

Localized growth modes, dynamic textures, and upper critical dimension for the Kardar-Parisi-Zhang equation in the weak noise limit

Hans C. Fogedby*

Institute of Physics and Astronomy,

University of Aarhus,

DK-8000, Aarhus C, Denmark

and

NORDITA, Blegdamsvej 17,

DK-2100, Copenhagen Ø, Denmark

Abstract

A nonperturbative weak noise scheme is applied to the Kardar-Parisi-Zhang equation for a growing interface in all dimensions. It is shown that the growth morphology can be interpreted in terms of a dynamically evolving texture of localized growth modes with superimposed diffusive modes. Applying Derrick's theorem it is conjectured that the upper critical dimension is four.

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There is a current interest in the general morphology and in particular the scaling properties of nonequilibrium models in statistical physics. There is, moreover, a need to develop general methods beyond perturbation theory, renormalization group theory, mode coupling theory, and numerical simulations, which permit an analysis of these often intractable problems.

The purpose of this Letter is two-fold. On the one hand, I should like to draw the attention of the community to the availability of a weak noise approach to stochastic equations driven by Gaussian noise which is based on a principle of least action and which allows a discussion of stochastic processes in terms of classical equations of motion and, moreover, provides the Arrhenius factor or weight of a specific kinetic transition; the method is summarized below. Secondly, I apply the weak noise approach to the Kardar-Parisi-Zhang (KPZ) equation for the kinetic growth of an interface in arbitrary dimensions and show that the growth morphology can be interpreted as a dynamical network of growth modes with superimposed diffusive modes. As a corollary, using Derrick's theorem, it is finally shown that the texture of growth modes does not persist above four dimensions, indicating that the upper critical dimension of the KPZ equation is $d = 4$. Details and further developments will be discussed elsewhere.

The variationally-based weak noise method dates back to Onsager [1] and has since reappeared as the Freidlin-Wentzel theory of large deviations [2–4] and as the weak noise saddle point approximation to the functional Martin-Siggia-Rose scheme [5, 6]. The point of departure is the Langevin equations for the set of stochastic variables w_n , $n = 1, \dots, N$, driven by white Gaussian noise [7, 8]

$$\frac{dw_p}{dt} = -\frac{1}{2}F_p - \frac{\Delta}{2}G_{mn}\nabla_m G_{pn} + G_{pn}\eta_n, \quad (1)$$

$$\langle \eta_n \eta_m \rangle(t) = \Delta \delta_{nm} \delta(t), \quad (2)$$

where $F_n(w_p)$ is the drift, $G_{nm}(w_p)$ is accounting for multiplicative noise, $\nabla_n = \partial/\partial w_n$ and Δ is the explicit noise strength; sums are performed over repeated indices. The associated Fokker-Planck equation for the probability distribution $P(x_n, t)$ then has the form [7, 8]

$$\frac{\partial P}{\partial t} = \frac{1}{2}\nabla_n[F_n + \Delta\nabla_m K_{mn}]P, \quad (3)$$

where the symmetrical noise matrix $K_{pm}(w_p) = G_{pn}(w_p)G_{mn}(w_p)$.

Introducing the WKB ansatz

$$P(w_n, t) \propto \exp[-S(w_n, t)/\Delta], \quad (4)$$

the Hamilton-Jacobi equation $\partial S/\partial t + H = 0$, $p_n = \nabla_n S$ follows to leading order in Δ with Hamiltonian

$$H = -\frac{1}{2}p_n F_n + \frac{1}{2}K_{nm}p_n p_m, \quad (5)$$

The Hamilton equations of motion are

$$\frac{dw_n}{dt} = -\frac{1}{2}F_n + K_{nm}p_m, \quad (6)$$

$$\frac{dp_n}{dt} = \frac{1}{2}p_m \nabla_n F_m - \frac{1}{2}p_m p_q \nabla_n K_{mq}, \quad (7)$$

determining classical orbits on the energy surfaces given by H in a classical (w_n, p_n) phase space. Finally, the action S is given by

$$S(w_n, T) = \int^{w_n, T} dt d^N w p_n \frac{dw_n}{dt} - HT. \quad (8)$$

The weak noise scheme bears the same relationship to stochastic fluctuations as the WKB approximation in quantum mechanics, associating the phase of a wave function with the action of a classical orbit. In addition to providing a classical orbit picture of stochastic fluctuations and thus allowing the use of dynamical system theory, the method also yields the Arrhenius factor $P \propto \exp(-S/\Delta)$ for a kinetic transition to w_n in time T . Here the action S serves as the weight in the same manner as the energy E in the Boltzmann factor $P \propto \exp(-E/kT)$ for equilibrium processes. This completes the brief review of the nonperturbative weak noise scheme; for details see e.g. Ref. [9]

The KPZ equation is a field theoretic Langevin equation describing the nonequilibrium growth of an interface [10–13]

$$\frac{\partial h}{\partial t} = \nu \nabla^2 h + \frac{\lambda}{2} \nabla h \cdot \nabla h - F + \eta, \quad (9)$$

$$\langle \eta \eta \rangle(\mathbf{r}, t) = \Delta \delta^d(\mathbf{r}) \delta(t). \quad (10)$$

Here $h(\mathbf{r}, t)$ is the height of the growth profile, ν a diffusion coefficient, λ a growth coefficient, F an imposed drift, and η a locally correlated white Gaussian noise of strength Δ . Dynamic renormalization group (DRG) studies [11, 14, 15] indicates that the KPZ equation conforms

to the dynamical scaling hypothesis with long time -long distance correlations $\langle hh \rangle(\mathbf{r}, t) = r^{2\zeta} \Phi(t/r^z)$, characterized by roughness exponent ζ , dynamic exponent z and scaling function Φ . The KPZ equation is, moreover, invariant under the Galilean transformation, $\mathbf{r} \rightarrow \mathbf{r} - \lambda \mathbf{u}^0 t$, $h \rightarrow h + \mathbf{u}^0 \cdot \mathbf{r}$, $F \rightarrow F + (\lambda/2) \mathbf{u}^0 \cdot \mathbf{u}^0$, and the slope $\nabla h = \mathbf{u} \rightarrow \mathbf{u} + \mathbf{u}^0$ which, implying the scaling law $\zeta + z = 2$, delimits the KPZ universality class. In $d = 1$ the stationary distribution of h is $P(h) \propto \exp[-(\nu/\Delta) \int dx (\nabla h)^2]$ [16], yielding $\zeta = 1/2$ and $z = 2 - \eta = 3/2$. In $d \geq 2$ the DRG implies a kinetic phase or roughness transition at a finite λ from a smooth phase with $z = 2$, and $\zeta = (2 - d)/2$, the linear Edwards-Wilkinson (EW) case for $\lambda = 0$ [17], to a rough phase with still debatable scaling exponents z and $\zeta = 2 - z$, see e.g. Refs. [18–21]. It has, moreover, been conjectured that $d = 4$ is an upper critical dimension beyond which the KPZ equation exhibits EW behavior [22–24].

Applying the nonlinear Cole-Hopf transformation [11] $w(\mathbf{r}, t) = \exp(\lambda h(\mathbf{r}, t)/2\nu)$ the KPZ equation (9) takes the form

$$\frac{\partial w}{\partial t} = \nu \nabla^2 w - \frac{\lambda}{2\nu} w F + \frac{\lambda}{2\nu} \eta, \quad (11)$$

with multiplicative noise. In a moving frame w is changed according to $w \rightarrow w \exp[(\lambda/2\nu) \mathbf{u}^0 \cdot \mathbf{r}]$. Note that in the noiseless case for $\eta = 0$ (11) reduces to the linear diffusion equation permitting a rather complete analysis of the deterministic KPZ equation for a relaxing interface [11, 25]. The Cole-Hopf equation (11) with multiplicative noise forms the basis for the mapping of the KPZ equation to a model of directed polymers (DP) in a quenched random medium which by means of the replica method relates the roughness transition to a pinning transition in the DP model [26, 27].

In recent work I have applied the weak noise method to the KPZ equation or, equivalently, the noisy Burgers equation in $d = 1$ [9, 28]. Here the method readily yields a consistent Galilean invariant dynamical picture of a growing interface in terms of propagating domain walls with superimposed diffusive modes. The localized domain walls account for the growth; the diffusive modes form a subdominant background governed by EW dynamics and not contributing to the growth. The method, moreover, identifies scaling exponents, universality classes, and associates the dynamic exponent z with the domain wall dispersion law or the Hurst exponent $H = 1/z$ with the anomalous superdiffusion of growth modes, see also Ref. [12].

Drawing on the insight gained in $d = 1$, the weak noise method is here applied to the

KPZ equation in arbitrary dimensions [21]. Unlike the $d = 1$ case, where the local slope field $u = \nabla h$ is the natural variable, the Cole-Hopf diffusive field w is more convenient for analysis in $d > 1$. Referring to the general weak noise scheme and making the assignment, $w_n \rightarrow w$, $p_n \rightarrow p$, $K_{nm} \rightarrow (\lambda/2\nu)^2 w^2 \delta^d(\mathbf{r} - \mathbf{r}')$, and $F_n \rightarrow -2[\nu \nabla^2 w - (\lambda/2\nu)wF]$, the Hamiltonian density is given by

$$\mathcal{H} = p[\nu \nabla_n \nabla_n - \nu k^2]w + \frac{1}{2}k_0^2 w^2 p^2, \quad (12)$$

and the canonical field equations take the form

$$\frac{\partial w}{\partial t} = \nu \nabla^2 w - \nu k^2 w + k_0^2 w^2 p, \quad (13)$$

$$\frac{\partial p}{\partial t} = -\nu \nabla^2 p + \nu k^2 p - k_0^2 p^2 w. \quad (14)$$

Here $k = (\lambda F/2\nu^2)^{1/2}$ and $k_0 = \lambda/2\nu$ define two characteristic inverse length scales. Using the equation of motion (13) the action is

$$S(w, T) = \frac{1}{2}k_0^2 \int_{w_i, 0}^{w, T} d^d x dt w^2 p^2, \quad (15)$$

and $P(w, T) \propto \exp(-S(w, T)/\Delta)$ yields the transition probability.

The action and as a result the equations of motion are invariant under the combined Galilei transformation $w \rightarrow w \exp[(\lambda/2\nu)\mathbf{u}^0 \cdot \mathbf{r}]$, $p \rightarrow p \exp[-(\lambda/2\nu)\mathbf{u}^0 \cdot \mathbf{r}]$. The equations of motion determine orbits in the canonical (w, p) phase space from an initial configuration w_i to a final configuration w traversed in time T with the noise field p as a slaved variable. The orbits lie on the constant energy surfaces $H = \int d^d x \mathcal{H}$. The action evaluated along the orbit then yields the transition probability $P(w_i \rightarrow w, T)$. The formalism is symplectic (canonical) and it is an easy task to perform canonical transformations to the height field h or the local slope field $\mathbf{u} = \nabla h$.

The growth of the interface is due to the propagation of localized modes across the system. As in the $d = 1$ case the first task is thus to identify the relevant excitations and connect them in a dynamical network in order to obtain a consistent growth morphology. In the static limit the field equations (13) and (14) assume the symmetrical form: $\nu \nabla^2 w = \nu k^2 w - k_0^2 w^2 p$ and $\nu \nabla^2 p = \nu k^2 p - k_0^2 p^2 w$. On the noiseless manifold, $p = 0$, and the noisy manifold, $p = \nu w$, the static equations reduce to the linear diffusion equation and the nonlinear Schrödinger equation (NLSE), respectively,

$$\nabla^2 w = k^2 w \quad \text{for } p = 0, \quad (16)$$

$$\nabla^2 w = k^2 w - k_0^2 w^3 \quad \text{for } p = \nu w. \quad (17)$$

The diffusion equation (16) admits the radially symmetric solution $w_+(r) \propto r^{1-d/2} I_{1-d/2}(kr)$, where $I(z)$ is the modified Bessel function [29]. For small r , $w_+(r) \rightarrow \text{cst.}$ For large r , $w_+(r) \propto r^{(1-d)/2} \exp(kr)$, yielding the asymptotic height field $h_+(r) = (2\nu/\lambda)((1-d)/2) \ln r + kr \approx (2\nu/\lambda)kr$, and the outward-pointing vector slope field $\mathbf{u}_+(r) = (2\nu/\lambda)k\mathbf{r}/r$ of constant magnitude $(2\nu/\lambda)k$. In $d = 1$, $w_+(x) \propto \cosh(kx)$, giving rise to the height field $h_+(x) = (2\nu/\lambda) \ln \cosh(kx)$, and the slope field $u_+(x) = \nabla h = (2\nu/\lambda)k \tanh(kx)$, i.e., the right hand domain wall solution of the static Burgers equation $\nu \nabla^2 u = -\lambda u \nabla u$ [30, 31]. Since the solution w_+ lives on the noiseless manifold $p = 0$ and the action $S_+ = 0$ it carries no dynamics.

The NLSE (17) admits a radially symmetric bound state $w_-(r)$ falling off as $w_-(r) \propto r^{(1-d)/2} \exp(-kr)$ for large r [32–34]. At the origin $w_-(0) = a_d k/k_0$, where a_d depends on the dimension. Numerically, $a_1 = 1.41(\sqrt{2})$, $a_2 = 2.21$, $a_3 = 4.34$. In the limit $d \rightarrow 4$ the amplitude diverges, the width vanishes, and the bound state disappears for $d \geq 4$. The asymptotic height field $h_-(r) = -(2\nu/\lambda)kr$ and the inward-pointing slope field $\mathbf{u}_- = -(2\nu/\lambda)k\mathbf{r}/r$ of constant magnitude $(2\nu/\lambda)k$. In $d = 1$ the NLSE admits the soliton solution $w_-(x) = a_1(k/k_0) \cosh^{-1}(kr)$ yielding the height field $h_-(r) = -(2\nu/\lambda) \ln \cosh(kr)$ and slope field $u_- = \nabla h_- = -(2\nu/\lambda)k \tanh(kr)$, i.e., the noise-induced left hand domain wall solution [35, 36]. Here the solution w_- is associated with the noisy manifold $p = \nu w$ and carries according to (15) the action $S_- = (k_0^2/2)T \int d^d x w_-^4$. In Fig. 1 the radial solutions of the NLSE are depicted with parameter choice $k = 1$ and $k_0 = 1$ in $d = 1, 2, 3, 3.5$.

At large distances the slope fields \mathbf{u}_\pm associated with w_\pm approach vector fields of constant magnitude $2(\nu/\lambda)k$ and the boundary condition $\mathbf{u} = 0$, corresponding to a flat interface, is implemented by combining a set of modes with appropriately chosen amplitudes k . In a charge language, connecting modes with positive ($k > 0$) and negative charges ($k < 0$) and enforcing charge neutrality $\sum_i k_i = 0$, the boundary condition is automatically enforced.

The construction of the network is implemented in terms of the slope field \mathbf{u} . Assigning a dilute network of static modes at positions $\mathbf{r}_i^0, i = 1 \dots$ with charges k_i the total slope field is $\mathbf{u}(\mathbf{r}) = (2\nu/\lambda) \sum_i \nabla w_i(|\mathbf{r} - \mathbf{r}_i^0|)/w_i(|\mathbf{r} - \mathbf{r}_i^0|)$, $\sum_i k_i = 0$. In the vicinity of the mode position \mathbf{r}_l^0 the slope field is shifted by $\mathbf{u}_l^0 = (2\nu/\lambda) \sum_{i \neq l} \nabla w_i(|\mathbf{r}_l^0 - \mathbf{r}_i^0|)/w_i(|\mathbf{r}_l^0 - \mathbf{r}_i^0|)$ and the l -th mode is assigned the velocity $\mathbf{v}_l = -\lambda \mathbf{u}_l^0$. Using the asymptotic expressions for the modes and introducing a core radius ϵ or order k^{-1} the *self-consistent* dilute dynamical network

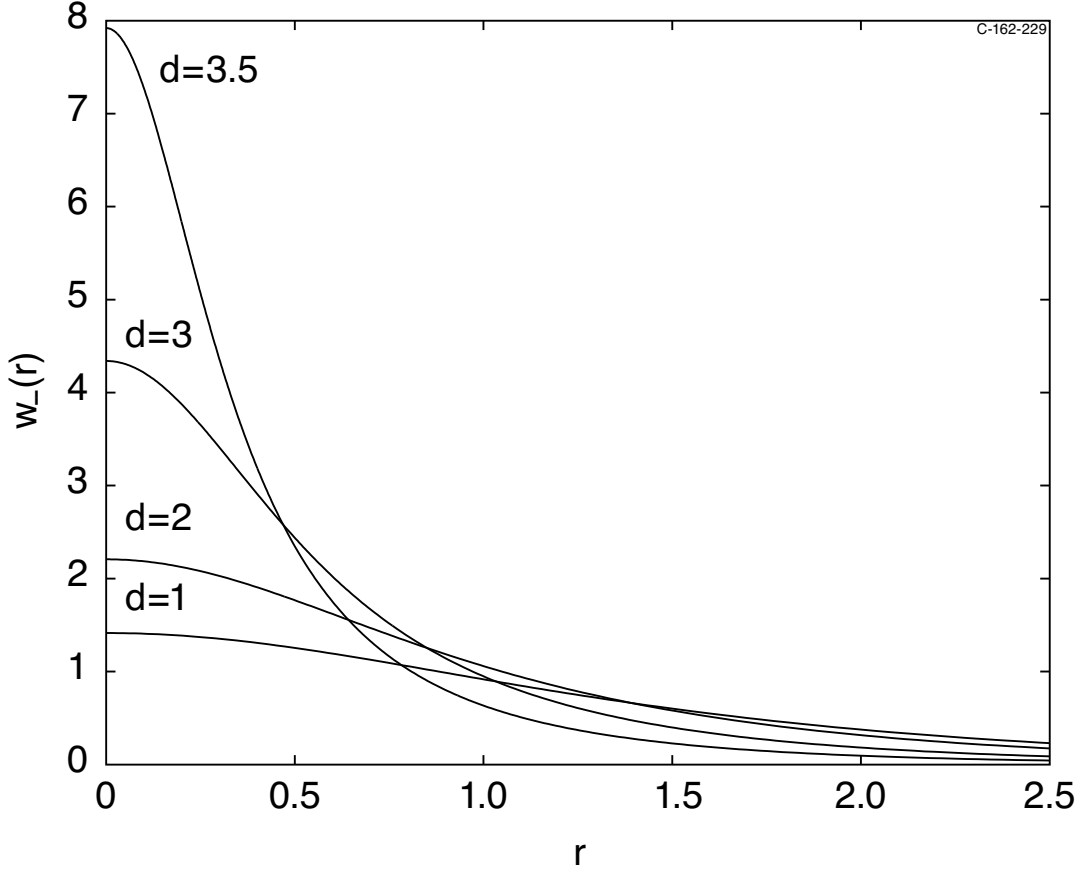


FIG. 1: The radially symmetric bound states of the NLSE are depicted for $k = k^0 = 1$ in $d = 1, d = 2, d = 3, d = 3.5$.

constituting the growth of an interface together with the associated action is given by

$$h(\mathbf{r}, t) = \frac{2\nu}{\lambda} \sum_i k_i \sqrt{(\mathbf{r} - \mathbf{r}_i(t))^2 + \epsilon}, \quad (18)$$

$$\mathbf{r}_i(t) = \int_0^t \mathbf{v}_i(t') dt' + \mathbf{r}_i^0, \quad \sum_i k_i = 0, \quad (19)$$

$$\mathbf{v}_i(t) = -2\nu \sum_{l \neq i} k_l \frac{\mathbf{r}_i(t) - \mathbf{r}_l(t)}{|\mathbf{r}_i(t) - \mathbf{r}_l(t)|}, \quad (20)$$

$$S = \sum_{i, k_i < 0} S_i, \quad S_i = \frac{k_0^2}{2} T \int d^d x w_i^4. \quad (21)$$

In Fig. 2 we depict a 3D snapshot of a 4-node growth morphology in $d = 2$. In the linear case for $k_0 = 0$ the equations of motion (13) and (14) admit extended diffusive modes with dispersion $\omega = \nu k^2$, corresponding to the EW universality class, as discussed in $d = 1$ case in Refs. [9, 28]. The linear modes do not contribute to the growth and are superimposed

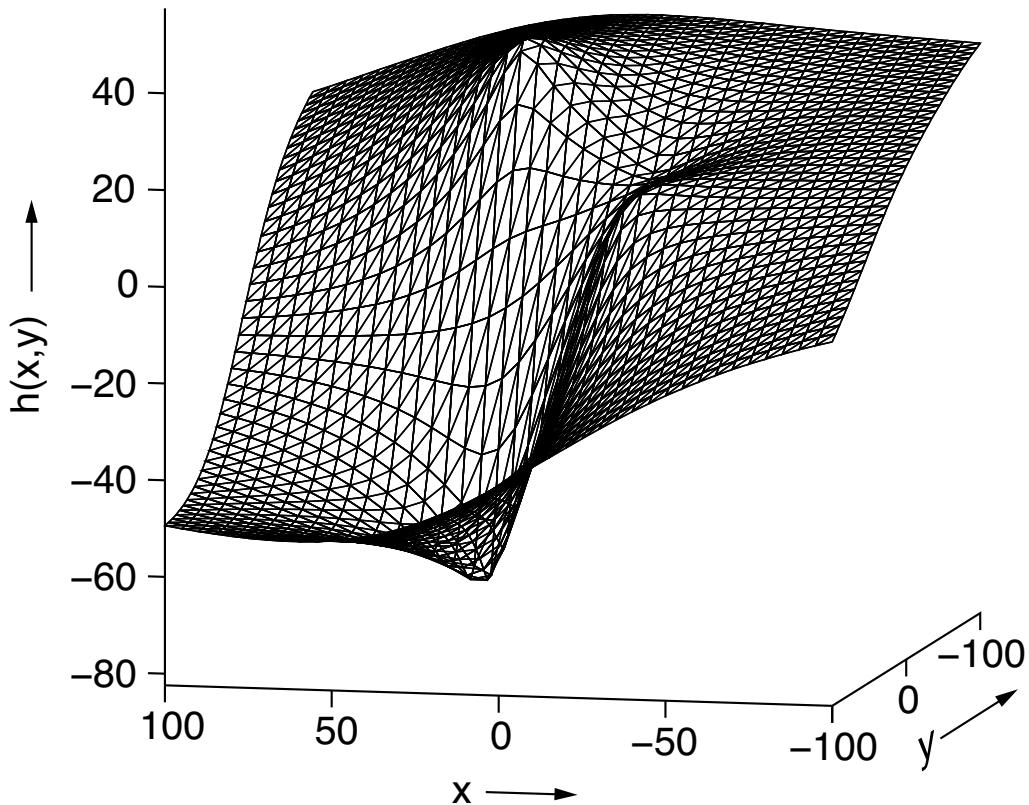


FIG. 2: 3D plot of a 4-node height profile with nodes at $(\pm 20, \pm 20)$ and charges 2.0, -1.5 , -1.0 , and 0.5 (in units of $2\nu/\lambda$)

on the dynamical network in the KPZ case.

On the basis of mode coupling and DP theory [22–24] it has been argued that the upper critical dimension for the KPZ equation is $d_c = 4$. In the present context d_c is associated with the absence of growth modes above $d = 4$ as indicated by the numerics of the NLSE. Here I present a more rigorous argument for the existence of an upper critical dimension using Derrick’s theorem [34, 37] based on constrained minimization.

The NLSE (17) can be derived from the variation of the free energy $F = K + (1/2)k^2N - k_0^2I$, where $K = (1/2) \int d^d x (\nabla w)^2$ is the deformation energy, $N = \int d^d x w^2$ the norm, and $I = \int d^d x w^4$ the interaction, i.e., $\delta F/\delta w = 0$ yields $\nabla^2 w = k^2 w - k_0^2 w^3$. Moreover, multiplying the NLSE by w and integrating over space yields the first identity: $-2K = k^2 N - k_0^2 I$. Under the scale transformation $w(\mathbf{r}) \rightarrow w(\mu\mathbf{r})$ one infers $K \rightarrow \mu^{d-2}K$, $N \rightarrow$

$\mu^d N$, and $I \rightarrow \mu^d I$ and subject to constrained minimization $dF/d\mu|_{\mu=1} = 0$ the second identity: $(d-2)K + (k^2/2)dN - (k_0^2/4)dI = 0$. Eliminating K from the identities one infers $k^2 N = (k_0^2/4)(4-d)I$. Since $N > 0$ and $I > 0$ it follows that $d < 4$ in order for a bound state to exist with finite norm.

In the present Letter I have summarized a general nonperturbative variationally-based weak noise approach to white noise driven stochastic processes. The method has, moreover, been applied to the KPZ equation in arbitrary dimensions yielding a Galilean invariant self-consistent dynamical network of modes accounting for the kinetic growth of the interface. In $d = 1$ the results agree with earlier findings, i.e, a network of matched domain walls whose dispersion yields the dynamic exponent $z = 3/2$; in $d > 1$, the detailed scaling properties remain to be worked out. Finally, based on the dynamical network representation and constrained minimization I have given an argument for $d = 4$ as the upper critical dimension for the KPZ equation.

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* Electronic address: `fogedby@phys.au.dk`

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