Turbulent Transport, Dissipation & Drag

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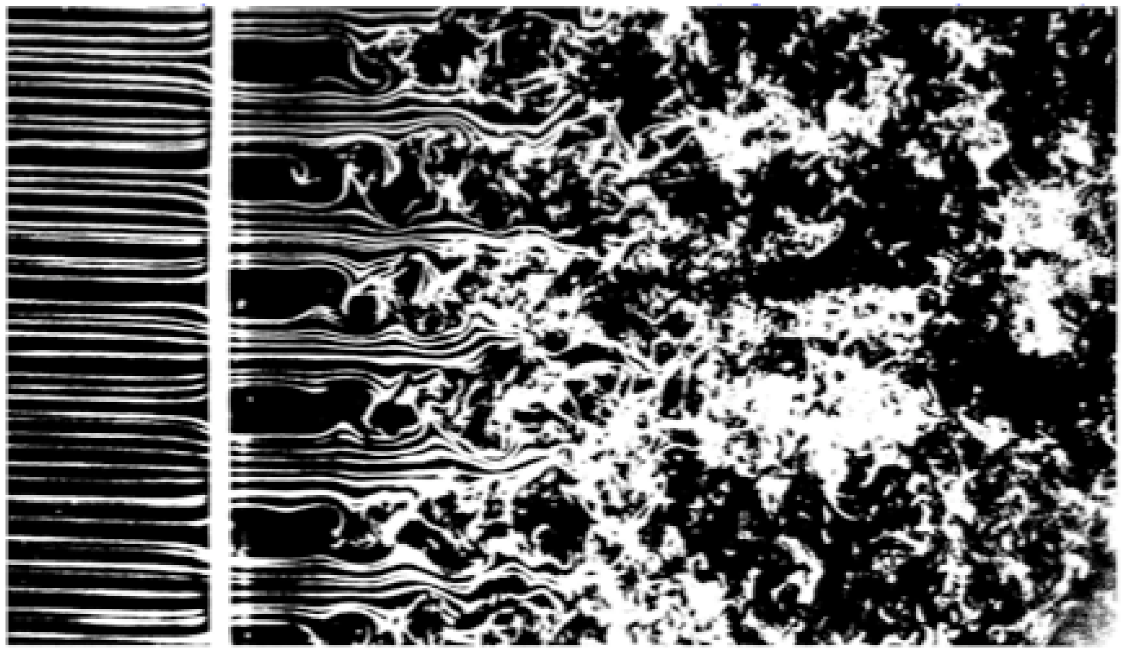
Outline:

1. Phenomenology, physics & philosophy
2. Mathematical models & methods
3. Conclusions & conundrums

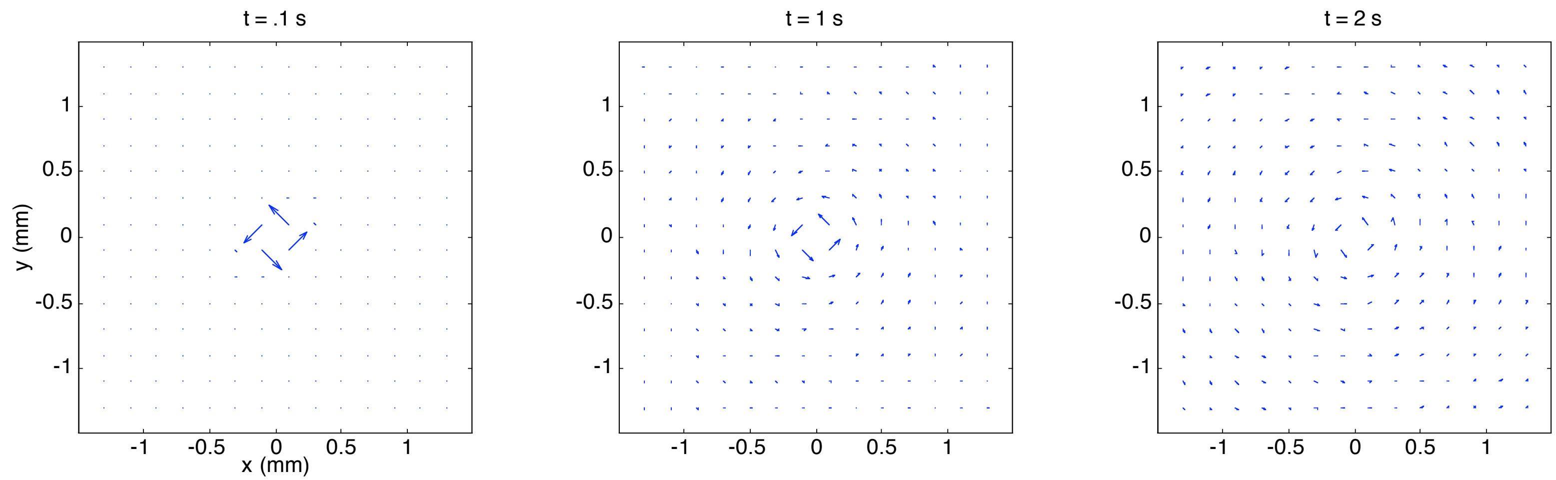
Phenomenology, physics & philosophy



Key features of turbulent flows:

- **Qualitative:** Turbulence enhances *mixing* and thus also transport of mass, heat and momentum in a fluid.
- **Quantitative:** Turbulent transport can display *universal* features: effective transport coefficients may be *independent* of material parameters (molecular transport coefficients) of the fluid.

Viscosity: molecular transport coefficient of momentum



Spot size grows diffusively, $r^2 \approx \nu t$.

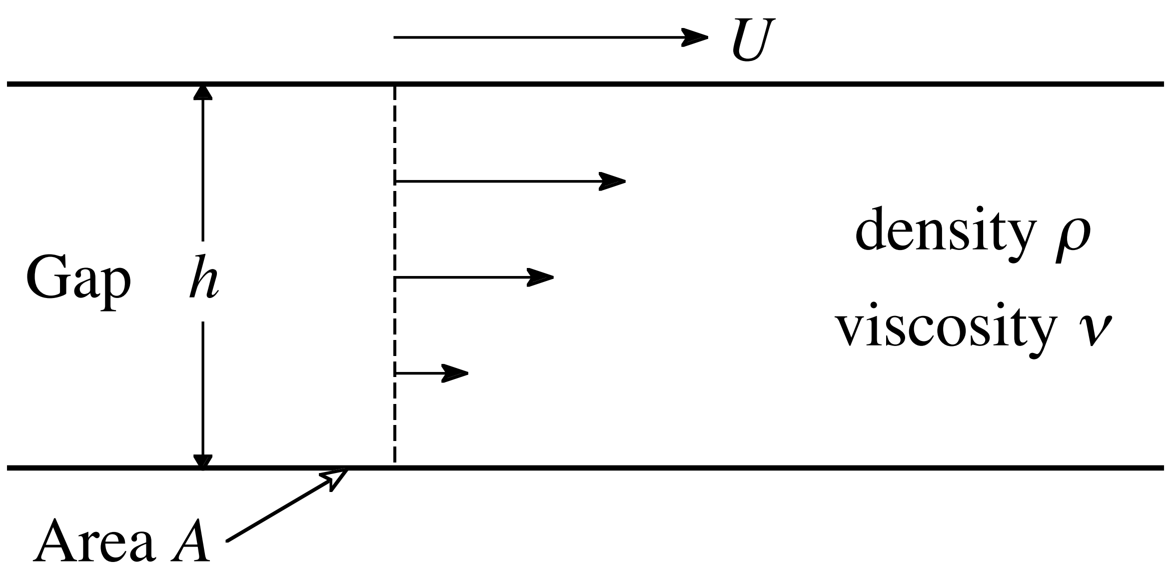
Momentum diffusion coefficient:

ν = kinematic viscosity

Viscosities of familiar fluids:

Fluid	Density (ρ)	Kinematic Viscosity (ν)
Water (0^{o} C)	1.0×10^{3} kg/m^{3}	1.8×10^{-6} m^{2}/s
Water (20^{o} C)	1.0×10^{3} kg/m^{3}	1.0×10^{-6} m^{2}/s
Air (0^{o} C)	1.3 kg/m^{3}	1.3×10^{-5} m^{2}/s
Air (20^{o} C)	1.2 kg/m^{3}	1.5×10^{-5} m^{2}/s
Glycerine (20^{o} C)	1.3×10^{3} kg/m^{3}	1.1×10^{-3} m^{2}/s

Viscosity is the *friction* and the *dissipation* coefficient



Force required to maintain *laminar* flow: $F = \rho \cdot \nu \cdot (U/h) \cdot A$

Power required to maintain *laminar* flow: $P = F \times U = \rho A h \cdot \nu \cdot U^2/h^2$

General relationship: $P_{laminar} \sim$ mass × viscosity × (stirring rate)^2

Example: what is the maximum speed V of your car?

Suppose engine power P = 100 horsepower ≈ 75,000 W.

- If drag due to air is *laminar* friction, then $P = P_{laminar}$.
- Use *spherical approximation* for the car so $P_{laminar} = P_{Stokes} = 6\pi\, r\, \rho_{air}\, \nu_{air}\, V^2$
- Therefore, $V_{max} \approx (P/[6\pi\rho_{air}\nu_{air}r])^{1/2}$
- Use $r = 1\ m$ as radius of the sphere.
- $V_{max} = 14{,}000\ m/s = 30{,}000\ mph$!
- *Note:* speed of sound = 350 m/s = 750 mph

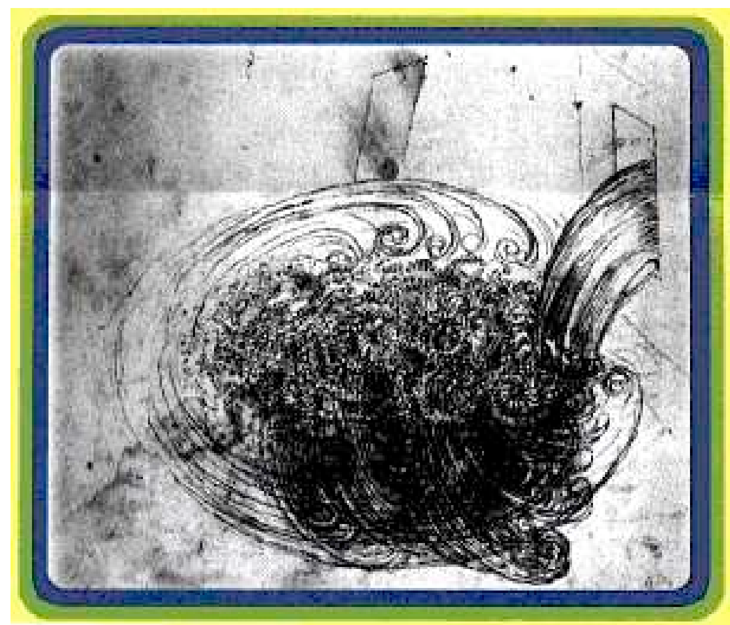
Laminar vs. Turbulent flows:

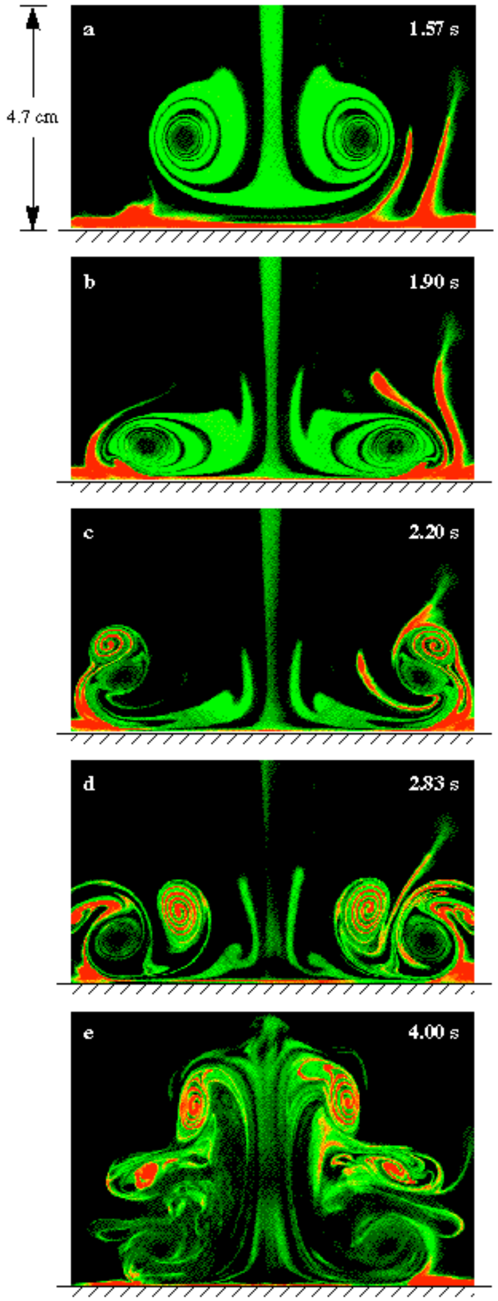
- Laminar flows appear at high viscosity or low stirring rates in small domains.
- Turbulence appears at low viscosity or high stirring rates in large systems.
- Measure of susceptibility to instabilities leading to turbulence is the *Reynolds number*: $\boldsymbol{Re} = Uh/\nu$.

The turbulent cascade process:

small flow features—“*eddies*”—appear spontaneously …

da Vinci's words "... The small eddies are almost numberless, and large things are rotated only by large eddies and not by small ones, and small things are turned by both small eddies and large"





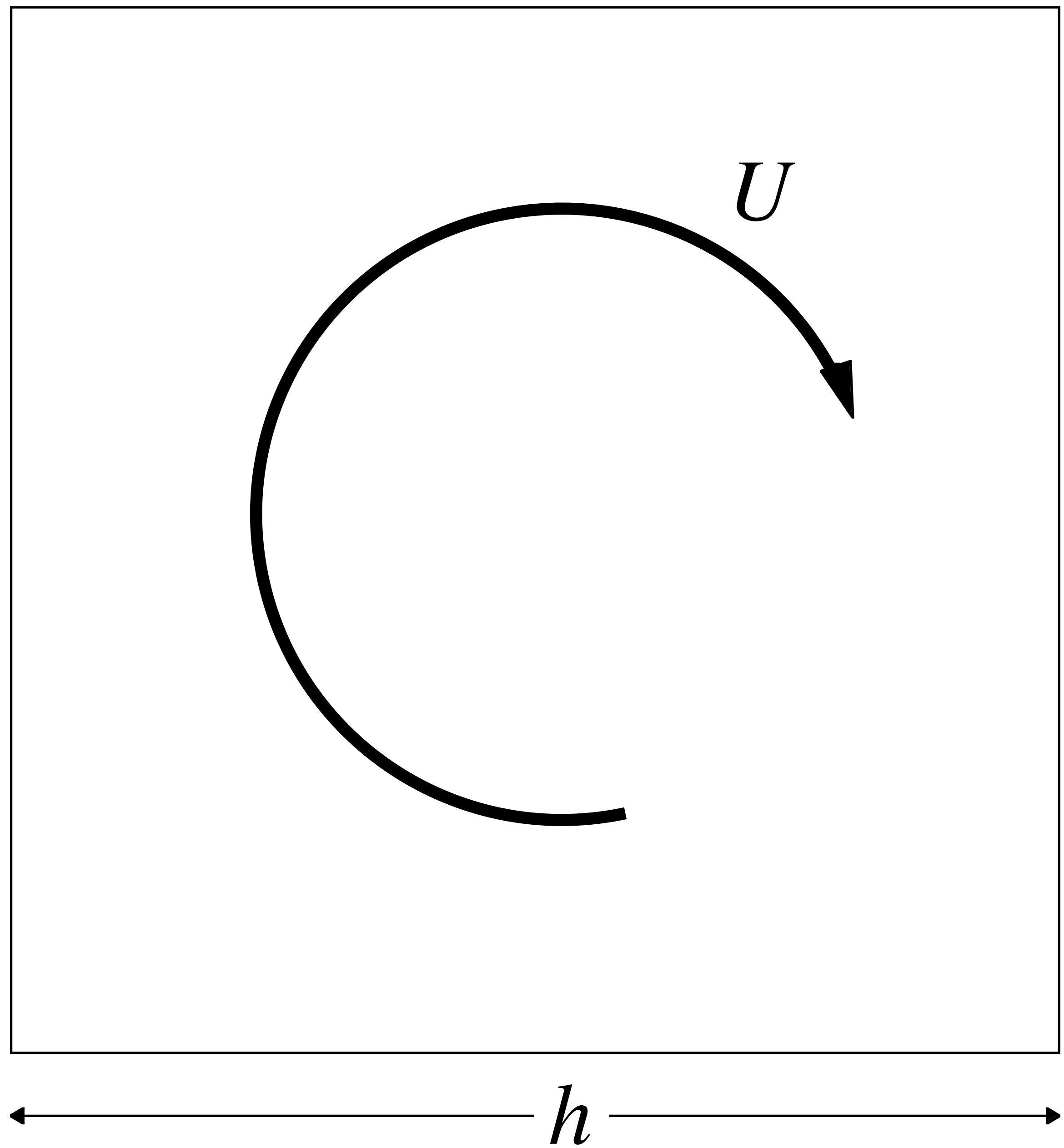
Big whorls have little whorls,
Which feed on their velocity,
And little whorls have lesser whorls,
And so on to viscosity.

Lewis F. Richardson
Cambridge University
circa 1922

Zeroth Law of Turbulence:

If, in a turbulent flow experiment, everything is held constant except the kinematic viscosity which is lowered to zero, then the power consumed does *not* vanish, but has a finite lower limit.

(Richardson, Taylor, Kolmogorov: first half of the 20th century.)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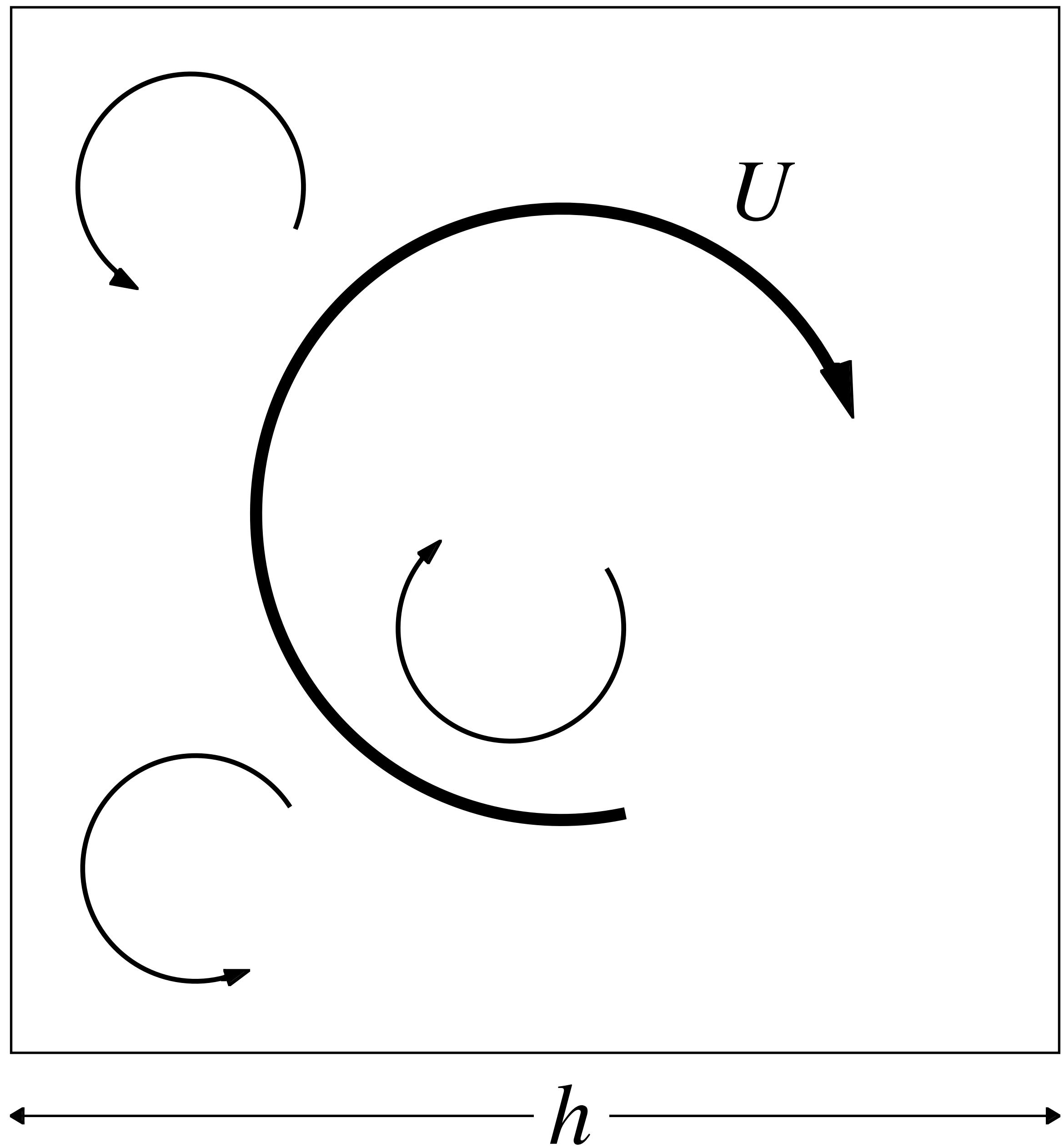
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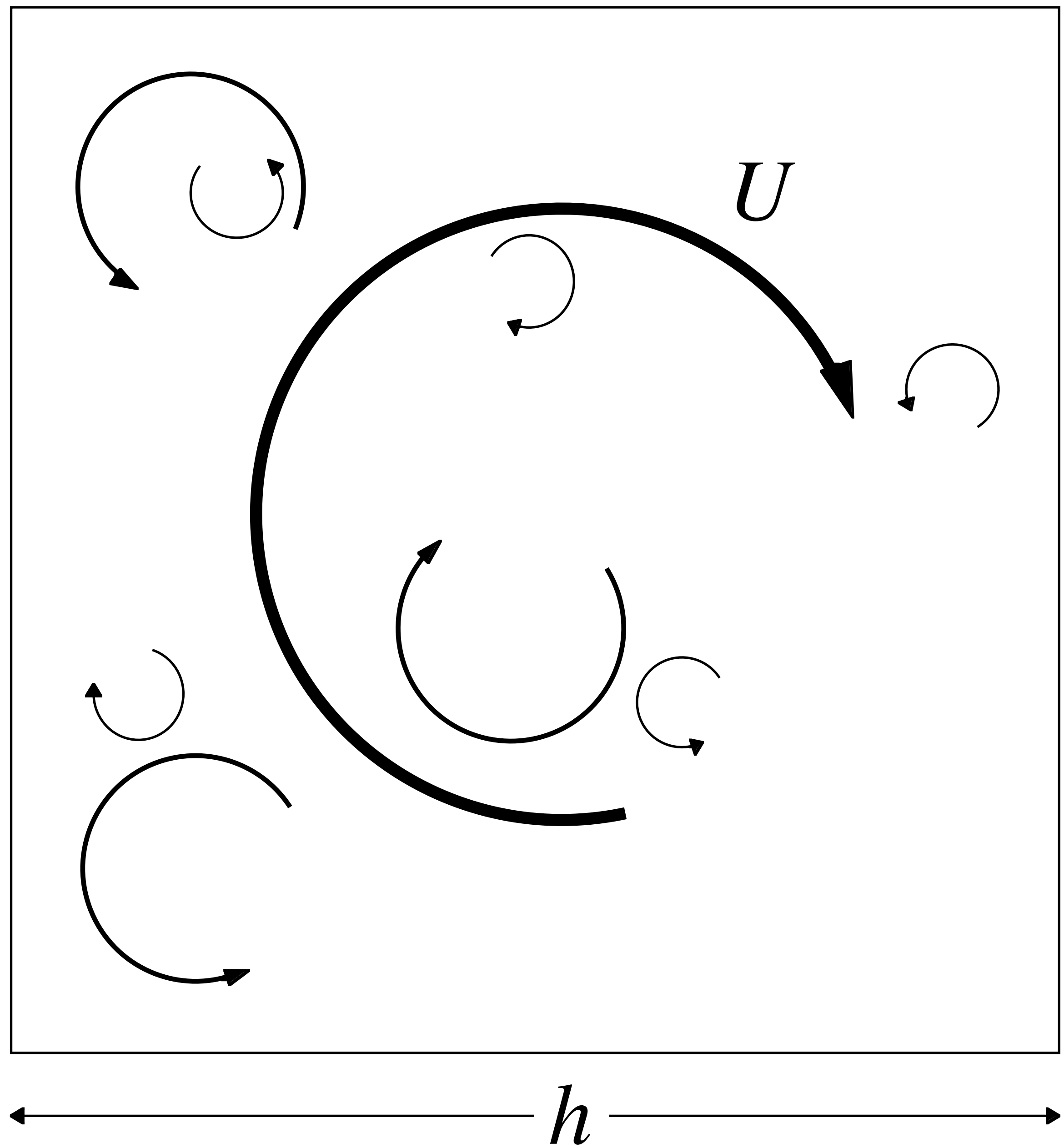
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Kolmogorov 1941 theory: focus on energy/mass spectrum $E(k)$

$$U^2 \equiv \left\langle |\mathbf{u}|^2 \right\rangle = 2 \times \int_{(\text{large scale})^{-1}}^{(\text{small scale})^{-1}} E(k)\,dk \qquad [E] = L^3/T^2$$

→ Energy is input at large ("outer") scale ℓ

→ Energy is dissipated at small scale η (to be determined)

→ Energy is transferred via inviscid mechanisms between ℓ and η.

→ Energy dissipation rate (flux) per unit mass $\varepsilon \equiv$ Power/mass; $[\varepsilon] = L^2/T^3$

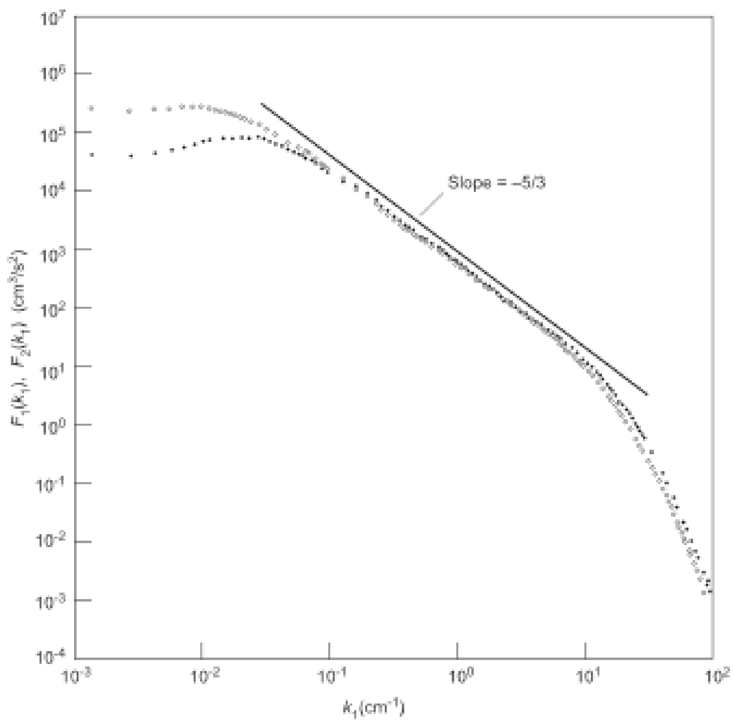
→ Assume that for $\ell^{-1} << k << \eta^{-1}$ viscosity plays *no* role, only k & ε determine E:

→ $E(k) \approx C_K\, \varepsilon^{2/3}\, k^{-5/3}$ in the "inertial" range $\ell^{-1} << k << \eta^{-1}$

Reality check …

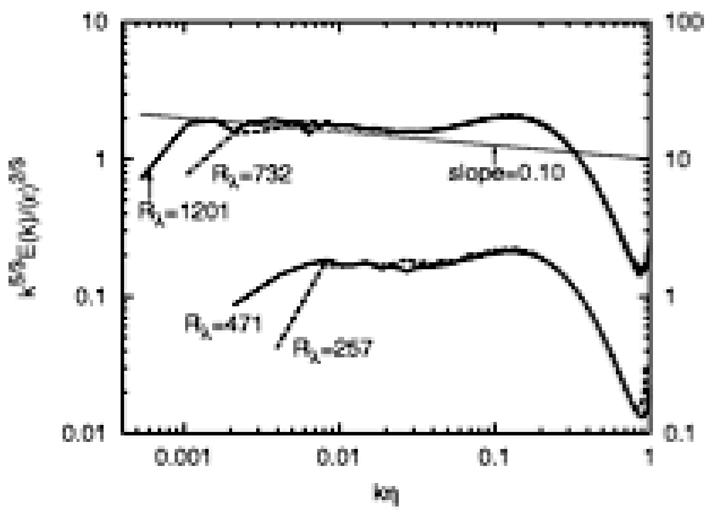
Experiment

Champagne, 1978



Direct numerical simulation (DNS)

Kaneda *et al*, 2002



More from Kolmogorov theory …

Assuming the small "dissipation scale" η depends only on ε & ν,

$$\eta \propto \left(\frac{\nu^3}{\varepsilon}\right)^{1/4}$$

So then as $\nu \rightarrow 0$ …

$$U^2 \approx 2 \times \int_{2\pi\ell^{-1}}^{2\pi\eta^{-1}} E(k)dk \approx 2C_K\varepsilon^{2/3} \int_{2\pi\ell^{-1}}^{\infty} k^{-5/3}dk \propto (\varepsilon\ell)^{2/3}$$

$$\Rightarrow \varepsilon \propto \frac{U^3}{\ell}$$

Recap:

- At low Re, $\varepsilon \sim \nu U^2/\ell^2$

$$c_D(Re) \equiv \varepsilon\ell/U^3 \sim \nu/U\ell = 1/Re$$

- At high Re, $\varepsilon \sim U^3/\ell$

$$c_D(Re) = \varepsilon\ell/U^3 \sim \mathcal{O}(1)$$

- What else could it be? How can we construct units of power out of U, ℓ & ρ without using ν?

- ***General relationship*:**

$$P_{turbulent} \sim \text{mass} \times \text{geometrical factors} \times (\text{stirring rate})^3$$

Reality check …

Experiments:

On the scaling of the turbulence energy dissipation rate

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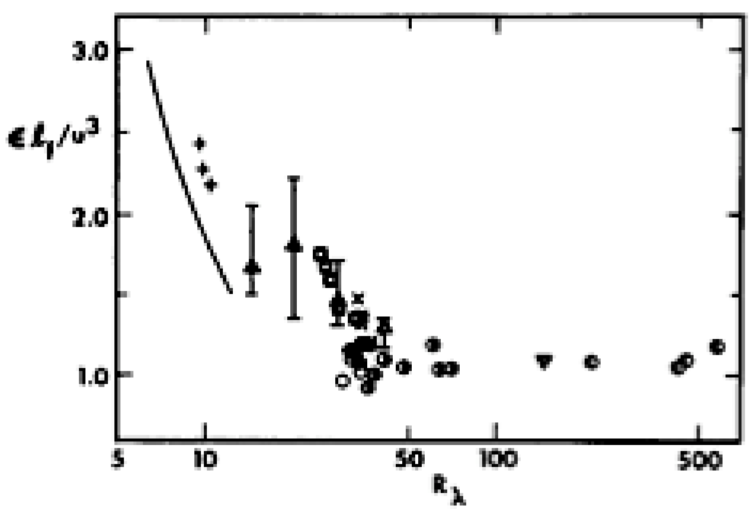
(Received 29 November 1983; accepted 23 February 1984)



Direct numerical simulations:

An update on the energy dissipation rate in isotropic turbulence

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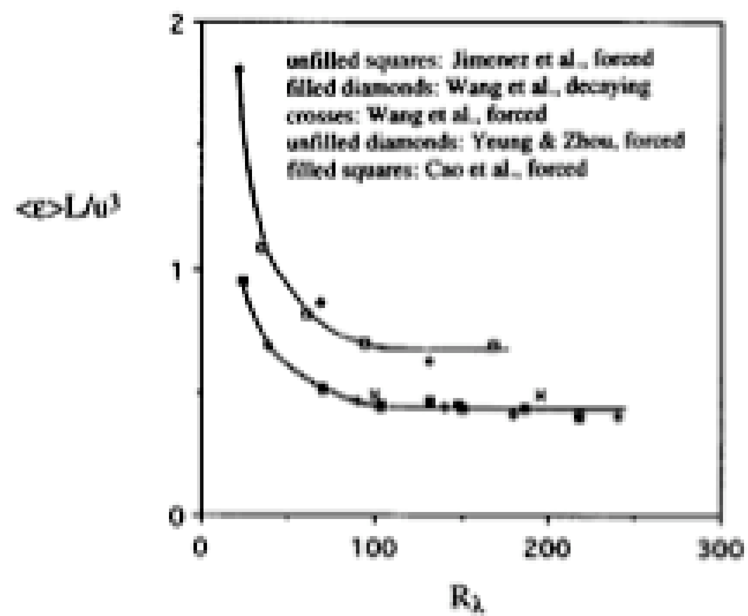
(Received 29 July 1997; accepted 23 October 1997)



Example: What is the maximum speed V of your car?

Suppose engine power P = 100 horsepower ≈ 75,000 W.

- If drag due to air is turbulent dissipation, then $P = P_{turbulent} = c_D \rho_{air} A V^3$.
- c_D is the drag coefficient, depends only only on the *shape* of the car.
- $V_{max} \approx (P/ c_D \rho_{air} A)^{1/3}$
- Use $A = 1\ m^2$ and $c_D = .2$ for guesstimate:
- $V_{max} = 66\ m/s \approx$ 140 *mph*
- Compare laminar estimate $V_{max} \approx$ Mach 40!

Mathematical models & methods

Navier-Stokes equations:

$$\partial_t \vec{u} + \vec{u} \cdot \nabla \vec{u} + \tfrac{1}{\rho} \nabla p = \nu \Delta \vec{u} + \tfrac{1}{\rho} \vec{f}$$

$$\nabla \cdot \vec{u} = 0$$

- $p(\boldsymbol{x},t)$ is pressure
- $\boldsymbol{f}(\boldsymbol{x},t)$ is body force (appropriately "nice")
- plus initial condition $\boldsymbol{u}_0(\boldsymbol{x})$ (appropriately "nice")
- plus boundary conditions in domain Ω (appropriately "nice")
- At best, at high *Re* we can hope for *weak* solutions in 3-dimensions
- $\boldsymbol{u}(t,\boldsymbol{x}) \in L^\infty([0,\infty), L^2(\Omega))$ i.e., finite bulk kinetic energy at each time t
- $\boldsymbol{u}(t,\boldsymbol{x}) \in L^2([0,\infty), H^1(\Omega))$ i.e., finite space-time averaged dissipation rate

Question: how much (if any) of the turbulence lore follows *rigorously* from the Navier-Stokes equations?

- Must formulate mathematical setting appropriately and define quantities precisely.
- General observation (over recent decades): turbulent scalings are often *limits* for solutions of the Navier-Stokes equations.
- Simplest example: body-forced turbulence in the absence of boundaries (model for homogeneous, isotropic turbulence).

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Energy dissipation in body-forced turbulence

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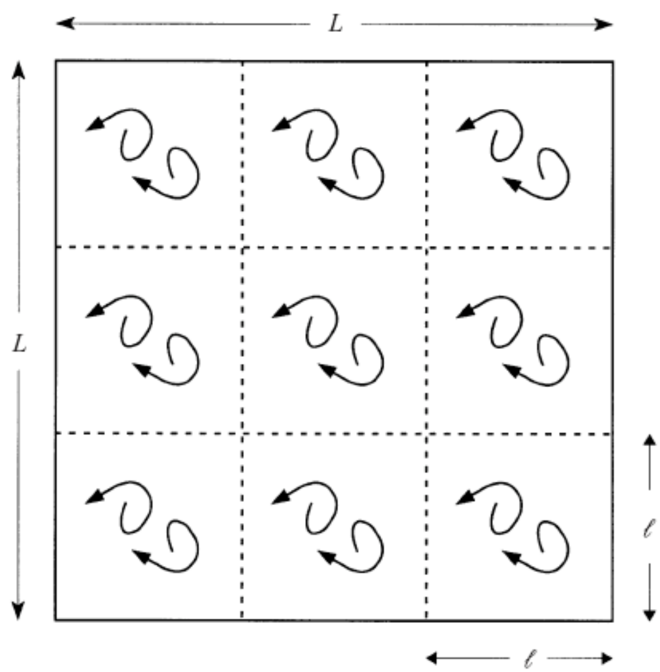


Ω : 3-d torus = $[0, L]^3$ with periodic BCs.

Body force $\boldsymbol{f}(\boldsymbol{x})$ periodic on scale $\ell \leq L$.

WLOG $\operatorname{div} \boldsymbol{f} = 0$ and $\int_{\Omega} \boldsymbol{f} = 0$.

Write $\boldsymbol{f}(\boldsymbol{x}) = F\, \boldsymbol{\Phi}(\ell^{-1}\boldsymbol{x})$.

$\boldsymbol{\Phi}(\boldsymbol{y})$ is the *shape* of the force, periodic on $[0, 1]^3$.

Facts, definitions and questions

- Solution has mean zero if initial data does: $\langle \vec{u} \rangle = 0$

- Mean kinetic energy defines turbulent velocity scale:

$$U^2 \equiv \langle |\vec{u}|^2 \rangle$$

- Power balance (from averaging **u**·*NS-eq*):

$$\varepsilon \equiv \left\langle \nu |\nabla \vec{u}|^2 \right\rangle = \left\langle \tfrac{1}{\rho} \vec{f} \cdot \vec{u} \right\rangle \quad (\leq \text{for weak solutions})$$

- U and ε are *emergent* quantities, properties of solutions.

- Are just ε, U, ν, and ℓ simply related?

- … without reference to F, ρ and L?

Theorem: if $\mathbf{\Phi} \in L^2([0,\ 1]^3)$, then there exist constants c_1 and c_2, uniform in ν, ℓ, L, F & $\boldsymbol{u}_0(\boldsymbol{x})$, such that

$$\varepsilon \ \le\ c_1 \nu \frac{U^2}{\ell^2} + c_2 \frac{U^3}{\ell}$$

- Remark 1: c_1 and c_2 depend on homogeneous ratios of norms of the shape function $\mathbf{\Phi}$ (see proof, next slide).
- Remark 1(a): this "shape" dependence is not unexpected due to vagueness of definition of ℓ in the scaling theory.
- Remark 2: defining $Re = U\ell/\nu$, theorem says

$$c_D(Re) \ =\ \varepsilon\ell/U^3 \ \le\ {}^{c_1}/_{Re} + c_2$$

Proof

First consider the power balance for weak solutions:

$$\varepsilon \equiv \left\langle \nu |\nabla \vec{u}|^2 \right\rangle \le \left\langle \tfrac{1}{\rho} \vec{f} \cdot \vec{u} \right\rangle = \tfrac{1}{\rho} F \left\langle \vec{\Phi} \cdot \vec{u} \right\rangle \le \tfrac{1}{\rho} F \left\| \vec{\Phi} \right\|_2 U$$

Then let smooth, divergence-free vector field $\boldsymbol{\Psi}(\boldsymbol{y})$ be periodic on $[0,1]^3$

such that $\int \boldsymbol{\Phi}(\boldsymbol{y}) \boldsymbol{\Psi}(\boldsymbol{y}) d^3 y \neq 0$. Project *NS-eq* onto $\boldsymbol{\Psi}(\boldsymbol{\ell}^{-1}\boldsymbol{x})$:

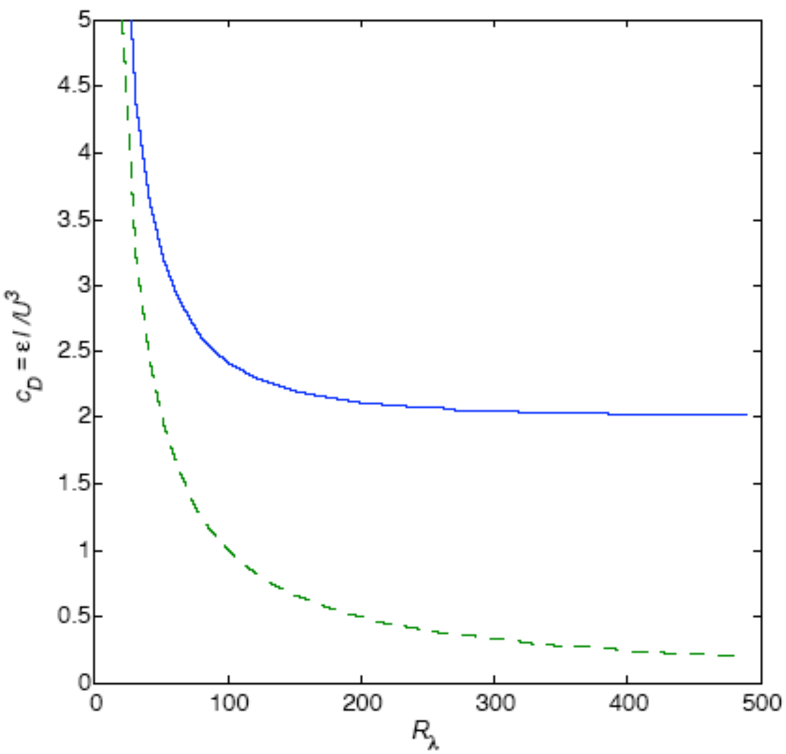
$$\left\langle \vec{u} \cdot \left(-\vec{\nabla} \vec{\Psi} \right) \cdot \vec{u} \right\rangle = \left\langle \nu \Delta \vec{\Psi} \cdot \vec{u} \right\rangle + \tfrac{1}{\rho} F \left\langle \vec{\Phi} \cdot \vec{\Psi} \right\rangle$$

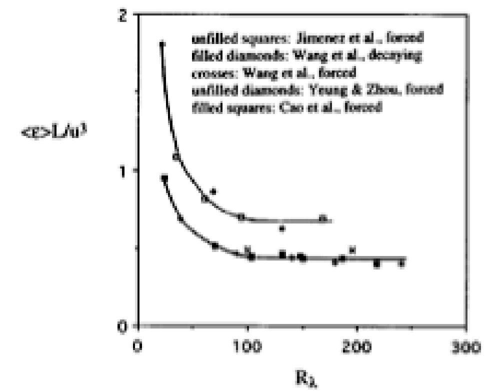
Estimate ${}^1/_\rho\, F$, sort out ℓs, and insert into power balance:

$$\tfrac{1}{\rho} F \left| \left\langle \vec{\Phi} \cdot \vec{\Psi} \right\rangle \right| \le \nu \left\| \Delta_{\vec{y}} \vec{\Psi} \right\|_2 \frac{U}{\ell^2} + \left\| \vec{\nabla}_{\vec{y}} \vec{\Psi} \right\|_\infty \frac{U^2}{\ell} \qquad \Rightarrow$$

$$\varepsilon \le \frac{\left\| \Delta_{\vec{y}} \vec{\Psi} \right\|_2 \left\| \vec{\Phi} \right\|_2}{\left| \left\langle \vec{\Phi} \cdot \vec{\Psi} \right\rangle \right|} \nu \frac{U^2}{\ell^2} + \frac{\left\| \vec{\nabla}_{\vec{y}} \vec{\Psi} \right\|_\infty \left\| \vec{\Phi} \right\|_2}{\left| \left\langle \vec{\Phi} \cdot \vec{\Psi} \right\rangle \right|} \frac{U^3}{\ell}$$

Upper & lower bounds (sketch)





Dissipation in Stokes flow, with $c_D \sim Re^{-1}$, is a *lower* bound.

Remarks

- Coefficients c_1 and c_2 can be optimized over choices of projector $\mathbf{\Psi}$. Even better, define *min-max* variational problem for the bounds.

- Soluble for a simple geometry: forced flow in a slippery channel … analytically (at ∞ *Re*) with Eckhardt & Schumacher (*JFM* 2003); numerically (finite *Re*) with Petrov & Lu (*J. Turbulence* in review).

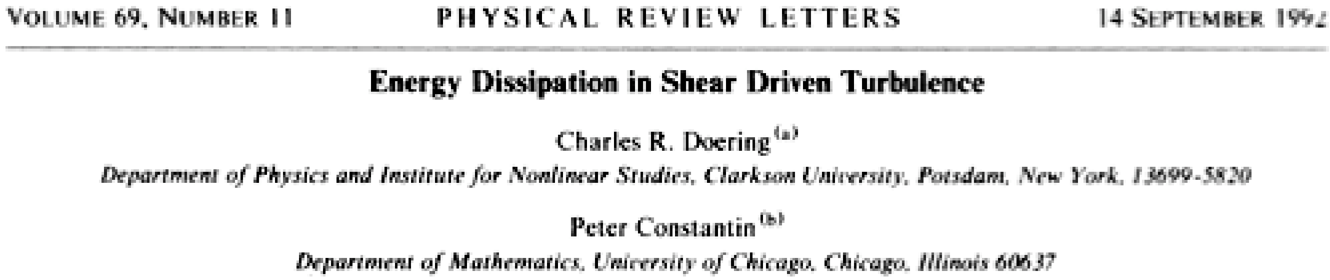
- If $\mathbf{\Phi} \notin L^2$ then scaling may change … with only $\mathbf{\Phi} \in H^{-1}$ best estimate is $\varepsilon \leq c_3 U^4/\nu$, consistent with theories of Vascillicos *et al* (*JFM* 2004) and Biferale *et al* (*PRL* 2004).

- For *boundary-driven* flows U is given (not emergent): dissipation-rate estimation has a *long* history beginning with Hopf in 1941 …

  … continuing with Lou Howard's & Fritz Busse's work in 60s & 70s …

  … with a resurgence in the 90s including many new developments by collaborators & colleagues including Peter Constantin, Siegfried Grossmann *et al*, Xiaoming Wang, Rich Kerswell …

  … with many new applications by many others.



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Energy Dissipation in Shear Driven Turbulence

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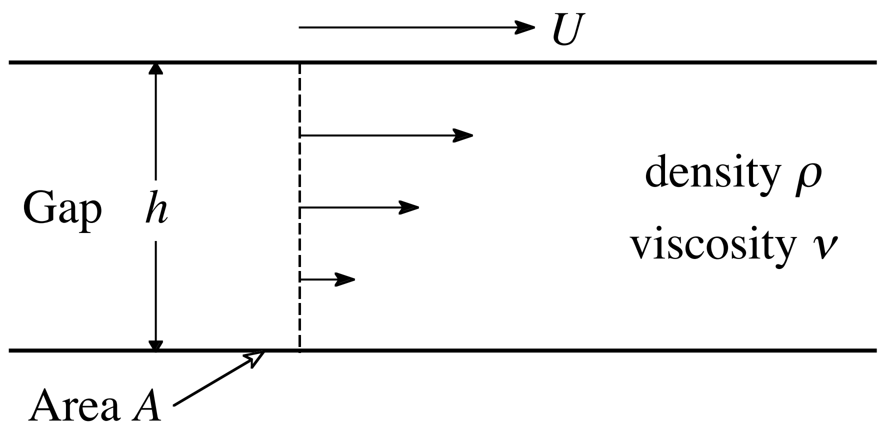
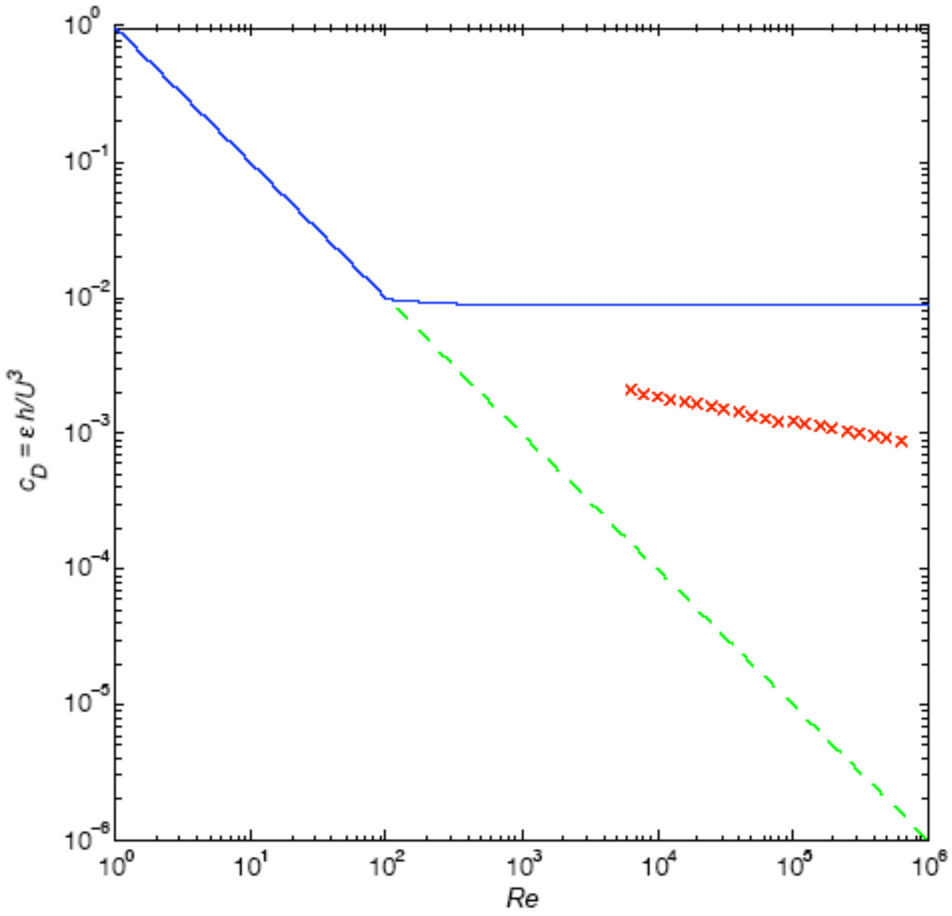


Theorem: the power ***P*** required to maintain any solution of the incompressible Navier-Stokes equations, *laminar* or *turbulent*, is bounded according to

$$\rho\, \nu\, U^2 A/h \;\leq\; P = \rho A h \varepsilon \;\leq\; c_D\, \rho\, A\, U^3$$

with $c_D < .09$ (Plasting & Kerswell, *JFM* 2003: $c_D < .0086$)

Reality check …



- Upper bound
- Lower bound
- Real data from Lathrop, Feinberg & Swinney (1992)
- "*Logarithmic friction law*"

$$c_D \sim 1/[\ln(Re)]^2$$

as $Re \to \infty$ for smooth walls.

Conclusions and comments

- Fully developed turbulent dissipation and drag becomes independent of viscosity (modulo logarithms) … diffusive transport of momentum becomes a physical *property of the flow*, *not of the fluid*!

  The molecular transport coefficients are essentially "renormalized" to zero ***but they cannot be zero!***

  Implication: your car won't go any faster——and you won't get any better gas mileage——if the air gets any more slippery!

- Turbulent transport of mass and heat become independent of the material parameters (modulo caveats) … "eddy diffusion" & "eddy conductivity" are *properties of the flow*, *not properties of the fluid*!

- Turbulent transport is essential ingredient in natural phenomena (e.g., *atmosphere & ocean science, astrophysics*) and engineering applications (e.g., *drag in pipe flows, mixing in chemical production, heat transfer*).

- There are still many open problems to work on …

Conundrum

- For many cases analysis *and* experiment agree that $\varepsilon_{\text{turbulent}} \sim U^3/\ell$.

- But analysis has ***not*** established such a bound for open flows …

  … such as turbulence past an object (*e*.*g*., past a car or sphere).

- As of Snowbird 2005, we must resort to Hopf-1941 estimates …

  … for drag coefficient of the form $c_D \sim Re\, e^{Re}$.

- For a car travelling at 60mph, $Re \approx 3\times10^6$.

- There's ***room for improvement***!

- With special thanks to the National Science Foundation PHY (Mathematical Physics) DMS (Applied Mathematics),

- and to DOE (via Los Alamos),

- and to all my collaborators and colleagues from whom I've learned so much …

The End