_{t,\Delta t} = \mathbf{R}_{t,\Delta t}\tilde{\tilde{\mathbf{C}}}_t$$

A. If C are *upper triangular with non-zero diagonal*, one can verify that

$$\tilde{\tilde{\Delta}}_{t,\Delta t}\,,\tilde{\Delta}_{t,\Delta t} \xrightarrow{\Delta t \to \pm\infty} \mathrm{diag}\left(e^{\pm\Delta t\,\lambda_1}, e^{\pm\Delta t\,\lambda_2}, \cdots, e^{\pm\Delta t\,\lambda_N}\right)$$

B. By simple manipulations

$$\mathbf{R}_{t,\Delta t} = \tilde{\mathbf{C}}_{t+\Delta t}\,\Delta_{t,\Delta t}\tilde{\mathbf{C}}_t^{-1} = \tilde{\tilde{\mathbf{C}}}_{t+\Delta t}\,\Delta_{t,\Delta t}\tilde{\tilde{\mathbf{C}}}_t^{-1}$$

$$\Rightarrow \left[\tilde{\tilde{\mathbf{C}}}_{t+\Delta t}^{-1}\tilde{\mathbf{C}}_{t+\Delta t}\right] = \tilde{\tilde{\Delta}}_{t,\Delta t}\left[\tilde{\tilde{\mathbf{C}}}_t^{-1}\tilde{\mathbf{C}}_t\right]\tilde{\Delta}_{t,\Delta t}^{-1}$$

(*by matrix components*)

$$\Rightarrow \left[\tilde{\tilde{\mathbf{C}}}^{-1}_{t+\Delta t}\tilde{\mathbf{C}}_{t+\Delta t}\right]_{\mu\nu} \to \exp\left[\Delta t\left(\lambda_\mu - \lambda_\nu\right)\right]\left[\tilde{\tilde{\mathbf{C}}}_t^{-1}\tilde{\mathbf{C}}_t\right]_{\mu\nu}$$

$$\Rightarrow \left[\tilde{\tilde{\mathbf{C}}}^{-1}_{t+\Delta t}\tilde{\mathbf{C}}_{t+\Delta t}\right]_{\mu\nu} \approx \begin{cases} 0 & \mu > \nu \\ \exp\left[\Delta t\left(\lambda_\mu - \lambda_\nu\right)\right] & \mu < \nu \qquad \lambda_\mu - \lambda_\nu > 0 \\ \phi_\mu & \mu = \nu \end{cases}$$

If we follow the reversed dynamics

$$\tilde{\tilde{\mathbf{C}}}^{-1}_{t+\Delta t}\tilde{\mathbf{C}}_{t+\Delta t} \xrightarrow{\Delta t \to -\infty} \Phi \qquad (\textit{diagonal matrix})$$

$$\Rightarrow \tilde{\mathbf{C}}_{t+\Delta t} \xrightarrow{\Delta t \to -\infty} \tilde{\tilde{\mathbf{C}}}_{t+\Delta t}\Phi$$

All random initial conditions converge to the same ones, apart a prefactor

Thus this reversed dynamics converges to covariant vectors for almost any initial condition

Covariant Lyapunov Vectors properties

- They coincide with *stable* and *unstable* manifolds

- They are invariant under time reversal.

$$\vec{v}_+^{\,j} = \vec{v}_-^{\,N-j+1} \qquad \lambda^+_j = -\lambda^-_{N-j+1}$$

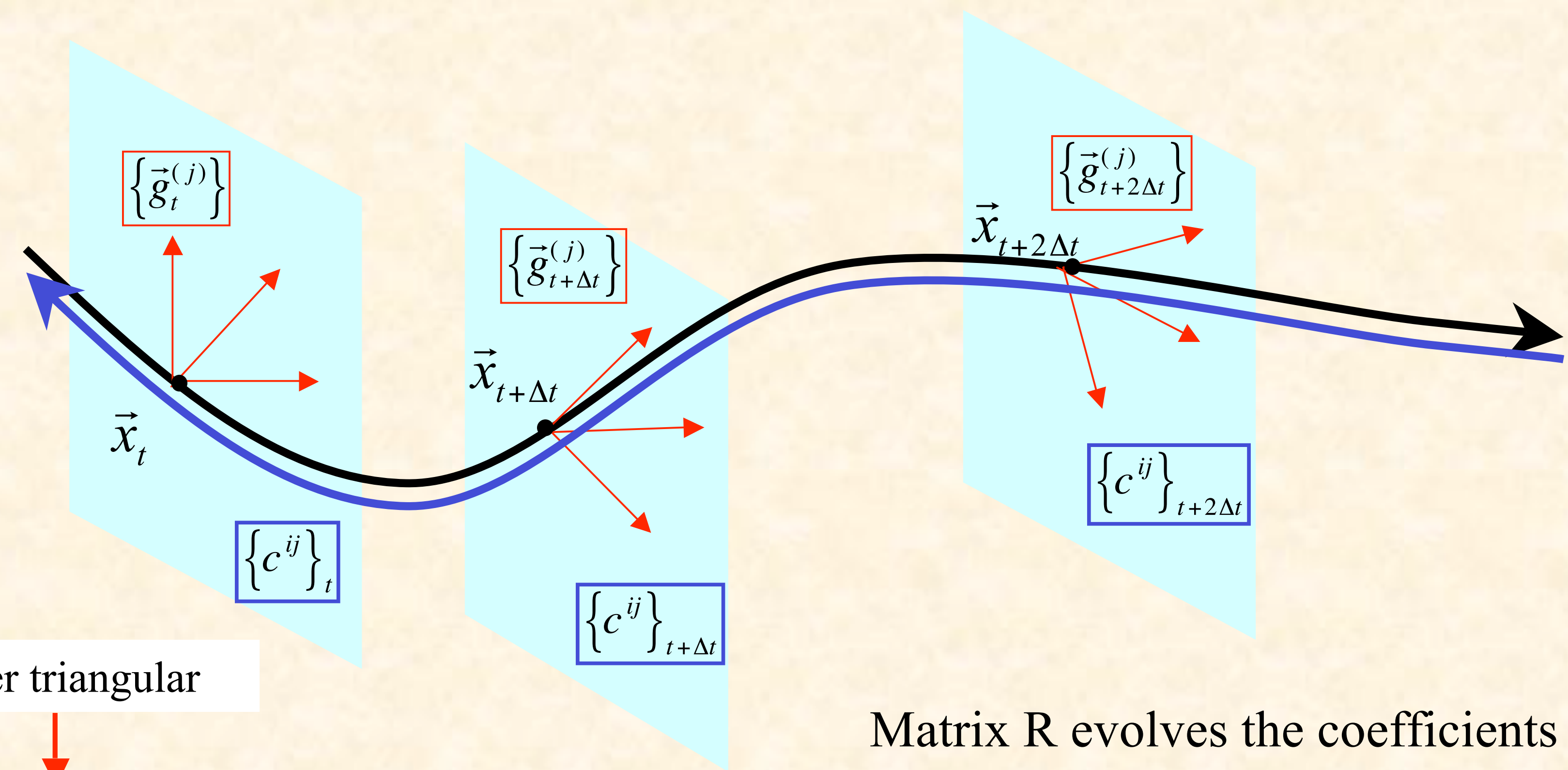
- They are covariant with dynamics and do yield correct growth factors (LEs):

$$\mathbf{M}_{t,\Delta t}\vec{v}_t^{\,j} = \gamma_j \vec{v}_{t+\Delta t}^{\,j} \qquad \left\langle \ln\left\| \mathbf{M}_{t,\Delta t}\vec{v}_t^{\,j} \right\| \right\rangle = \lambda_j$$

- They are norm independent and, for time reversible systems, coincide with the Oseledec splitting (*Ruelle 1979*)

- They can be computed for non time reversible systems too by following backward a stored forward trajectory

The stable algorithm for covariant Lyapunov Vectors



Upper triangular

$$[\mathbf{C}_t]_{ij} = c_t^{ij} = \left(\vec{g}_t^{\,i} \cdot \vec{v}_t^{\,j}\right)$$

$$\vec{v}_t^{\,j} = \sum_{i=1}^{j} c_t^{ij} \vec{g}_t^{\,i}$$

Matrix R evolves the coefficients C according to tangent dynamics

$$\left(\mathbf{R}_{t,\Delta t}\right)^{-1} \mathbf{C}_{t+\Delta t} \Delta_{t,\Delta t} = \mathbf{C}_t$$

A Simple recipe

- Start from a random initial condition.

- Run a forward transient to obtain convergence of GS vectors

- Continue your phase space trajectory continuously storing the QR decomposition of tangent space.

- Run a final backward transient only storing the R matrices from QR

- Generate a random upper triangular matrix C

- Evolve C backward by inverting R matrices along the backward transient

- Convergence to CLV coefficients is ruled by difference between nearest LEs

- Once backward transient has been done and CLV coefficients are converged, continue to move backward along trajectories. CLV can be recovered as V=QC

- Some further tricks to ease memory storage in RAM are possible

On Wolfe & Samelson (2007): vector *n-th* out of *N*

$$\phi_n = \sum_{i=n}^{N} \langle \hat{\xi}_i, \phi_n \rangle \hat{\xi}_i,$$

$$\phi_n = \sum_{j=1}^{n} \langle \hat{\eta}_j, \phi_n \rangle \hat{\eta}_j,$$

$$\sum_{j=1}^{n} \langle \hat{\eta}_j, \phi_n \rangle \hat{\eta}_j = \sum_{i=n}^{N} \langle \hat{\xi}_i, \phi_n \rangle \hat{\xi}_i$$

$$\langle \hat{\eta}_k, \phi_n \rangle = \sum_{j=1}^{n} \left[\sum_{i=n}^{N} \langle \hat{\eta}_k, \hat{\xi}_i \rangle \langle \hat{\xi}_i, \hat{\eta}_j \rangle \right] \langle \hat{\eta}_j, \phi_n \rangle \quad k \le n.$$

since $\sum_{k=1}^{N} \langle f_i, e_k \rangle \langle e_k, f_j \rangle = \delta_{ij}.$

$$\sum_{j=1}^{n} \sum_{i=1}^{n-1} \langle \hat{\eta}_k, \hat{\xi}_i \rangle \langle \hat{\xi}_i, \hat{\eta}_j \rangle \langle \hat{\eta}_j, \phi_n \rangle = 0 \quad k \le n.$$

$$\mathbf{D}^{(n)} \mathbf{y}^{(n)} = 0,$$

where $y_k^{(n)} = \langle \hat{\eta}_k, \phi_n \rangle \quad k = 1, 2, \ldots, n,$

$$D_{kj}^{(n)} = \sum_{i=1}^{n-1} \langle \hat{\eta}_k, \hat{\xi}_i \rangle \langle \hat{\xi}_i, \hat{\eta}_j \rangle \quad k, j \le n.$$

n - 1 forward and *n* backward GSV are needed to compute the kernel

Some applications

- Angles between CLV or linear combinations of CLV: hyperbolicity.
- Localization properties
- Hydrodynamic modes …
- Data assimilation algorithms ?

1. Localization properties in spatially extended systems

•Localization properties of vector *j* can be characterized by the inverse participation ratio

$$Y_2(j) = \frac{\left\langle \sum_i \left(\alpha_i^j\right)^4 \right\rangle}{\left(\sum_i \left(\alpha_i^j\right)^2\right)^2}$$

where

$$\alpha_i^j = \begin{cases} \delta x_i \\ \sqrt{\delta q_i^{\,2} + \delta p_i^{\,2}} \end{cases}$$

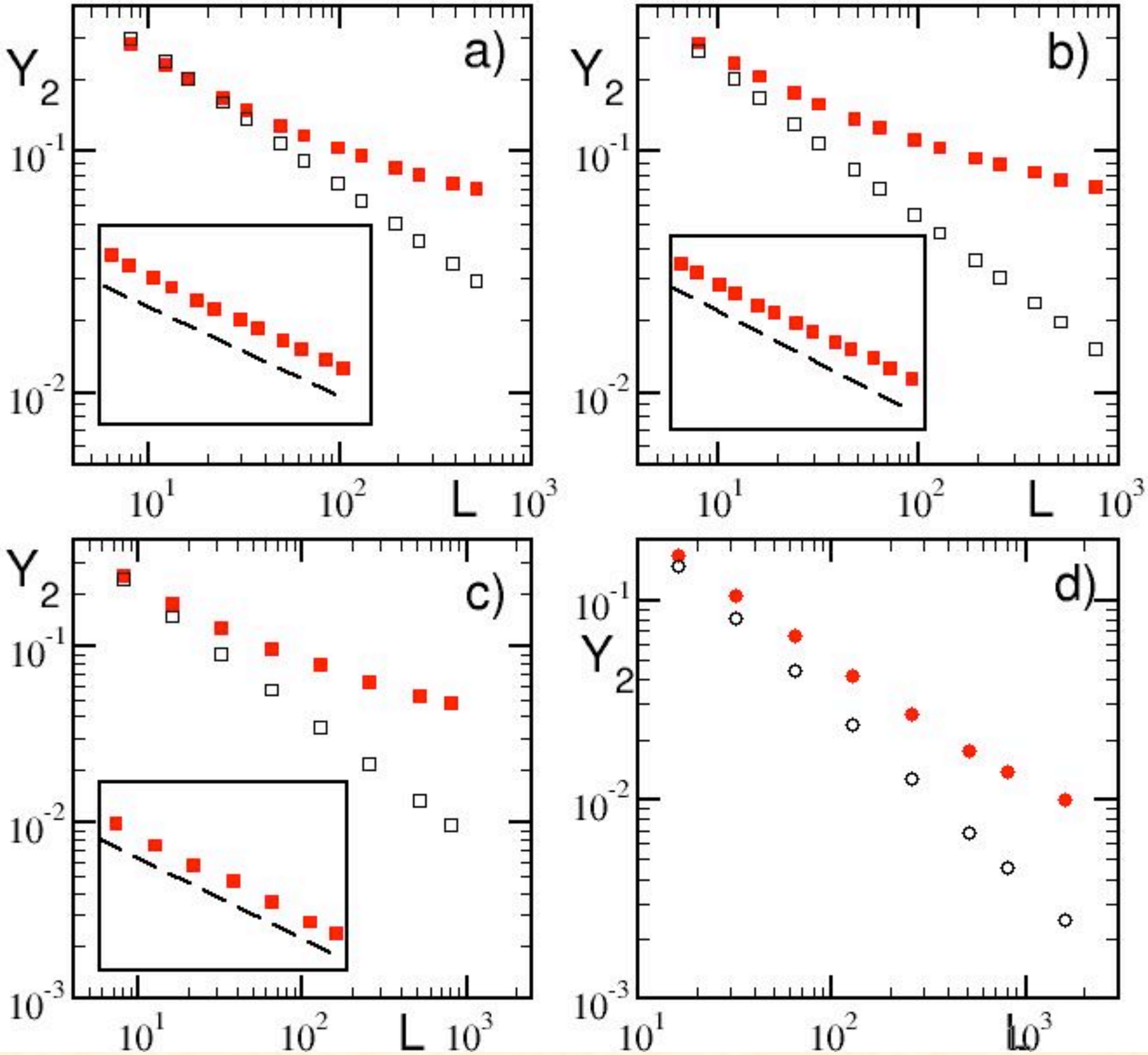
• Localized: nonvanishing Y_2

$$Y_2(j) \approx 1/\ell + L^{-\gamma}$$

• Delocalized: vanishing Y_2

$$Y_2(j) \approx 1/L$$

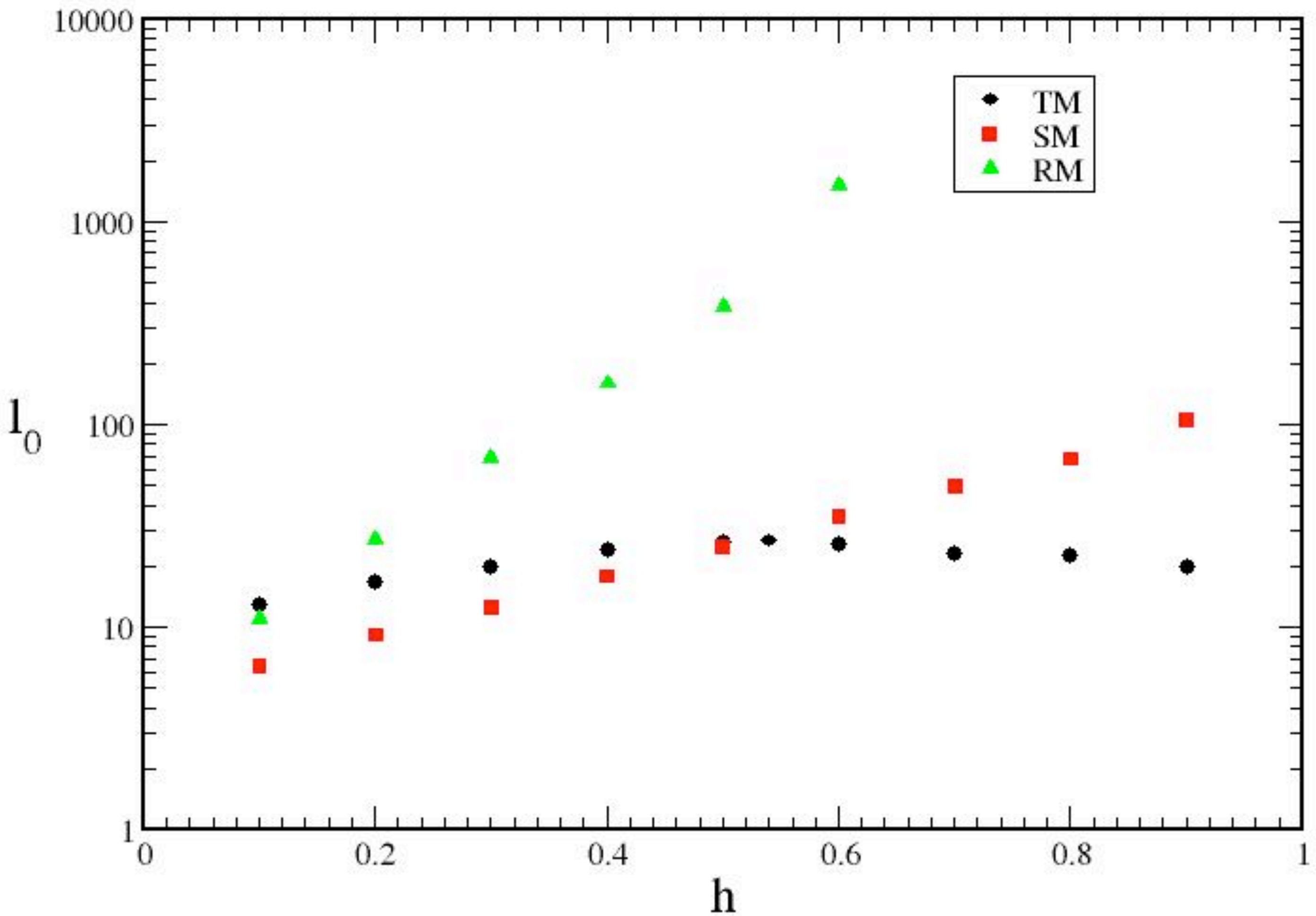
Localization in spatially extended systems – Numerical results



a) CML of Tent maps
b) Simplectic maps
c) Rotors
d) FPU

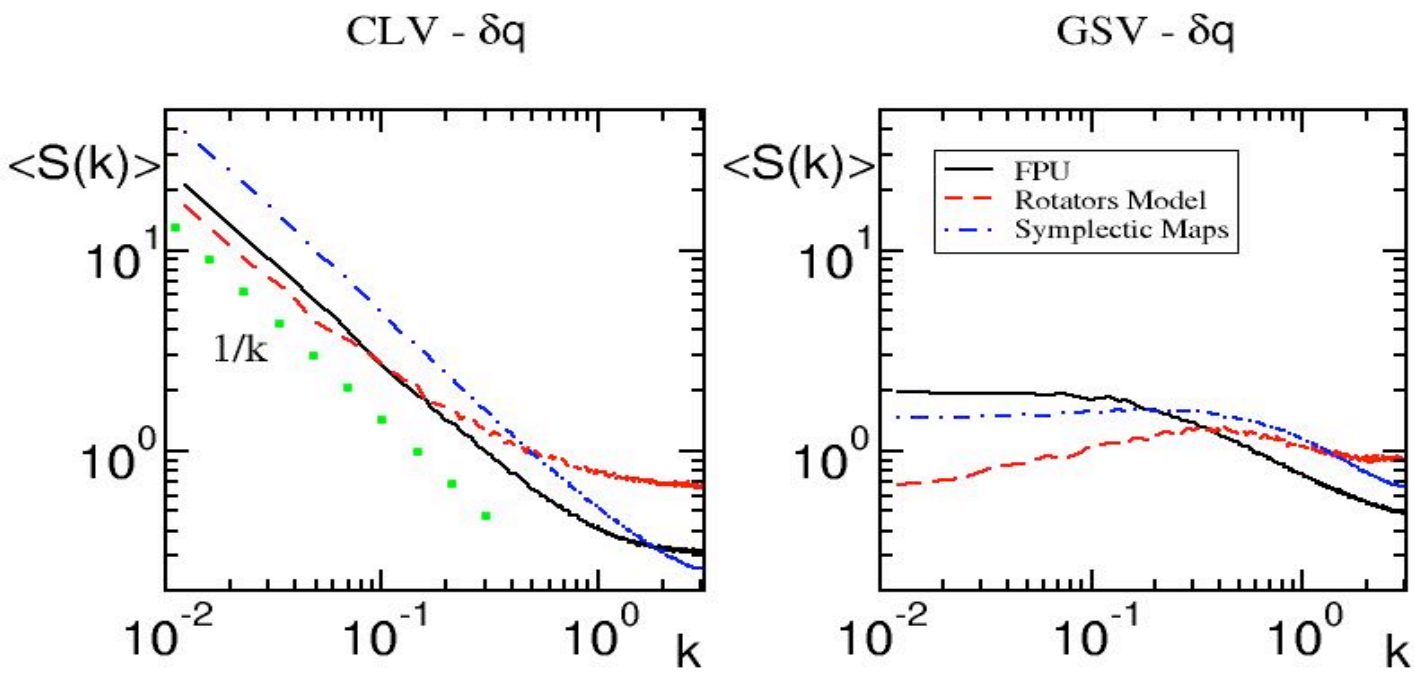
$$h = i/L = 0.2$$

Localization length



Fourier analysis of "last positive" vector

$$S(k) = \left| \sum_m \delta q_m e^{imk} \right|^2 \qquad k = 2\pi \frac{j}{L}$$



2. Density of Hyperbolicity "violations"

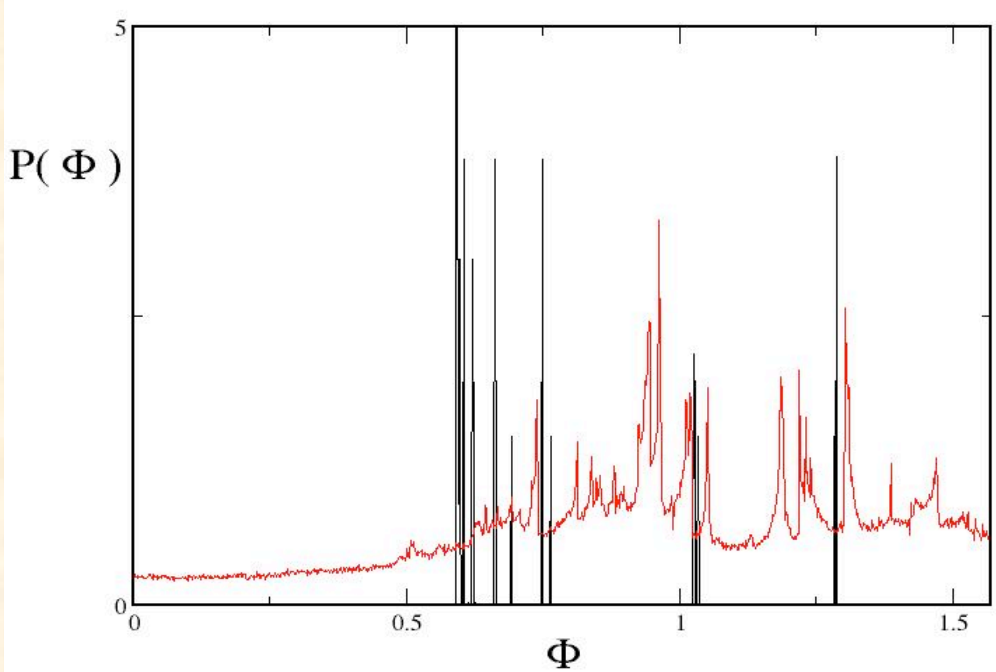
$$\Phi_n = \cos^{-1}\left(\left|\vec{v}_n^{(1)} \cdot \vec{v}_n^{(2)}\right|\right)$$

• Hénon Map

$$x_{n+1} = 1 - 1.4\, x_n^{\;2} + 0.3\, x_{n-1}$$

• Lozi Map

$$x_{n+1} = 1 - 1.4\left|x_n\right| + 0.3\, x_{n-1}$$



More then 2 dimensions, linear combinations between vectors should be considered

(Kuptsov & Kuznetsov ArXiv:0812.4823 (2009))

$$\vec{u}_n^{(u)} = \sum_{i=1}^{u} \beta_n^i \vec{v}_n^{(i)} \qquad \vec{u}_n^{(s)} = \sum_{i=u+1}^{N} \alpha_n^i \vec{v}_n^{(i)} \qquad w_n = \vec{u}_n^{(u)} \cdot \vec{u}_n^{(s)}$$

$$\mathbf{C}_n = \Big(\underbrace{\vec{c}_n^{\,1} \; \Big| \ldots \; \Big| \vec{c}_n^{\,u}}_{\mathbf{U}:\ \lambda_i > 0} \; \Big| \underbrace{\vec{c}_n^{\,u+1} \; \Big| \ldots \; \Big| \vec{c}_n^{\,u+s}}_{\mathbf{S}:\ \lambda_i < 0} \Big)$$

$$\mathbf{U}_n = \mathbf{Q}^{(u)}{}_n \, \mathbf{R}^{(u)}{}_n \qquad \mathbf{S}_n = \mathbf{Q}^{(s)}{}_n \, \mathbf{R}^{(s)}{}_n$$

$w_n^{(1)}$ largest singular value of $\mathbf{Q}_n^{(s)^T} \mathbf{Q}_n^{(s)}$

$$\Phi_n = \arccos\left(w_n^{(1)}\right)$$

Minimum angle between stable and unstable manifold

- CML of Tent maps

$$x^i_{t+1} = (1-2\varepsilon) f\left(x^i_t\right) + \varepsilon\left[f\left(x^{i+1}_t\right) + f\left(x^{i-1}_t\right)\right]$$

$$f(x) = \begin{cases} a\,x & 0 \le x < 1/a \\ \dfrac{a}{1-a}(x-1) & 1 \ge x \ge 1/a \end{cases}$$

- Symplectic Maps

$$p^i_{t+1} = p^i_t + \mu\left[g\left(q^{i+1}_t - q^i_t\right) - g\left(q^i_t - q^{i-i}_t\right)\right]$$

$$q^i_{t+1} = q^i_t + p^i_{t+1}$$

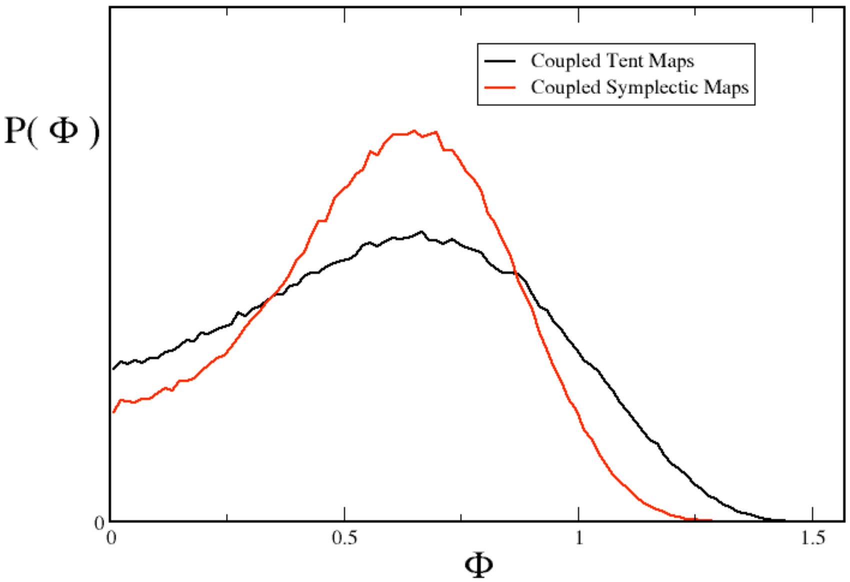
$$g(x) = \frac{1}{2\pi}\sin(2\pi x)$$

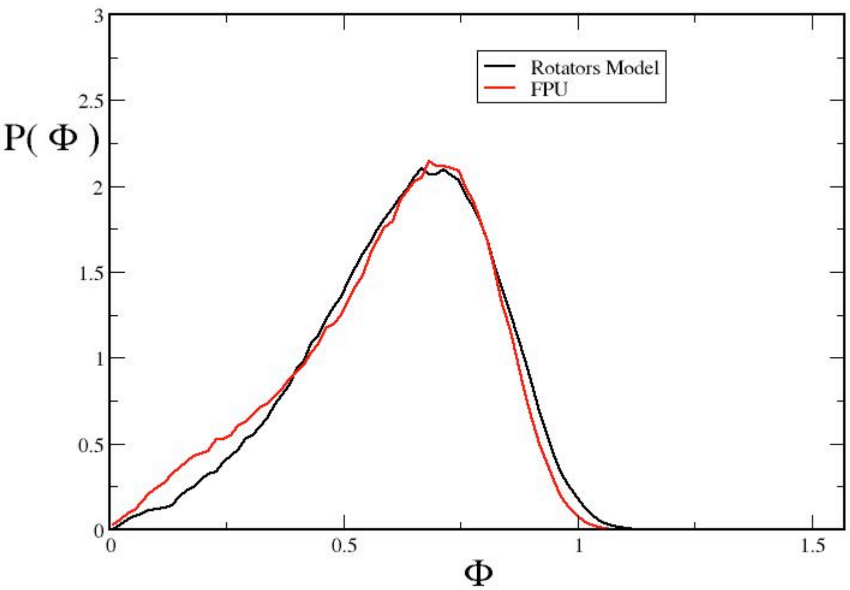
- Continuous time Hamiltonian systems

$$\ddot{q}_i = F\left(q_{i+1} - q_i\right) - F\left(q_i - q_{i-1}\right)$$

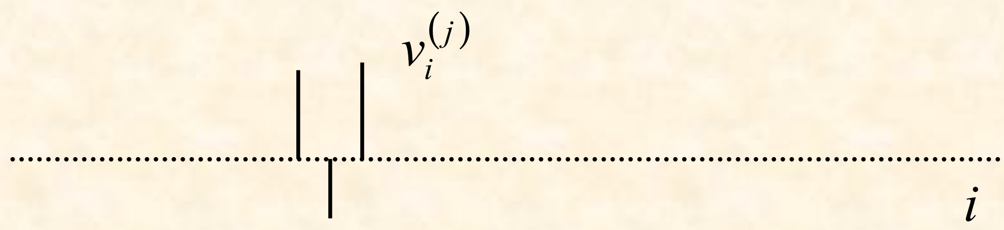
- Rotators $F(x) = \sin(x)$

- FPU $F(x) = x + x^3$

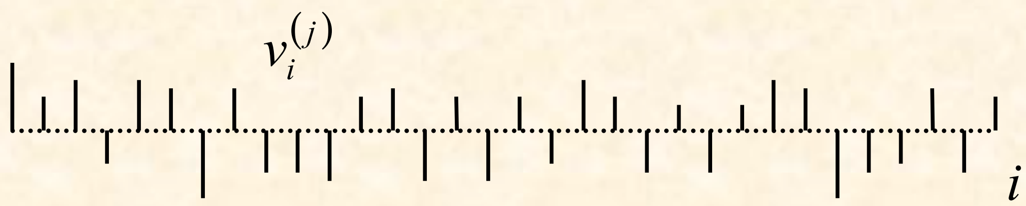




CLV as a tool to characterize collective modes



- Localized, extensive covariant Lyapunov vectors corresponding to microscopic dynamics



- Delocalized, nonextensive covariant Lyapunov vectors corresponding to collective modes

Conclusions

- Covariant Lyapunov Vectors are the right vectorial quantities to analyze spatiotemporal dynamics. Our dynamical algorithm is much more efficient then previous numerical methods.

- They are covariant with dynamics, invariant under time reversal, norm independent and allow to compute LEs by ensamble averages

- For time reversible systems they coincide with Oseledec splitting

- CLVs yield drastically different behavior with respect to GSV (where orthonormalization induced "noise" disrupt dynamical properties) for what concerns spatially extended systems.

- CLVs allow to numerically test (deviations from) hyperbolicity in dynamical systems.

- We would like to discuss possible applications to geophysical problems, like data assimilation

- More on applications to came on Thursday talks: see Takeuchi's

*Phys Rev Lett **99**, 130601 (2007).*

THANK YOU…

Phys Rev Lett ***99****, 130601 (2007).*