New bounds for the Kuramoto-Sivashinsky equation

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Abstract

We show that every L-periodic mean-zero solution u of the Kuramoto-Sivashinsky equation is on average $o(L)$ for $L \gg 1$, in the sense that for any $T > 0$ the space average of $|u(t)|$ is $O(L/T)$ for any $t > T$ and any L sufficiently large. For this we argue that on large spatial scales, the solution behaves like an entropy solution of the inviscid Burgers equation. The analysis of this non-standard perturbation of the Burgers equation is based on a "div-curl" argument.

1 Introduction

1.1 The Kuramoto-Sivashinky equation

The Kuramoto-Sivashinsky equation

$$u_t + \left(\tfrac{1}{2}\, u^2\right)_x + u_{xx} + u_{xxxx} = 0 \qquad \text{(KS)}$$

has been introduced as a model of nonlinear evolution of linearly unstable interfaces in various contexts, such as laminar flame fronts [17], pattern formation in reaction-diffusion systems [11], viscous flows of liquid films down an inclined plane [18], growth and roughening of crystal layers [8] (see [5], [22] for further references). In all these cases, a relevant setting is that of L-periodic initial data with zero average:

$$u_0(x+L) = u_0(x) \text{ for all } x \in \mathbf{R}, \quad \int_0^L u_0\, dx = 0. \qquad \text{(IC)}_L$$

Of course, (KS) preserves periodicity and zero average. We think of $L \gg 1$.

In order to understand the nature of the evolution, it is convenient to

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think of the linear and the nonlinear part separately:

$$u_t + u_{xx} + u_{xxxx} = 0, \tag{1.1}$$
$$u_t + \left(\tfrac{1}{2}u^2\right)_x = 0. \tag{1.2}$$

As can be seen by Fourier transformation

$$\hat{u}_t \;=\; (k^2 - k^4)\,\hat{u}, \tag{1.3}$$

(1.1) amplifies the long wave-length part of the energy spectrum $|\hat{u}|^2$ but damps the short wave-length end. The most unstable wave-length is $O(1)$ and has a growth rate of $O(1)$. Hence solutions of (1.1) generically grow exponentially. On the other hand, solutions of the inviscid Burgers equation (1.2) with non-trivial initial data satisfying $(\mathrm{IC})_L$ will invariably develop a discontinuity in finite time. Since (1.2) preserves the average energy density

$$\mathcal{E}_L(u) \;:=\; \tfrac{1}{L}\int_0^L u^2\,dx$$

(until the formation of singularities), this means that it transports energy from the long wave-lengths to the short wave-lengths. Hence the effect of (1.2) is likely to balance the exponential growth due to (1.1).

1.2 (KS) versus the "Burgers-Sivashinsky" equation

The "Burgers-Sivashinsky" equation (cf. [2, 9])

$$u_t + \left(\tfrac{1}{2}\,u^2\right)_x - u - u_{xx} = 0 \tag{BS}$$

has an analogous interpretation in Fourier space for fixed L, since under Fourier transformation its linear part reads as

$$\hat{u}_t \;=\; (1 - k^2)\,\hat{u}. \tag{1.4}$$

Instead, for large L, the *vanishing* of (1.3) at long wave-length makes an essential difference: a naive way to see this is to rescale both equations to a fixed system size in such a way that the energy transport mechanism (i.e., (1.1)) and the growth rate $O(1)$ of the most unstable mode are preserved; that is, $LU(t,X) = u(t,LX)$, which gives for the dispersion relations (1.3), (1.4),

$$\left.\begin{array}{ll}(\mathrm{KS}): & \hat{U}_t = \frac{k^2}{L^2}(1-\frac{k^2}{L^2})\hat{U} \;\stackrel{L\uparrow\infty}{\longrightarrow}\; 0\\ (\mathrm{BS}): & \hat{U}_t = (1-\frac{k^2}{L^2})\hat{U} \;\stackrel{L\uparrow\infty}{\longrightarrow}\; \hat{U}\end{array}\right\}\text{loc. unif. for } k\in[0,\infty). \tag{1.5}$$

Equation (1.5) shows that on one hand, in order to prevent exponential growth, the nonlinear term in (BS) has to transport energy all the way down from long to short wave-lengths; which indeed it does, since it has been shown by Goodman [9] that

$$\limsup_{t\to\infty} \mathcal{E}_L(u(t)) \;\le\; C\,L^2 \tag{1.6}$$

for all L-periodic solutions with zero average, that is

$$\text{space average of } |u| \;=\; O(L). \tag{1.7}$$

Note that this bound is sharp, since (BS)-(IC)$_L$ admits periodic steady states u_L for any L, to which solutions stabilize, and for those one can predict by matched asymptotic analysis that $\mathcal{E}_L(u_L) \sim \frac{2}{3}\,L^2$.

On the other hand, in (KS) the task of the nonlinear term is somewhat easier, since in view of (1.5) it essentially has to take care only of wave-lengths $k = O(L)$. Indeed, numerical simulations [1, 12, 19, 23] suggest that the average energy density $\mathcal{E}_L$ is bounded independently of the system size L. As a matter of facts, numerical simulations also display a spatio-temporal chaotic behavior (cf. [3, 5]) of solutions. Hence, occasional large deviations might occur. Still, it seems reasonable to expect a universal bound at least for the space-time averaged energy density $\frac{1}{T}\int_0^T \mathcal{E}_L(u(t))dt$. This conjecture is supported by a universal bound on all stationary solutions of (KS)-(IC)$_L$, due to Michelson [13].

As we shall discuss below, in tackling this long-standing open question the vanishing of the linear operator at long wave-lengths does not seem to have been exploited so far. It is the purpose of this paper to introduce a new strategy, where such property will be used in an essential way.

1.3 Known bounds

Quite some efforts have been made to prove bounds for (KS)-(IC)$_L$. The challenge comes from the fact that the energy identity

$$\frac{d}{dt}\mathcal{E}_L(u(t)) \;=\; \frac{1}{L}\int_0^L (u_x^2 - u_{xx}^2)\,dx \tag{1.8}$$

is oblivious of the Burgers term, and thus provides only exponential bounds (cf. Lemma 3.1). The breakthrough is due to Nicolaenko, Scheurer and Temam [15], who obtained a uniform bound on $\mathcal{E}_L(u)$ for odd initial data; the oddness assumption was later removed by Il'yashenko [10] (with a weaker bound) and by Goodman [9] (with the same bound). Collet, Eckmann, Epstein and Stubbe [4] proved the best bound available at present: A universal constant C exists such that any solution u of (KS)-(IC)$_L$ satisfies

$$\limsup_{t\to\infty} \mathcal{E}_L(u(t)) \;\leq\; C\, L^{\frac{11}{5}}. \tag{1.9}$$

This means that

$$\text{space average of } |u| \;=\; O(L^{\frac{11}{10}}). \tag{1.10}$$

The strategy in [15, 9, 4] is based on an ingenious Lyapunov-type argument: In short, for a generic L-periodic function s one has

$$\frac{d}{dt}\int_0^L (u-s)^2\,dx \;=\; -\tfrac{1}{2}\int_0^L s_x\,u^2\,dx + \int_0^L \left[u_x^2 - u_{xx}^2 + (s+s_{xx})u_{xx}\right]\,dx,$$

and for a careful choice of s one finds for some $C>1$:

$$\frac{d}{dt}\int_0^L (u-s)^2\,dx \;\leq\; -C^{-1}\int_0^L (u-s)^2\,dx + \underbrace{C\int_0^L s_{xx}^2\,dx}_{\leq C\,L^{\frac{16}{5}}}.$$

The more careful the choice of s is, the better the exponent (under the brace, the estimate for the choice of s performed in [4]).

1.4 New bounds

The main result of this paper is the following improvement of (1.10):

$$\text{space average of } |u| \;=\; o(L). \tag{1.11}$$

More precisely, we will prove:

THEOREM 1 *A universal constant $C>0$ exists such that for any $T>0$ there exists an $L_0 = L_0(T)$ such that*

$$\mathcal{E}_L(u(t)) \;\leq\; C\,\frac{L^2}{t^2}\,, \quad t\in[0,T]$$

for all $L>L_0$ and all solutions u of (KS)-(IC)$_L$.

By translation invariance in time of (KS), Theorem 1 immediately implies:

COROLLARY 1 *A universal constant $C > 0$ exists such that for any $T > 0$ there exists an $L_0 = L_0(T)$ such that*

$$\sup_{t>T} \mathcal{E}_L(u(t)) \;\le\; C\,\frac{L^2}{T^2}$$

for all $L > L_0$ and all solutions u of (KS)-(IC)$_L$.

In turn, Corollary 1 yields (1.11) in the sense of:

$$\lim_{L\to\infty} \sup \left\{ \limsup_{t\to\infty} \frac{1}{L}\int_0^L \left(\frac{u(t)}{L}\right)^2 dx\,,\quad u \text{ solution of (KS)-(IC)}_L \right\} \;=\; 0.$$

Corollary 1 allows to improve those bounds which follow from an estimate of the absorbing set in L^2. For instance, the Hausdorff dimension of the global attractor $\mathcal{A}$ for odd initial data (cf. [21, Theorem VI.5.1]) and the dimension of inertial manifolds $\mathcal{M}$ (cf. [16, Corollary 5]):

$$d_H(\mathcal{A}) \;=\; o(L^{\frac{5}{4}}), \quad \dim(\mathcal{M}) \;=\; o(L^{\frac{12}{5}}).$$

Apart from the interest of gaining an order of magnitude of one tenth (cf. (1.10) versus (1.11)), the main merit of our work may be to introduce a different strategy, which is more adapted to (KS) compared with the aforementioned Lyapunov-type argument. Indeed, as Goodman pointed out, the Lyapunov-type argument may also be applied to (BS), yielding the sharp bound (1.7), and in this sense it does not seem to distinguish between an equation with a non-vanishing dispersion law (and simple large time behavior), as (BS), and one with vanishing dispersion law (and highly complex spatio-temporal dynamics), as (KS). Comparing (1.7) with (1.11), we see that this is the case for our strategy, which will be discussed now.

2 Strategy for Theorem 1

2.1 (KS) as a perturbation of Burgers equation

Our strategy is inspired by (1.5). It is based on the following two observations:

(a) On large x and u scales, solutions of (KS) behave like entropy solutions of the inviscid Burgers equation, that is

$$\left.\begin{array}{rcl} u_t + \left(\frac{1}{2}\, u^2\right)_x & = & 0 \\ \left(\frac{1}{2}\, u^2\right)_t + \left(\frac{1}{3}\, u^3\right)_x & \leq & 0 \end{array}\right\} \quad \text{in } \mathcal{D}'((0,\infty)\times \mathbf{R}); \tag{B}$$

(b) solutions of (B)-(IC)$_L$ decay.

In order to see (a), we observe that (KS) can be rewritten as

$$u_t + (\tfrac{1}{2}\, u^2 + \underbrace{u_x + u_{xxx}}_{(\star)})_x = 0,$$

$$\left(\tfrac{1}{2}\, u^2\right)_t + (\tfrac{1}{3}\, u^3 + \underbrace{u\, u_{xxx} - u_x\, u_{xx}}_{(\star\star)})_x = \tfrac{1}{4}\, u^2 - \left(\tfrac{1}{2}\, u^2 + u_{xx}\right)^2 \;\leq\; \underbrace{\tfrac{1}{4}\, u^2}_{(\star\star\star)}.$$

We want to treat the terms $(\star)$, $(\star\star)$, $(\star\star\star)$ as perturbations of (B). In order to exploit the mean-value zero condition, it is natural to rescale length according to

$$x \;=\; L\,\hat{x}. \tag{2.1}$$

To ensure control of the term $(\star\star\star)$, we can not pass to a longer time scale; we thus leave time unrescaled:

$$t \;=\; \hat{t}. \tag{2.2}$$

To preserve Burgers equation, we are poised to rescale u according to

$$u \;=\; L\,\hat{u}. \tag{2.3}$$

With this rescaling we obtain

$$\hat{u}_{\hat{t}} + (\tfrac{1}{2}\,\hat{u}^2 + L^{-2}\,\hat{u}_{\hat{x}} + L^{-4}\,\hat{u}_{\hat{x}\hat{x}\hat{x}})_{\hat{x}} = 0, \tag{2.4}$$

$$\left(\tfrac{1}{2}\,\hat{u}^2\right)_{\hat{t}} + (\tfrac{1}{3}\,\hat{u}^3 + L^{-4}\,\hat{u}\,\hat{u}_{\hat{x}\hat{x}\hat{x}} - L^{-4}\,\hat{u}_{\hat{x}}\,\hat{u}_{\hat{x}\hat{x}})_{\hat{x}} \leq \tfrac{1}{4}\,\hat{u}^2. \tag{2.5}$$

We will show that the formal limit $L \uparrow \infty$ in (2.4) and (2.5) can be rigorously justified in the following sense:

PROPOSITION 2.1 *Let $\{L_\nu\}_{\nu\uparrow\infty}$ such that $L_\nu \stackrel{\nu\uparrow\infty}{\to} \infty$, and let u_ν be a solution of (KS)-(IC)$_{L_\nu}$. Rescale t, x, u_ν according to (2.1)-(2.3). A subsequence (again labeled by ν) and a 1-periodic function $\hat{u} \in L^4_{\rm loc}((0,\infty)\times\mathbf{R})$ with zero average exist such that*

$$\hat{u}_\nu \;\stackrel{\nu\uparrow\infty}{\longrightarrow}\; \hat{u} \quad \textit{in } L^4_{\rm loc}((0,\infty)\times\mathbf{R}). \tag{2.6}$$

Moreover, $\hat u$ is a solution of

$$\left.\begin{array}{rcl} \hat u_t + \left(\frac12 \hat u^2\right)_{\hat x} & = & 0 \\ \left(\frac12 \hat u^2\right)_{\hat t} + \left(\frac13 \hat u^3\right)_{\hat x} & \le & \frac14 \hat u^2 \end{array}\right\} \quad \text{in } \mathcal{D}'((0,\infty)\times\mathbf{R}).$$

Then we may appeal to the following result, recently proved by De Lellis, Otto and Westdickenberg [6], which shows that the right-hand side $\frac14 \hat u^2$ in the entropy equation has no effect:

THEOREM 2.2 *[6, Corollary 2.5] Let $\Omega \subset \mathbf{R}^2$ be open, let $\hat f$ be a non-negative measure with vanishing $\mathcal{H}^1$-density at every point of Ω, and let $\hat u \in L^4_{\mathrm{loc}}(\Omega)$ be a solution of*

$$\left.\begin{array}{rcl} \hat u_{\hat t} + \left(\frac12 \hat u^2\right)_{\hat x} & = & 0 \\ \left(\frac12 \hat u^2\right)_{\hat t} + \left(\frac13 \hat u^3\right)_{\hat x} & \le & \hat f \end{array}\right\} \quad \text{in } \mathcal{D}'(\Omega).$$

Then $\hat u$ is a solution of (B).

Proposition 2.1 and Theorem 2.2 make (a) precise, whereas (b) is quantified by the following proposition:

PROPOSITION 2.3 *A universal constant $C_0 > 0$ exists such that*

$$\int_0^1 \hat u^2(\hat t)\, d\hat x \;\le\; \frac{C_0}{\hat t^2} \quad \text{for all } \hat t > 0 \tag{2.7}$$

for all solutions $\hat u$ of (B)-(IC)$_1$.

As shown in Section 8, the combination of Proposition 2.1, Proposition 2.3 and (1.8) yields Theorem 1 for $1 \le t \le T$ via an indirect argument, whereas the following intermediate result provides in particular the appropriate bound in the initial layer $0 < t < 1$:

PROPOSITION 2.4 *A universal constant $C > 0$ exists such that*

$$\int_0^L u^2(t)\, dx \;\le\; C\, L^3 \left(1 + \frac{1}{t^2}\right) \quad \text{for all } t > 0$$

for all solutions u of (KS)-(IC)$_L$.

2.2 Strategy for Propositions 2.1, 2.3 and 2.4

In order to complete the picture, we have to outline the argument for:

(i) uniform integrability of u (cf. Lemma 5.1);

(ii) bound in the initial layer (cf. Proposition 2.4);

(iii) identification of the limit $\hat{u}$ (cf. the strong convergence result in Proposition 2.1);

(iv) decay of the limit (cf. Proposition 2.3).

The main difficulty comes from the non-standard character of the perturbation ($\star$), ($\star\star$), ($\star\star\star$), which on one hand does not preserve maximum principle, and on the other hand has a destabilizing effect for low modes. Indeed, for diffusive perturbations of (B) which preserve maximum principle, such as the viscid Burgers

$$\hat{u}_{\hat{t}} + \left(\tfrac{1}{2}\,\hat{u}^2\right)_{\hat{x}} - L^{-2}\,\hat{u}_{\hat{x}\hat{x}} \;=\; 0,$$

items (i) to (iv), and in fact Theorem 1 itself, would all follow from Oleinik's estimate, $\hat{u}_{\hat{x}} \le \hat{t}^{-1}$. Here, we have to introduce an alternative approach: It is essentially based on a "div-curl" argument for Burgers inspired by the ideas of Di Perna [7] and Murat [14], Tartar [20]. Let us give an idea of this approach by using it to prove (iv): Similar arguments, adapted to the perturbed equation, will be employed to prove (i)-(iii).

Proof of Proposition 2.3: In view of (B), the time-space curl of the vector field $\binom{-\frac{1}{2}\hat{u}^2}{\hat{u}}$ vanishes. Hence there is a function $\hat{h}$ such that

$$\begin{pmatrix}\hat{h}_{\hat{t}}\\ \hat{h}_{\hat{x}}\end{pmatrix} \;=\; \begin{pmatrix}-\frac{1}{2}\,\hat{u}^2\\ \hat{u}\end{pmatrix}. \tag{2.8}$$

Note that $\hat{h}$ is 1-periodic since $\hat{u}$ is 1-periodic with zero average. Furthermore, since

$$\sup_{\hat{x}\in[0,1]} \hat{h}_0 - \inf_{\hat{x}\in[0,1]} \hat{h}_0 \;\le\; \tfrac{1}{2}\int_0^1 |\hat{h}_{0\,\hat{x}}|\,d\hat{x} \;\le\; \tfrac{1}{2}\left(\int_0^1 \hat{u}_0^2\,d\hat{x}\right)^{\frac{1}{2}},$$

without loss of generality we may assume that

$$-\tfrac{1}{2}\left(\int_0^1 \hat{u}_0^2\,d\hat{x}\right)^{\frac{1}{2}} \;\le\; \hat{h}_0 \;\le\; 0, \tag{2.9}$$

which in view of $\hat{h}_{\hat{t}} \le 0$ strengthens to

$$\hat{h} \;\le\; 0. \tag{2.10}$$

Now we use the fact that we control the time-space divergence of the vector field $\binom{\frac{1}{2}\hat{u}^2}{\frac{1}{3}\hat{u}^3}$:

$$\begin{aligned}
\tfrac{1}{12}\int_0^T\int_0^1 \hat{u}^4\,d\hat{x}d\hat{t} \;&=\; \int_0^T\int_0^1 \begin{pmatrix} -\frac{1}{2}\,\hat{u}^2 \\ \hat{u} \end{pmatrix}\cdot\begin{pmatrix} \frac{1}{2}\,\hat{u}^2 \\ \frac{1}{3}\,\hat{u}^3 \end{pmatrix}\,d\hat{x}d\hat{t} \\
&\overset{(2.8)}{=} \int_0^T\int_0^1 \begin{pmatrix} \hat{h}_{\hat{t}} \\ \hat{h}_{\hat{x}} \end{pmatrix}\cdot\begin{pmatrix} \frac{1}{2}\,\hat{u}^2 \\ \frac{1}{3}\,\hat{u}^3 \end{pmatrix}\,d\hat{x}d\hat{t} \\
&=\; \int_0^1 \tfrac{1}{2}\,\underbrace{(\hat{h}\,\hat{u}^2)(T)}_{\le 0\ \ by\ (2.10)}\;d\hat{x} - \int_0^1 \tfrac{1}{2}\,\hat{h}_0\,\hat{u}_0^2\,d\hat{x} \\
&\quad - \int_0^T\int_0^1 \underbrace{\hat{h}\,\Big[\Big(\tfrac{1}{2}\,\hat{u}^2\Big)_{\hat{t}} + \Big(\tfrac{1}{3}\,\hat{u}^3\Big)_{\hat{x}}\Big]}_{\ge 0\ \ by\ (B)\ and\ (2.10)}\;d\hat{x}d\hat{t}.
\end{aligned}$$

Since T is arbitrary, we see that

$$\int_0^\infty\int_0^1 \hat{u}^4\,d\hat{x}d\hat{t} \le -6\int_0^1 \hat{h}_0\,\hat{u}_0^2\,d\hat{x} \;\overset{(2.9)}{\le}\; 3\,\left(\int_0^1 \hat{u}_0^2\,d\hat{x}\right)^{\frac{3}{2}}.$$

Evoking translation invariance in time we conclude that

$$\int_{\hat{s}}^\infty\int_0^1 \hat{u}^4\,d\hat{x}d\hat{t} \le 3\,\left(\int_0^1 \hat{u}^2(\hat{s})\,d\hat{x}\right)^{\frac{3}{2}} \quad \text{for all } \hat{s} > 0. \tag{2.11}$$

This can be seen as the following differential inequality:

$$\begin{aligned}
f(\hat{s}) := \int_{\hat{s}}^\infty\int_0^1 \hat{u}^4\,d\hat{x}d\hat{t} \;&\overset{(2.11)}{\le}\; 3\left(\int_0^1 \hat{u}^2(\hat{s})\,d\hat{x}\right)^{\frac{3}{2}} \\
\le\; 3\left(\int_0^1 \hat{u}^4(\hat{s})\,d\hat{x}\right)^{\frac{3}{4}} \;&=\; 3\,(-f'(\hat{s}))^{\frac{3}{4}},
\end{aligned}$$

which can be rewritten as

$$1 \;\le\; -C\,\frac{f'(\hat{s})}{f^{\frac{4}{3}}(\hat{s})} \;=\; C\,\left(\frac{1}{f^{\frac{1}{3}}(\hat{s})}\right)'$$

with a generic universal constant C. Integrating this inequality, we obtain

$$\int_{\hat{s}}^{\infty}\int_0^1 \hat{u}^4\, d\hat{x}d\hat{t} \;=\; f(\hat{s}) \;\leq\; \frac{C}{\hat{s}^3}.$$

Since by (B) and the periodicity of $\hat{u}$ we have

$$\frac{d}{d\hat{t}}\int_0^1 \hat{u}^2(\hat{t})\, d\hat{x} \;\leq\; 0,$$

we conclude

$$\begin{aligned}\left(\int_0^1 \hat{u}^2(2\hat{s})\, d\hat{x}\right)^2 &\leq \frac{1}{\hat{s}}\int_{\hat{s}}^{2\hat{s}}\left(\int_0^1 \hat{u}^2(\hat{t})\, d\hat{x}\right)^2 d\hat{t}\\ &\leq \frac{1}{\hat{s}}\int_{\hat{s}}^{\infty}\int_0^1 \hat{u}^4\, d\hat{x}d\hat{t} \;\leq\; \frac{C}{\hat{s}^4}.\end{aligned}$$

∎

Plan of the paper. Except for the final Section 8 where we prove Theorem 1, the remainder of the paper is devoted to the proof of Propositions 2.1 and 2.4, and is organized as follows. In Section 3 we derive basic consequences of the energy estimate. Section 4 is at the core of our proof: We exploit the div-curl argument to obtain two integral identities involving the L^4-norm of u. The first integral identity is used in Section 5 to prove the counterpart of (2.11), which in turn is used in Section 6 to prove Proposition 2.4 via an ode argument. The second integral identity is used in Section 7 to complete the proof of Proposition 2.1 by a compensated compactness argument.

Notations. We shall systematically omit the spatial domain of integration when it coincides with a period of the integrand. The notation $f \lesssim g$ means that a universal constant $C > 0$ exists such that $f \leq C\, g$.

3 Energy estimate

We start with simple consequences of the energy estimate:

LEMMA 3.1 *Let u be an L-periodic solution of (KS). Then for all $s \geq 0$*

and $T > 0$*:*

$$\sup_{t\in(s,s+T)} \int u^2(t)\,dx \lesssim e^{\frac{T}{2}} \int u^2(s)\,dx, \tag{3.1}$$

$$\int_s^{s+T} \int u_x^2\,dx dt \lesssim e^{\frac{T}{2}} \int u^2(s)\,dx, \tag{3.2}$$

$$\int_s^{s+T} \int u_{xx}^2\,dx dt \lesssim e^{\frac{T}{2}} \int u^2(s)\,dx, \tag{3.3}$$

$$\left(\int_s^{s+T} \int |u_x|^3\,dx dt\right)^{\frac{2}{3}} \lesssim e^{\frac{T}{2}} \int u^2(s)\,dx. \tag{3.4}$$

Proof of Lemma 3.1: By translation invariance in time, it suffices to consider $s = 0$. We obtain by integration by parts (using the periodicity of u)

$$\frac{d}{dt}\int \tfrac{1}{2}\,u^2\,dx \;=\; \int u_x^2\,dx - \int u_{xx}^2\,dx. \tag{3.5}$$

Note that

$$\begin{aligned} \int u_x^2\,dx &= -\int u\,u_{xx}\,dx \\ &\le \left(\int u^2\,dx\right)^{\frac{1}{2}} \left(\int u_{xx}^2\,dx\right)^{\frac{1}{2}} \;\le\; \left\{ \begin{array}{l} \frac{1}{4}\int u^2\,dx + \int u_{xx}^2\,dx \\ \frac{1}{2}\int u^2\,dx + \frac{1}{2}\int u_{xx}^2\,dx \end{array} \right\} \end{aligned} \tag{3.6}$$

Using (3.6) in (3.5) we see that

$$\frac{d}{dt}\int \tfrac{1}{2}\,u^2\,dx \;\le\; \tfrac{1}{2}\int \tfrac{1}{2}\,u^2\,dx,$$

and integrating in time we get (3.1). Integrating (3.5) yields

$$\begin{aligned} \int \tfrac{1}{2}\,u^2(T)\,dx + \int_0^T \int u_{xx}^2\,dx dt &\le \int \tfrac{1}{2}\,u_0^2\,dx + \int_0^T \int u_x^2\,dx dt \\ &\overset{(3.6)}{\le} \int \tfrac{1}{2}\,u_0^2 + \tfrac{1}{2}\int_0^T \int u_{xx}^2\,dx dt + \int_0^T \int \tfrac{1}{2}\,u^2\,dx dt, \end{aligned}$$

so that

$$\int_0^T \int u_{xx}^2\,dx dt \;\le\; \int u_0^2\,dx + \int_0^T \int u^2\,dx dt \;\overset{(3.1)}{\lesssim}\; e^{\frac{T}{2}} \int u_0^2\,dx,$$

which is (3.3). Using (3.1) and (3.3) in (3.6) we obtain (3.2).

To prove (3.4), we observe that

$$\sup_{x\in(0,L)} |u_x| \ \lesssim\ \left(\int u_x^2\,dx\right)^{\frac{1}{4}} \left(\int u_{xx}^2\,dx\right)^{\frac{1}{4}}.$$

Hence

$$\begin{aligned}\int |u_x|^3\,dx \ &\lesssim\ \left(\int u_x^2\,dx\right)^{1+\frac{1}{4}} \left(\int u_{xx}^2\,dx\right)^{\frac{1}{4}}\\ &\overset{(3.6)}{\lesssim}\ \left(\int u^2\,dx\right)^{\frac{1}{2}} \left(\int u_x^2\,dx\right)^{\frac{1}{4}} \left(\int u_{xx}^2\,dx\right)^{\frac{3}{4}}.\end{aligned}$$

This yields

$$\begin{aligned}\int_0^T\int |u_x|^3\,dxdt \lesssim &\left(\sup_{t\in(0,T)}\int u^2(t)\,dx\right)^{\frac{1}{2}} \left(\int_0^T\int u_x^2\,dxdt\right)^{\frac{1}{4}}\\ &\times\left(\int_0^T\int u_{xx}^2\,dxdt\right)^{\frac{3}{4}},\end{aligned}$$

and (3.4) follows from (3.1)-(3.3). ∎

4 The div-curl lemma

This section is essential to our proof. We use a div-curl argument to obtain two integral identities involving the L^4-norm of u. The first one will be used in Lemma 5.1 to gain control of such norm, and in turn of the initial layer, as stated in Proposition 2.4. The second one will be needed in Proposition 2.1 in order to prove compactness in L^4 under the rescaling (2.1)-(2.3).

LEMMA 4.1 *Let u be a solution of (KS)-(IC)$_L$. (a) For h such that*

$$\begin{pmatrix} h_t \\ h_x \end{pmatrix} = \begin{pmatrix} -\left(\frac{1}{2}\,u^2 + u_x + u_{xxx}\right) + \frac{1}{L}\int \frac{1}{2}\,u^2\,dx \\ u \end{pmatrix}$$

and all $\zeta \in C_c^\infty((0,\infty))$, we have

$$\begin{aligned}&\tfrac{1}{12}\int_0^\infty\int u^4\,dx\,\zeta\,dt + \tfrac{1}{L}\int_0^\infty\left(\int \tfrac{1}{2}\,u^2\,dx\right)^2\zeta\,dt + \int_0^\infty\int u_x^3\,dx\,\zeta\,dt\\ &= -\int_0^\infty\int\left(u_x^2 - u_{xx}^2\right)\,h\,dx\,\zeta\,dt - \int_0^\infty\int \tfrac{1}{2}\,u^2\,h\,dx\,\zeta_t\,dt;\end{aligned}$$

(b)] for h such that

$$\begin{pmatrix} h_t \\ h_x \end{pmatrix} = \begin{pmatrix} -\left(\frac{1}{2}\,u^2 + u_x + u_{xxx}\right) \\ u \end{pmatrix}$$

and all $\zeta \in C_c^\infty((0,\infty))$, we have

$$\begin{aligned} &\tfrac{1}{12}\int_0^\infty\!\!\int u^4\,dx\,\zeta\,dt + \int_0^\infty\!\!\int u_x^3\,dx\,\zeta\,dt \\ &\quad = \int_0^\infty\!\!\int \left(\tfrac{1}{2}u + u_{xx}\right)^2 h\,dx\,\zeta\,dt - \int_0^\infty\!\!\int \tfrac{1}{4}\,u^2\,h\,dx\,\zeta\,dt \\ &\qquad - \int_0^\infty\!\!\int \tfrac{1}{2}\,u^2\,h\,dx\,\zeta_t\,dt. \end{aligned}$$

Proof of Lemma 4.1: To combine both parts, we consider

$$\begin{pmatrix} h_t \\ h_x \end{pmatrix} = \begin{pmatrix} -\left(\frac{1}{2}\,u^2 + u_x + u_{xxx}\right) + g \\ u \end{pmatrix} \tag{4.1}$$

with a spatially constant g. We now reproduce the argument from Proposition 2.3. It follows from (KS) that

$$\left(\tfrac{1}{2}\,u^2\right)_t + \left(\tfrac{1}{3}\,u^3 + u\,u_x + u\,u_{xxx} - u_x\,u_{xx}\right)_x = u_x^2 - u_{xx}^2. \tag{4.2}$$

Introducing the vector field

$$V = \begin{pmatrix} \frac{1}{2}\,u^2 \\ \frac{1}{3}\,u^3 + u\,u_x + u\,u_{xxx} - u_x\,u_{xx} \end{pmatrix},$$

we can rewrite this equation as

$$\begin{pmatrix} \partial_t \\ \partial_x \end{pmatrix} \cdot V = u_x^2 - u_{xx}^2. \tag{4.3}$$

So on one hand, we obtain by integration by parts in t and x (using the periodicity of h and u)

$$\begin{aligned} \int_0^\infty\!\!\int V \cdot \begin{pmatrix} h_t \\ h_x \end{pmatrix} dx\,\zeta\,dt &\overset{(4.3)}{=} -\int_0^\infty\!\!\int \tfrac{1}{2}\,u^2\,h\,dx\,\zeta_t\,dt \\ &\quad - \int_0^\infty\!\!\int \left(u_x^2 - u_{xx}^2\right) h\,dx\,\zeta\,dt. \end{aligned}$$

On the other hand, a direct computation yields

$$\begin{aligned}
&\int_0^\infty \int V \cdot \begin{pmatrix} h_t \\ h_x \end{pmatrix} dx\,\zeta\,dt \\
&\overset{(4.1)}{=} \int_0^\infty \int \tfrac{1}{12}\,u^4\,dx\,\zeta\,dt + \int_0^\infty \int \tfrac{1}{2}\,u^2\,g\,dx\,\zeta\,dt \\
&\quad + \int_0^\infty \int \left(\tfrac{1}{2}\,u^2\,u_x + \tfrac{1}{2}\,u^2\,u_{xxx} - u\,u_x\,u_{xx}\right) dx\,\zeta\,dt.
\end{aligned}$$

An integration by parts in x shows that

$$\int \left(\tfrac{1}{2}\,u^2\,u_x + \tfrac{1}{2}\,u^2\,u_{xxx} - u\,u_x\,u_{xx}\right) dx \;=\; -2\int u\,u_x\,u_{xx}\,dx \;=\; \int u_x^3\,dx.$$

Therefore

$$\begin{aligned}
&\tfrac{1}{12}\int_0^\infty \int u^4\,dx\,\zeta\,dt + \int_0^\infty \int \tfrac{1}{2}\,u^2\,g\,dx\,\zeta\,dt + \int_0^\infty \int u_x^3\,dx\,\zeta\,dt \\
&\quad = -\int_0^\infty \int \left(u_x^2 - u_{xx}^2\right) h\,dx\,\zeta\,dt - \int_0^\infty \int \tfrac{1}{2}\,u^2\,h\,dx\,\zeta_t\,dt.
\end{aligned}$$

The choice of $g(t) = \frac{1}{L}\int \frac{1}{2}\,u^2\,dx$ in (4.1) yields part (a). Part (b) follows from choosing $g = 0$ and observing that

$$\begin{aligned}
-\int \left(u_x^2 - u_{xx}^2\right) h\,dx &\overset{(4.1)}{=} \int \left(u\,u_{xx} + u_{xx}^2\right) h\,dx + \int u^2\,u_x\,dx \\
&= \int \left(\tfrac{1}{2}u + u_{xx}\right)^2 h\,dx - \int \tfrac{1}{4}\,u^2\,h\,dx.
\end{aligned}$$

∎

5 Uniform integrability

Generalizing the argument from Proposition 2.3, we derive an a-priori estimate in L^4 from the div-curl Lemma 4.1.

LEMMA 5.1 *Let $L \gtrsim 1$, and let u be a solution of (KS)-(IC)$_L$. Then for all $s \geq 0$ and $T > 0$:*

$$\int_s^{s+T} \int u^4\,dxdt \;\lesssim\; L^{\frac{1}{2}} \left(e^{\frac{T}{2}} \int u^2(s)\,dx\right)^{\frac{3}{2}}. \tag{5.1}$$

PROOF OF LEMMA 5.1: By translation invariance in time, it is sufficient to consider the case $s = 0$. Let h be as in part (a) of Lemma 4.1. For fixed T, we normalize h such that

$$\int_0^T \int h\,dxdt \;=\; 0. \tag{5.2}$$

Since

$$\frac{d}{dt}\int h\,dx \;=\; -\int \left(\tfrac{1}{2}\,u^2 + u_x + u_{xxx}\right)\,dx + \int \tfrac{1}{2}\,u^2\,dx \;=\; 0,$$

(5.2) strengthens to

$$\int h\,dx \;=\; 0 \quad \text{for all } 0 \le t \le T.$$

Hence

$$\sup_{x\in(0,L)} |h| \;\le\; \int |h_x|\,dx \;=\; \int |u|\,dx \;\le\; \left(L\int u^2\,dx\right)^{\frac{1}{2}},$$

and in turn

$$\sup_{\substack{x\in(0,L)\\ t\in(0,T)}} |h| \;\le\; \left(L \sup_{t\in(0,T)} \int u^2(t)\,dx\right)^{\frac{1}{2}}. \tag{5.3}$$

Choosing (after an approximation argument) $\zeta = \chi_{(0,T)}$ in part (a) of Lemma 4.1, we see that

$$\begin{aligned}\tfrac{1}{12}\int_0^T\!\!\int u^4\,dx dt + \tfrac{1}{L}\int_0^T \left(\int \tfrac{1}{2}\,u^2\,dx\right)^2 dt + \int_0^T\!\!\int u_x^3\,dx dt \\ = -\int_0^T\!\!\int \left(u_x^2 - u_{xx}^2\right)\,h\,dx dt + \int \tfrac{1}{2}\,u^2\,h\,dx\Big|_0^T.\end{aligned}$$

In view of (5.3), this yields the estimate

$$\begin{aligned}&\tfrac{1}{12}\int_0^T\!\!\int u^4\,dx dt \\ &\le \left(L \sup_{t\in(0,T)} \int u^2(t)\,dx\right)^{\frac{1}{2}} \left(\int_0^T\!\!\int \left(u_x^2 + u_{xx}^2\right)\,dx dt + \sup_{t\in(0,T)}\int u^2(t)\,dx\right) \\ &+ \int_0^T\!\!\int |u_x|^3\,dx dt.\end{aligned}$$

Appealing to Lemma 3.1 completes the proof of (5.1). ∎

6 The initial layer

As established in Proposition 2.3, solutions of (B) can be bounded *independently* of the initial data. This means that even if a solution makes a large excursion, it will be pushed back in a time which is independent of the excursion. In this section we prove Proposition 2.4, which states that solutions of (KS) have the same behavior. We proceed, as in the proof of Proposition 2.3, via an ode argument based on Lemma 5.1.

PROOF OF PROPOSITION 2.4: For a given $0 < T \leq 1$, we introduce the time average

$$g(s) \;=\; \int_s^{s+T} \int u^2 \, dxdt.$$

We first notice that g is comparable with $\int u^2\,dx$ on time intervals of size T. Indeed, by Lemma 3.1 we have for all $s \geq 0$

$$\begin{aligned} T \int u^2(s+T)\,dx \;&=\; \int_s^{s+T} \int u^2(s+T)\,dxdt \\ &\overset{(3.1)}{\lesssim} \int_s^{s+T} e^{\frac{s+T-t}{2}} \int u^2(t)\,dxdt \;\lesssim\; g(s). \end{aligned} \tag{6.1}$$

According to Lemma 5.1, we have

$$g(s) \leq L^{\frac12}\,T^{\frac12} \left(\int_s^{s+T} \int u^4\,dxdt\right)^{\frac12} \overset{(5.1)}{\lesssim} L^{\frac34}\,T^{\frac12} \left(\int u^2(s)\,dx\right)^{\frac34}.$$

We write

$$\begin{aligned} \int u^2(s)\,dx \;&=\; \left(\int u^2(s)\,dx - \int u^2(s+T)\,dx\right) + \int u^2(s+T)\,dx \\ &=\; -g'(s) + \int u^2(s+T)\,dx \\ &\overset{(6.1)}{\leq} -g'(s) + C\,T^{-1}\,g(s). \end{aligned}$$

This yields the differential inequality

$$g(s) \;\lesssim\; L^{\frac34}\,T^{\frac12} \left(-g'(s) + C\,T^{-1}\,g(s)\right)^{\frac34}.$$

Introducing the function

$$h(s) \;=\; e^{-\frac{C\,s}{T}}\,g(s),$$

we rewrite the differential inequality as

$$e^{\frac{Cs}{T}}\, h(s) \;\lesssim\; L^{\frac{3}{4}}\, T^{\frac{1}{2}} \left(-e^{\frac{Cs}{T}}\, h'(s)\right)^{\frac{3}{4}},$$

respectively

$$\left(\frac{3}{h^{\frac{1}{3}}(s)}\right)' \;=\; -\frac{h'(s)}{h^{\frac{4}{3}}(s)} \;\gtrsim\; \frac{e^{\frac{Cs}{3T}}}{L\,T^{\frac{2}{3}}}.$$

An integration yields

$$\frac{1}{h^{\frac{1}{3}}(s)} \;\geq\; \frac{1}{h^{\frac{1}{3}}(s)} - \frac{1}{h^{\frac{1}{3}}(0)} \;\gtrsim\; \frac{T^{\frac{1}{3}}}{C\,L}\left(e^{\frac{Cs}{3T}} - 1\right),$$

that is

$$h(s) \lesssim \frac{L^3}{T\,(e^{\frac{Cs}{3T}} - 1)^3}.$$

Translating back to g and using (6.1) we gather

$$\int u^2(s+T)\,dx \;\lesssim\; \frac{1}{T}\,g(s) \;\lesssim\; \frac{L^3}{T^2}\,\frac{e^{\frac{Cs}{T}}}{\left(e^{\frac{Cs}{3T}} - 1\right)^3} \tag{6.2}$$

for all $0 < T \leq 1$ and $s \geq 0$.

Let $t > 0$; to conclude we distinguish two cases. If $t \leq 1$ we choose $s = T = \frac{t}{2}$ in (6.2), yielding

$$\int u^2(t)\,dx \;\lesssim\; \frac{L^3}{t^2}.$$

If $t > 1$ we choose $T = \frac{1}{2}$ and $s = t - \frac{1}{2}$ in (6.2), so that

$$\int u^2(t)\,dx \;\lesssim\; L^3,$$

and the proof of Proposition 2.4 is complete. ∎

7 Compactness

In this section we prove Proposition 2.1. To this aim we shall need the following Hölder continuity property for the function h defined in Lemma 4.1 (b).

LEMMA 7.1 *Let u be a solution of (KS)-(IC)$_L$, let h as in Lemma 4.1 (b), and let*

$$M \;=\; M(T,s) \;=\; e^{\frac{T}{2}} \int u^2(s)\,dx.$$

Then

$$|h(t_1,x_1)-h(t_2,x_2)| \lesssim M^{\frac{1}{2}}\,|x_1-x_2|^{\frac{1}{2}} + M^{\frac{2}{3}}\,|t_1-t_2|^{\frac{1}{3}} + M^{\frac{1}{2}}\,|t_1-t_2|^{\frac{1}{4}} + M^{\frac{1}{2}}\,|t_1-t_2|^{\frac{1}{8}}$$

for all x_1, $x_2 \in \mathbf{R}$, $s \geq 0$, $T > 0$ and t_1, $t_2 \in (s, s+T)$.

PROOF OF LEMMA 7.1: By translation invariance, it is enough to consider the case $s = 0$. Hölder continuity in space is easily established: Indeed, according to the definition of h we have for $t_0 \in (0,T)$ and x_1, $x_2 \in [0,L]$

$$\begin{aligned} |h(t_0,x_1)-h(t_0,x_2)| \;&\leq\; \int_{x_1}^{x_2} |u(t_0)|\,dx \\ &\leq\; |x_1-x_2|^{\frac{1}{2}} \left(\sup_{t\in(0,T)} \int u^2(t)\,dx\right)^{\frac{1}{2}} \\ &\overset{(3.1)}{\lesssim}\; M^{\frac{1}{2}}\,|x_1-x_2|^{\frac{1}{2}}. \end{aligned} \tag{7.1}$$

For Hölder continuity in time we use a standard interpolation argument. We introduce the mollifiers

$$\varphi_\delta(s) \;=\; \frac{1}{\delta}\,\varphi\left(\frac{s}{\delta}\right), \text{ where } \varphi(s) = C^{-1}\,\chi_{(-1,1)}\,e^{-\frac{1}{1-s^2}}, \; \int_{\mathbf{R}} \varphi(s)\,ds \;=\; 1.$$

We will optimize in δ later. Since

$$\begin{aligned} \left|\int \varphi_\delta(x-x_0)\,h(t,x)\,dx - h(t,x_0)\right| \;&\overset{(7.1)}{\lesssim}\; M^{\frac{1}{2}} \int \varphi_\delta(x-x_0)\,|x-x_0|^{\frac{1}{2}}\,dx \\ &\lesssim\; M^{\frac{1}{2}}\delta^{\frac{1}{2}}, \end{aligned}$$

we have

$$\begin{aligned} |h(t_1,x_0)-h(t_2,x_0)| &\lesssim M^{\frac{1}{2}}\delta^{\frac{1}{2}} + \left|\int \varphi_\delta(x-x_0)\,[h(t_1,x)-h(t_2,x)]\,dx\right| \\ &= M^{\frac{1}{2}}\delta^{\frac{1}{2}} + \left|\int_{t_1}^{t_2}\int \varphi_\delta(x-x_0)\,h_t\,dxdt\right|. \end{aligned}$$

We estimate the integral on the right-hand side as follows:

$$
\begin{aligned}
&\left|\int_{t_1}^{t_2}\int \varphi_\delta(x-x_0)\, h_t\, dxdt\right| \\
&\quad = \left|\int_{t_1}^{t_2}\int \varphi_\delta(x-x_0)\, \left(\tfrac{1}{2}\, u^2 + u_x + u_{xxx}\right)\, dxdt\right| \\
&\quad \lesssim |t_1-t_2| \left(\sup_{t\in(0,T)} \int u^2(t)\, dx\right) \sup|\varphi_\delta| \\
&\qquad + |t_1-t_2| \left(\sup_{t\in(0,T)} \int u^2(t)\, dx\right)^{\frac{1}{2}} \left(\int \left((\varphi_{\delta x})^2 + (\varphi_{\delta xxx})^2\right)\, dx\right)^{\frac{1}{2}} \\
&\quad \overset{(3.1)}{\lesssim} |t_1-t_2| \left(\delta^{-1}\, M + \delta^{-\frac{3}{2}}\, M^{\frac{1}{2}} + \delta^{-\frac{7}{2}}\, M^{\frac{1}{2}}\right).
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&|h(t_1,x_0) - h(t_2,x_0)| \\
&\quad \lesssim M^{\frac{1}{2}}\delta^{\frac{1}{2}} + |t_1-t_2| \left(\delta^{-1}\, M + \delta^{-\frac{3}{2}}\, M^{\frac{1}{2}} + \delta^{-\frac{7}{2}}\, M^{\frac{1}{2}}\right). \qquad (7.2)
\end{aligned}
$$

Lemma 7.1 follows by selecting δ so that the first term on the right-hand side of (7.2) is equal to the largest of the other ones, that is:

$$
\delta \;=\; \max\left\{|t_1-t_2|^{\frac{2}{3}}\, M^{\frac{1}{3}},\; |t_1-t_2|^{\frac{1}{2}},\; |t_1-t_2|^{\frac{1}{4}}\right\}.
$$

∎

Now we are ready to prove Proposition 2.1.

Proof of Proposition 2.1: We work on a fixed domain $\hat t \in (T^{-1}, T)$, $T > 1$, and denote by C a generic constant independent of ν (but possibly depending on T). Statements on convergence are to be understood possibly upon selection of subsequences, all of them labeled again by $\nu \uparrow \infty$. Proposition 2.4 implies that

$$
\int u_\nu^2(T^{-1})\, dx \;\lesssim\; L_\nu^3\, T^2 \;\le\; C\, L_\nu^3. \qquad (7.3)
$$

As motivated in Section 2.1, we rescale variables according to

$$
x \;=\; L_\nu\, \hat x, \quad u \;=\; L_\nu\, \hat u, \quad t \;=\; \hat t. \qquad (7.4)
$$

The estimate from Lemma 5.1 in form of

$$\int_{T^{-1}}^{T}\int u_\nu^4\,dx\,dt \;\lesssim\; L_\nu^{\frac12}\left(e^{\frac{T}{2}}\int u_\nu^2(T^{-1})\,dx\right)^{\frac32} \overset{(7.3)}{\leq} C\,L_\nu^5,$$

upon the rescaling (7.4) translates into

$$\int_{T^{-1}}^{T}\int \hat u_\nu^4\,d\hat x\,d\hat t\;\leq C.$$

Hence

$$\left.\begin{array}{lll} \hat u_\nu \rightharpoonup \hat u & \text{in } L^4\left((T^{-1},T);L^4_{\text{per}}([0,1])\right) \\ \hat u_\nu^2 \rightharpoonup \overline{\hat u^2} & \text{in } L^2\left((T^{-1},T);L^2_{\text{per}}([0,1])\right) \\ \hat u_\nu^3 \rightharpoonup \overline{\hat u^3} & \text{in } L^{\frac43}\left((T^{-1},T);L^{\frac43}_{\text{per}}([0,1])\right) \\ \hat u_\nu^4 \overset{*}{\rightharpoonup} \overline{\hat u^4} & \text{in } \left(C^0\left([T^{-1},T];C^0_{\text{per}}([0,1])\right)\right)^* \end{array}\right\} \tag{7.5}$$

and $\hat u$ has zero average. Upon the rescaling (7.4), (KS) turns into

$$\hat u_{\nu\hat t}+\left(\tfrac12\,\hat u_\nu^2\right)_{\hat x}+L_\nu^{-2}\,\hat u_{\nu\hat x\hat x}+L_\nu^{-4}\,\hat u_{\nu\hat x\hat x\hat x\hat x}\;=\;0.$$

Under the convergence (7.5), this yields the distributional identity

$$\hat u_{\hat t}+\left(\tfrac12\,\overline{\hat u^2}\right)_{\hat x}\;=\;0. \tag{7.6}$$

As a consequence of (KS) we have

$$\begin{aligned}\left(\tfrac12 u_\nu^2\right)_t+\left(\tfrac13 u_\nu^3\right)_x-2((u_{\nu x})^2)_{xx}+\left(\tfrac12 u_\nu^2\right)_{xxxx}&\\ =\tfrac14 u_\nu^2-\left(\tfrac12\,u_\nu+u_{\nu xx}\right)^2&.\end{aligned} \tag{7.7}$$

We consider the right-hand side of (7.7) as a measure,

$$\mu_\nu\;:=\;\tfrac14 u_\nu^2-\left(\tfrac12\,u_\nu+u_{\nu xx}\right)^2\;\leq\;\tfrac14 u_\nu^2, \tag{7.8}$$

and we rescale μ_ν so that the inequality (7.8) is preserved:

$$\hat\mu_\nu\;:=\;L_\nu^{-2}\,\mu_\nu\;\leq\;\tfrac14\,\hat u_\nu^2. \tag{7.9}$$

Since by Lemma 3.1,

$$\begin{aligned}\int_{T^{-1}}^{T}\int |\hat\mu_\nu|\,d\hat x d\hat t &= L_\nu^{-3}\int_{T^{-1}}^{T}\int |\mu_\nu|\,dxdt\\ &\lesssim L_\nu^{-3}\int_{T^{-1}}^{T}\int \left(u_\nu^2+(u_{\nu xx})^2\right)\,dxdt\\ &\le C\,L_\nu^{-3}\int u_\nu^2(T^{-1})\,dx\\ &\overset{(7.3)}{\le} C,\end{aligned}$$

we get

$$\hat\mu_\nu \overset{*}{\rightharpoonup} \hat\mu \quad \text{in } \left(C^0\left([T^{-1},T];C^0_{\mathrm{per}}([0,1])\right)\right)^*. \tag{7.10}$$

Hence, under the convergences (7.5) and (7.10), (7.9) turns into the distributional inequality

$$\hat\mu \le \tfrac14\,\overline{\hat u^2}. \tag{7.11}$$

Under the rescaling (7.4) and (7.9), (7.7) turns into

$$\left(\tfrac12\hat u_\nu^2\right)_{\hat t}+\left(\tfrac13\hat u_\nu^3\right)_{\hat x}-2\,L_\nu^{-2}\left((\hat u_{\nu x})^2\right)_{\hat x\hat x}+L_\nu^{-4}\left(\tfrac12\hat u_\nu^2\right)_{\hat x\hat x\hat x\hat x} = \hat\mu_\nu. \tag{7.12}$$

Notice that we have left the third term on the left-hand side partially unscaled, since for this term we may apply Lemma 3.1 yielding

$$\begin{aligned}\int_{T^{-1}}^{T}\int (\hat u_{\nu x})^2\,d\hat x d\hat t &= L_\nu^{-3}\int_{T^{-1}}^{T}\int (u_{\nu x})^2\,dxdt\\ &\le C\,L_\nu^{-3}\int u_\nu^2(T^{-1})\,dx\\ &\overset{(7.3)}{\le} C.\end{aligned}$$

Therefore (7.12) turns into the distributional equation

$$\left(\tfrac12\,\overline{\hat u^2}\right)_{\hat t}+\left(\tfrac13\,\overline{\hat u^3}\right)_{\hat x} = \hat\mu \tag{7.13}$$

under the convergences (7.5) and (7.10).

In the rest of the proof we focus on improving the weak convergence in (7.5) into strong convergence:

$$\hat u_\nu \longrightarrow \hat u \quad \text{in } L^4\left((T^{-1},T);L^4_{\mathrm{per}}([0,1])\right). \tag{7.14}$$

Indeed, once (7.14) is established, the distributional (in)equalities (7.6), (7.11) and (7.13) turn into

$$\left.\begin{array}{rcl} \hat u_t + \left(\frac12 \hat u^2\right)_{\hat x} &=& 0 \\ \left(\frac12 \hat u^2\right)_{\hat t} + \left(\frac13 \hat u^3\right)_{\hat x} &\le& \frac14 \hat u^2 \end{array}\right\} \quad \text{in } (T^{-1},T)\times \mathbf{R},$$

and the proof is complete. The argument which gives strong convergence (that is, rules out oscillations) is a standard compensated compactness argument ([7],[21]), again based on the div-curl structure. We use Lemma 4.1 (b): Let h_ν be defined by

$$\begin{pmatrix} h_{\nu t} \\ h_{\nu x} \end{pmatrix} = \begin{pmatrix} -\left(\frac12\, u_\nu^2 + u_{\nu x} + u_{\nu xxx}\right) \\ u_\nu \end{pmatrix},$$

normalized so that

$$\int_{T^{-1}}^{T}\int h_\nu \, dxdt = 0. \tag{7.15}$$

We rescale h_ν according to

$$\hat h_\nu = L_\nu^{-2}\, h_\nu, \tag{7.16}$$

so that $\hat h_\nu$ satisfies

$$\begin{pmatrix} \hat h_{\nu \hat t} \\ \hat h_{\nu \hat x} \end{pmatrix} = \begin{pmatrix} -\left(\frac12\, \hat u_\nu^2 + L_\nu^{-2}\, \hat u_{\nu \hat x} + L_\nu^{-4}\, \hat u_{\nu \hat x\hat x\hat x}\right) \\ \hat u_\nu \end{pmatrix}. \tag{7.17}$$

Under the rescaling (7.4) and (7.16), the estimate of Lemma 7.1 translates into

$$\begin{aligned} &|\hat h_\nu(\hat t_1,\hat x_1) - \hat h_\nu(\hat t_2,\hat x_2)| \\ &\quad \overset{(7.3)}{\le} C\left(|\hat x_1-\hat x_2|^{\frac12} + |\hat t_1-\hat t_2|^{\frac13} + L_\nu^{-\frac12}\left(|\hat t_1-\hat t_2|^{\frac14} + |\hat t_1-\hat t_2|^{\frac18}\right)\right) \end{aligned}$$

for all $\hat x_1,\, \hat x_2 \in \mathbf{R}$ and $\hat t_1,\, \hat t_2 \in (T^{-1},T)$. When combined with (7.15), this yields by the Ascoli-Arzelá Theorem

$$\hat h_\nu \longrightarrow \hat h \quad \text{in } C^0\left([T^{-1},T]; C^0_{\text{per}}([0,1])\right). \tag{7.18}$$

Using also the convergence (7.5), (7.17) turns into the distributional identity

$$\begin{pmatrix} \hat h_{\hat t} \\ \hat h_{\hat x} \end{pmatrix} = \begin{pmatrix} -\frac12\, \overline{\hat u^2} \\ \hat u \end{pmatrix}. \tag{7.19}$$

We now appeal to the integral identity in Lemma 4.1 (b), which upon the rescaling (7.5), (7.9) and (7.16) reads as

$$\begin{aligned}\tfrac{1}{12}\int_{T^{-1}}^{T}\int \hat u_\nu^4\,d\hat x\,\zeta\,d\hat t + L_\nu^{-5}\int_{T^{-1}}^{T}\int (u_{\nu x})^3\,dx\,\zeta\,dt \\ = -\int_{T^{-1}}^{T}\int \hat\mu_\nu\,\hat h_\nu\,d\hat x\,\zeta\,d\hat t - \int_{T^{-1}}^{T}\int \tfrac12\,\hat u_\nu^2\,\hat h_\nu\,d\hat x\,\zeta_{\hat t}\,d\hat t\end{aligned}$$

for all $\zeta\in C_c^\infty((T^{-1},T))$. Note that we have left the second term on the left-hand side unscaled in order to use Lemma 3.1, which gives

$$\begin{aligned}L_\nu^{-5}\int_{T^{-1}}^{T}\int |u_{\nu x}|^3\,dx\,\zeta\,dt \;&\lesssim\; C\,L_\nu^{-5}\left(\int u_\nu^2(T^{-1})\,dx\right)^{\frac32}\\ &\overset{(7.3)}{\le} C\,L_\nu^{-\frac12}.\end{aligned}$$

Passing to the limit as $\nu\uparrow\infty$ we obtain

$$\begin{aligned}&\tfrac{1}{12}\int_{T^{-1}}^{T}\int \zeta\,\overline{\hat u^4}(d\hat x,d\hat t)\\ &\quad= -\int_{T^{-1}}^{T}\int \hat\mu\,\hat h\,d\hat x\,\zeta\,d\hat t - \int_{T^{-1}}^{T}\int \tfrac12\,\overline{\hat u^2}\,\hat h\,d\hat x\,\zeta_{\hat t}\,d\hat t \qquad (7.20)\end{aligned}$$

under the convergences (7.5), (7.10) and (7.18). Since $\hat\mu$ and $\hat h$ are 1-periodic, an integration by parts on the first term on the right-hand side yields

$$\begin{aligned}-\int_{T^{-1}}^{T}\int \hat\mu\,\hat h\,d\hat x\,\zeta\,d\hat t \;&\overset{(7.13)}{=}\; -\int_{T^{-1}}^{T}\int \left[\left(\tfrac12\,\overline{\hat u^2}\right)_{\hat t}+\left(\tfrac13\,\overline{\hat u^3}\right)_{\hat x}\right]\hat h\,d\hat x\,\zeta\,d\hat t\\ &=\; \int_{T^{-1}}^{T}\int \tfrac12\,\overline{\hat u^2}\,\hat h_{\hat t}\,d\hat x\,\zeta\,d\hat t + \int_{T^{-1}}^{T}\int \tfrac12\,\overline{\hat u^2}\,\hat h\,d\hat x\,\zeta_{\hat t}\,d\hat t\\ &\quad+\int_{T^{-1}}^{T}\int \tfrac13\,\overline{\hat u^3}\,\hat h_{\hat x}\,d\hat x\,\zeta\,d\hat t\\ &\overset{(7.19)}{=}\; \int_{T^{-1}}^{T}\int \left[-\left(\tfrac12\,\overline{\hat u^2}\right)^2+\tfrac13\,\overline{\hat u^3}\,\hat u\right]d\hat x\,\zeta\,d\hat t\\ &\quad+\int_{T^{-1}}^{T}\int \tfrac12\,\overline{\hat u^2}\,\hat h\,d\hat x\,\zeta_{\hat t}\,d\hat t.\end{aligned}$$

Hence (7.20) turns into

$$\int_{T^{-1}}^{T}\int \zeta\,\overline{\hat u^4}(d\hat x,d\hat t) = \int_{T^{-1}}^{T}\int \left[-3\,\left(\overline{\hat u^2}\right)^2+4\,\overline{\hat u^3}\,\hat u\right]d\hat x\,\zeta\,d\hat t. \quad (7.21)$$

On the other hand, writing

$$(\hat{u}_\nu - \hat{u})^4 = \hat{u}_\nu^4 - 4\,\hat{u}_\nu^3\,\hat{u} + 6\,\hat{u}_\nu^2\,\hat{u}^2 - 4\,\hat{u}_\nu\,\hat{u}^3 + \hat{u}^4,$$

we obtain, using the weak convergence (7.5),

$$\begin{aligned}&\lim_{\nu\uparrow\infty}\int_{T^{-1}}^{T}\int(\hat{u}_\nu - \hat{u})^4\;d\hat{x}\,\zeta\,d\hat{t} \\ &\quad= \int_{T^{-1}}^{T}\int\zeta\,\overline{\hat{u}^4}(d\hat{x},d\hat{t}) + \int_{T^{-1}}^{T}\int\left(-4\,\overline{\hat{u}^3}\,\hat{u} + 6\,\overline{\hat{u}^2}\,\hat{u}^2 - 3\hat{u}^4\right)\,d\hat{x}\,\zeta\,d\hat{t}\end{aligned}\tag{7.22}$$

for all non-negative $\zeta \in C_c^\infty((T^{-1},T))$. Plugging (7.21) into (7.22) we see that

$$\begin{aligned}\lim_{\nu\uparrow\infty}\int_{T^{-1}}^{T}\int(\hat{u}_\nu - \hat{u})^4\;d\hat{x}\,\zeta\,d\hat{t} &= -3\int_{T^{-1}}^{T}\int\left(\left(\overline{\hat{u}^2}\right)^2 - 2\,\overline{\hat{u}^2}\hat{u}^2 + \hat{u}^4\right)\,d\hat{x}\,\zeta\,d\hat{t} \\ &= -3\int_{T^{-1}}^{T}\int(\overline{\hat{u}^2} - \hat{u}^2)^2\,d\hat{x}\,\zeta\,d\hat{t} \;\le\; 0,\end{aligned}$$

and the proof is complete. ∎

8 Proof of Theorem 1

If $t \le 2$ the result is already contained in Proposition 2.4. Hence we confine ourselves to the case $T > 2$, $t \in (2,T]$. In fact, it suffices to show:

Claim. *For any $T > 2$ there exists an $L_0(T)$ such that*

$$\int_{t-1}^{t}\int u^2\,dxdt \;\lesssim\; \frac{L^3}{(t-1)^2} \quad \textit{for all } t \in (2,T] \tag{8.1}$$

for all $L > L_0$ and all solutions u of (KS)-(IC)$_L$.

Indeed, then by (3.1) in Lemma 3.1 we have for $t \ge s$

$$\int u^2(t)\,dx \;\lesssim\; e^{\frac{t-s}{2}}\int u^2(s)\,dx.$$

Hence

$$\int u^2(t)\,dx \lesssim \int_{t-1}^{t}\int u^2\,dxds \overset{(8.1)}{\lesssim} \frac{L^3}{(t-1)^2} \overset{t>2}{\lesssim} \frac{L^3}{t^2}$$

for all $t \in (2,T]$, and Theorem 1 follows.

To prove the claim we set up an indirect argument, and assume that $T > 2$ exists such that for any $\nu \in \mathbf{N}$ there exist an L_ν, a solution u_ν of (KS)-(IC)$_{L_\nu}$ and a time $t_\nu \in (2, T]$ such that $L_\nu \overset{\nu\uparrow\infty}{\to} \infty$ and

$$\int_{t_\nu - 1}^{t_\nu} \int u_\nu^2 \, dx dt \; > \; C_0 \frac{L_\nu^3}{(t_\nu - 1)^2}.$$

Possibly upon selection of a subsequence, we may assume $t_\nu \to \hat{t}_0 \in [2, T]$. In view of Proposition 2.1 we have

$$\hat{u}_\nu(\hat{t}, \hat{x}) \; = \; \frac{1}{L_\nu} u_\nu(\hat{t}, L_\nu \hat{x}) \; \overset{\nu\uparrow\infty}{\longrightarrow} \; \hat{u}(\hat{t}, \hat{x}) \quad \text{in } L^4_{\text{loc}}((0, \infty) \times [0, 1]),$$

so that on one hand

$$\int_{\hat{t}_0 - 1}^{\hat{t}_0} \int \hat{u}^2 \, d\hat{x} d\hat{t} \; \geq \; C_0 \frac{1}{(\hat{t}_0 - 1)^2}. \tag{8.2}$$

In view of Proposition 2.1, Theorem 2.2 and Proposition 2.3, $\hat{u}$ satisfies

$$\int \hat{u}^2(\hat{t}) \, d\hat{x} \; \leq \; C_0 \frac{1}{\hat{t}^2} \quad \text{for all } \hat{t} > 0,$$

so that on the other hand

$$\int_{\hat{t}_0 - 1}^{\hat{t}_0} \int \hat{u}^2 \, d\hat{x} d\hat{t} \; \leq \; C_0 \frac{1}{\hat{t}_0(\hat{t}_0 - 1)},$$

in contradiction with (8.2). This proves the claim and the Theorem.

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