On the Non-Homogeneous Stationary Kuramoto-Sivashinsky Equation

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Abstract

In this note we give some explicit estimates for the L^∞-norm of the periodic solutions of the time-independent non-homogeneous Kuramoto-Sivashinsky equation. In particular, we give an estimate of the Michelson's upper bound of all periodic solutions of the time-independent homogeneous Kuramoto-Sivashinsky equation.

Key words: Kuramoto-Sivashinsky Equations; Global attractor; Michelson's constant

1 Introduction

The Kuramoto-Sivashinsky equation

$$\frac{\partial}{\partial t}U + \nabla^4 U + \nabla^2 U + \frac{1}{2}|\nabla U|^2 = 0 \tag{1.1}$$

has been independently discovered by Kuramoto and Tsuzuki [11], and by Sivashinsky [15] in the study of a reaction diffusion system and flame front propagation respectively as well as in the study of 2D Kolmogorov fluid flows [16].

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In fact in [16], one considered a slightly modified Kolmogorov flow $U = (v, w)$ given by the equations

$$\begin{cases} v_t + vv_x + wv_y &= -p_x + R^{-1}\nabla^2 v - \mu R^{-1} v - R^{-1}\sin y \\ w_t + vw_x + ww_y &= -p_y + R^{-1}\nabla^2 w - \mu R^{-1} w \\ v_x + v_y = 0, \end{cases} \tag{1.2}$$

where R stands for the Reynolds number and $\mu > 0$ is a small friction coefficient, under the rescaling $\mu = \lambda\epsilon^4$, $\xi = \epsilon x$, $\tau = \epsilon^4 t$ with a constant λ. The stream function $\Psi(x, y)$ is assumed to have the form

$$\Psi(x, y) = \cos y + A(\xi + c\eta, \tau) + O(\epsilon)$$

with another constant c. Then it is shown that $U(\zeta, s) = \frac{c}{\sqrt{2}} A(\sqrt{6}\zeta, 6\sqrt{2}s)$ satisfies the equation

$$U_s + U_{\zeta\zeta\zeta\zeta} + ((2 - (3c)^{-2}U_\zeta^2)U_\zeta)_\zeta - U_\zeta^2 + \frac{\lambda}{6}U = 0. \tag{1.3}$$

This remarkable equation becomes the one dimensional version of (1.1) in case $c \sim \infty$, $\lambda \sim 0$. In this case it can be written in the following way

$$u_t + u_{xxxx} + u_{xx} + uu_x = 0, \tag{1.4}$$

where $u = U'$. As argued in [16], this equation provides a good model for the weak turbulent effects observed for the flow in (1.2).

The equation (1.4) turned out to be a very fruitful subject of research. Periodic boundary conditions for (1.1) yield periodic boundary conditions for (1.4), i.e.

$$u(x, t) = u(x + L, t), \quad \forall x, \ \forall t, \tag{1.5}$$

where L is the period in the space variable and the supplementary condition

$$\int_0^L u(x)dx = 0. \tag{1.6}$$

Under these conditions the equation (1.4) has, once the period L is large enough, a very complicated global attractor $\mathcal{A}_L$ which seems to be amenable to computer based investigations. The complexity of $\mathcal{A}_L$ was observed in [8] and later in low-dimensional approximate inertial forms in [10]. This approach allowed the study of several global bifurcations of $\mathcal{A}_L$ [4]. Moreover, the mathematical study of the equation, initiated in [13] also leads to some very interesting results in [13], [5], [6], [2], [12], [7], [9] as well as challenging conjectures in [14], [3], [7]. For instance: Does there exist a universal constant K such that for any $u \in \mathcal{A}_L$ one has

$$|u(x)| \leq K, \quad \forall x \in \mathbb{R}. \tag{1.7}$$

An affirmative answer to this question would give immediate positive answers to the conjectures that the fractal dimension of the $\mathcal{A}_L$ scales as L [14] and (via [7]) that all elements of $\mathcal{A}_L$ are analytic functions in x with convergence radius everywhere larger that an absolute positive constant [3]. Both these conjectures have strong computational support.

In the remarkable paper [12], Michelson has taken a first step in proving (1.7) for all $u \in \mathcal{A}_L$, namely he has proven that there exists a constant K_{M} such that (1.7) holds for all stationary solutions of (1.4) with $K = K_{\mathrm{M}}$. The next step would be to prove that the set

$$\left\{\bar{u} = \int u\mu(du); \mu = \text{invariant probability measure on } \mathcal{A}_L,\ L > 0\right\} \tag{1.8}$$

is bounded in $L^\infty(\mathbb{R})$. To solve this problem one must study the non-homogeneous Kuramoto-Sivashinsky equation. In this note we extend Michelson's result to this latter equation and give a positive partial answer to the preceding problem. In fact, we study the non-homogeneous stationary Kuramoto-Sivashinsky equation

$$u''''(x) + u''(x) + u(x)u'(x) = f(x) \quad \forall x \in \mathbb{R} \tag{1.9}$$

with the periodic boundary conditions (1.5) and the supplementary condition (1.6).

Our main objective is to study the periodic solutions of (1.9) and their dependence on f. We will prove that the existence of periodic solutions puts some constraints on the function f. Our method is based on elementary estimates and is more direct than Michelson's. In particular, it yields an explicit estimate for the Michelson constant K_{M}, namely $K_{\mathrm{M}} \leq 92.2$ (see Theorem 5.3 below).

2 Integral Representation

Integrating equation (1.9), we get

$$u''' + u' + \frac{1}{2}u^2 = \frac{1}{2}F, \tag{2.1}$$

where

$$F(x) = 2\int_0^x f(y)dy + \text{const.} \tag{2.2}$$

Periodicity of u implies that F is also periodic with the same period L. Integrating over the period one more time, we obtain

$$\int_0^L F(x)dx = \int_0^L u^2(x)dx. \tag{2.3}$$

Let

$$F_{\min} = \min_x F(x), \qquad F_{\max} = \max_x F(x). \tag{2.4}$$

Equation (2.3) implies $F_{\max} \geq 0$. Let

$$c \geq \sqrt{F_{\max}}, \tag{2.5}$$

$$p(x) = \sqrt{c^2 - F(x)}, \; x \in \mathbb{R}. \tag{2.6}$$

Then the equation (2.1) can be written as

$$u''' + u' + \frac{1}{2}u^2 + \frac{1}{2}p^2 = \frac{1}{2}c^2, \tag{2.7}$$

where $p(x)$ is periodic.

Let $y := u - c$. Then we have that

$$y''' + y' + cy = -\frac{1}{2}\left(y^2 + p^2\right) \leq 0. \tag{2.8}$$

The solutions of the homogeneous part $y''' + y' + cy = 0$ are

$$y_1 = e^{-bx}, \quad y_2 = e^{\frac{b}{2}x}\cos\beta x, \quad \text{and} \quad y_3 = e^{\frac{b}{2}x}\sin\beta x,$$

where b is the positive solution of $b^3+b=c$ and $\beta=\sqrt{\frac{3}{4}b^2+1}$. By the change of variables

$$v(\xi)=\frac{1}{c}u\left(\frac{\xi}{\beta}\right),\quad q(\xi)=\frac{1}{c}p\left(\frac{\xi}{\beta}\right),\quad w(\xi)=v(\xi)-1, \tag{2.9}$$

the equation (2.7) becomes

$$v'''+\frac{1}{\beta^2}v'=\frac{1}{2}\delta(1-v^2-q^2), \tag{2.10}$$

where $\delta=\frac{c}{\beta^3}$. The equation (2.8) can be written for w as

$$w'''+\frac{1}{\beta^2}w'+\delta w=-\phi(\xi):=-\frac{1}{2}\delta(w^2+q^2)\leq 0. \tag{2.11}$$

Note that due to periodicity all functions in (2.10) and (2.11) are bounded on $\mathbb{R}$. Therefore we will first consider solutions of (2.10) and (2.11) which are bounded on $\mathbb{R}$ but not necessarily periodic.

The general solution for this equation is given by

$$\begin{aligned}
w(\xi) &= A(a)e^{-\frac{b}{\beta}(\xi-a)}+B(a)e^{\frac{b}{2\beta}(\xi-a)}\cos(\xi-a)\\
&+C(a)e^{\frac{b}{2\beta}(\xi-a)}\sin(\xi-a) \qquad\qquad (2.12)\\
&+A_0\int\limits_{\xi}^{a}\left[e^{-\frac{b}{\beta}(\xi-y)}-e^{\frac{b}{2\beta}(\xi-y)}\cos(\xi-y)+\frac{3b}{2\beta}e^{\frac{b}{2\beta}(\xi-y)}\sin(\xi-y)\right]\phi(y)dy,
\end{aligned}$$

where $A_0=\frac{\beta^2}{3b^2+1}$ and $\xi\leq a$.

Multiplying (2.12) by $e^{\frac{b}{\beta}(\xi-a)}$ and letting $\xi\to-\infty$, we obtain

$$A(a)=-A_0\psi(a):=-A_0\int\limits_{-\infty}^{a}e^{-\frac{b}{\beta}(a-y)}\phi(y)dy\leq 0, \tag{2.13}$$

with ϕ defined in (2.11).

We can now prove that w satisfies also the following equation:

$$w''-\frac{b}{\beta}w'+\frac{b^2+1}{\beta^2}w=-\psi\leq 0, \tag{2.14}$$

where ψ was defined in (2.13).

Indeed,

$$\begin{aligned} w(a) &= A(a) + B(a), \\ w'(a) &= -\frac{b}{\beta}A(a) + \frac{b}{2\beta}B(a) + C(a), \text{ and} \\ w''(a) &= \frac{b^2}{\beta^2}A(a) + \left(\frac{b^2}{4\beta^2} - 1\right) B(a) + \frac{b}{\beta}C(a), \end{aligned}$$

whence solving for $B(a)$ and $C(a)$ we obtain

$$A(a) = A_0 \left(w''(a) - \frac{b}{\beta}w'(a) + \frac{b^2+1}{\beta^2}w(a) \right).$$

This and the definition of ψ gives (2.14).

The general solution for (2.14) is

$$\begin{aligned} w(\xi) &= A_1(a)e^{\frac{b}{2\beta}(\xi-a)}\cos(\xi - a) + B_1(a)e^{\frac{b}{2\beta}(\xi-a)}\sin(\xi - a) \\ &\quad - \int_a^\xi e^{\frac{b}{2\beta}(\xi-y)}\sin(\xi - y)\psi(y)dy. \end{aligned} \tag{2.15}$$

Letting $a \to \infty$, we finally obtain the following:

Lemma 2.1 *Any solution w of the equation (2.11) bounded over all $\mathbb{R}$ is given by the following integral representation:*

$$w(\xi) = \int_\xi^\infty e^{\frac{b}{2\beta}(\xi-y)}\sin(\xi - y)\psi(y)dy, \tag{2.16}$$

where w and ψ are defined in (2.9) and (2.13) respectively.

In the next section we will show how this representation (2.16) leads to some global estimates for the bounded solution of the equations (2.10) and (2.11).

3 Global Estimates

We start by proving that the first derivative of the bounded solution w of (2.11) is also bounded.

Lemma 3.1 *For all $\xi \in \mathbb{R}$ we have*

$$\frac{b}{2\beta}w(\xi) + \frac{1}{\sqrt{\gamma}}w\left(\xi + \frac{\pi}{2}\right) \le w'(\xi) \le \frac{b}{2\beta}w(\xi) - \sqrt{\gamma}w\left(\xi - \frac{\pi}{2}\right), \quad (3.1)$$

where $\gamma(\beta) = e^{\frac{b\pi}{2\beta}}$.

PROOF. Recall that $\psi(y) \ge 0$, $y \in \mathbb{R}$. Therefore, from (2.16) we obtain

$$\begin{aligned}
w'(\xi) &= \int_{\xi}^{\infty} e^{\frac{b}{2\beta}(\xi-y)}\left[\frac{b}{2\beta}\sin(\xi-y)+\cos(\xi-y)\right]\psi(y)dy \\
&= \frac{b}{2\beta}w(\xi) + \int_{\xi}^{\xi+\frac{\pi}{2}} e^{\frac{b}{2\beta}(\xi-y)}\cos(\xi-y)\psi(y)dy \\
&+ e^{-\frac{b\pi}{4\beta}}\int_{\xi+\frac{\pi}{2}}^{\infty} e^{\frac{b}{2\beta}(\xi+\pi/2-y)}\cos\left((\xi+\pi/2)-y-\pi/2\right)\psi(y)dy \\
&\ge \frac{b}{2\beta}w(\xi) + \frac{1}{\sqrt{\gamma}}w\left(\xi+\frac{\pi}{2}\right),
\end{aligned}$$

and

$$\begin{aligned}
w'(\xi) &= \int_{\xi}^{\infty} e^{\frac{b}{2\beta}(\xi-y)}\left[\frac{b}{2\beta}\sin(\xi-y)+\cos(\xi-y)\right]\psi(y)dy \\
&= \frac{b}{2\beta}w(\xi) - \int_{\xi-\frac{\pi}{2}}^{\xi} e^{\frac{b}{2\beta}(\xi-y)}\cos(\xi-y)\psi(y)dy \\
&+ e^{\frac{b\pi}{4\beta}}\int_{\xi-\frac{\pi}{2}}^{\infty} e^{\frac{b}{2\beta}(\xi-\pi/2-y)}\cos\left((\xi-\pi/2)-y+\pi/2\right)\psi(y)dy \\
&\le \frac{b}{2\beta}w(\xi) - \sqrt{\gamma}w\left(\xi-\frac{\pi}{2}\right). \qquad \square
\end{aligned}$$

This lemma immediately implies the following global estimates for v.

Theorem 3.2 *Any bounded solution of (2.10) satisfies*

$$|v(\xi)| \le \frac{\gamma+1}{\gamma-1}, \quad \xi \in \mathbb{R}. \quad (3.2)$$

PROOF. From Lemma 3.1 it follows that

$$\gamma w\left(\xi-\frac{\pi}{2}\right)+w\left(\xi+\frac{\pi}{2}\right)\leq 0, \tag{3.3}$$

or, in terms of v,

$$\gamma v\left(\xi-\frac{\pi}{2}\right)+v\left(\xi+\frac{\pi}{2}\right)\leq 1+\gamma, \quad \xi\in\mathbb{R}. \tag{3.4}$$

Consider $v_1(\xi)=-v(-\xi)$, $q_1(\xi)=q(-\xi)$. Since (2.10) is valid if we replace $\{v,q\}$ with $\{v_1,q_1\}$, we also have

$$\gamma v\left(\xi+\frac{\pi}{2}\right)+v\left(\xi-\frac{\pi}{2}\right)\geq -(1+\gamma), \quad \xi\in\mathbb{R}. \tag{3.5}$$

Relations (3.4) and (3.5) imply (3.2). $\square$

Note that $\frac{\gamma+1}{\gamma-1}$ is bounded by $\frac{7}{5}$, which implies the following:

Corollary 3.3 *Any bounded solution u of (2.7) satisfies*

$$|u(x)|\leq \frac{7}{5}c. \tag{3.6}$$

Combining Lemma 3.1 and Theorem 3.2 we can now obtain L^∞ bounds for the first and second derivatives of v.

Lemma 3.4 *For v as in Theorem 3.2 we have*

$$-\frac{2\gamma}{\gamma-1}\left(\frac{b}{2\beta}+\frac{1}{\sqrt{\gamma}}\right)\leq v'\leq g(\beta):=\frac{2}{\gamma-1}\left(\frac{b}{2\beta}+\gamma\sqrt{\gamma}\right) \tag{3.7}$$

and

$$|v''|\leq \frac{b}{\beta}g(\beta)+\frac{b^2+1}{\beta^2}\frac{2\gamma}{\gamma-1}. \tag{3.8}$$

PROOF. Relation (3.7) follows from Theorem 3.2 and Lemma 3.1.

For (3.8) we apply (2.14), Theorem 3.2 and (3.7) to obtain

$$
\begin{aligned}
v'' &\leq \frac{b}{\beta}v' - \frac{b^2+1}{\beta^2}(v-1) \\
&\leq \frac{b}{\beta}g(\beta) + \frac{b^2+1}{\beta^2}\frac{2\gamma}{\gamma-1}.
\end{aligned}
$$

Using again the fact that $\{v_1, q_1\}$ defined in the proof of Theorem 3.2 is a solution of (2.10), we get

$$
v'' \geq -\frac{b}{\beta}g(\beta) - \frac{b^2+1}{\beta^2}\frac{2\gamma}{\gamma-1}. \qquad \square
$$

4 Universal Bounds

After the preparation done in the preceding sections we can now proceed to our main result.

Lemma 4.1 *Let $[\xi_1, \xi_2]$ be an interval such that $v'(\xi) > 0$ for $\xi \in (\xi_1, \xi_2)$, $v'(\xi_1) = v'(\xi_2) = 0$. Let h be an absolutely continuous non-negative function on the interval $[v_1, v_2]$, where $v_1 := v(\xi_1)$, $v_2 := v(\xi_2)$. Then*

$$
\int_{v_1}^{v_2} v^2 h(v)dv \geq \left(1 - q_0^2 - \frac{2g(\beta)}{\delta\beta^2}\right)\int_{v_1}^{v_2} h(v)dv + \frac{2}{\delta}\int_{\xi_1}^{\xi_2}(v')^2 v'' h'(v)d\xi, \quad (4.1)
$$

where $q_0 = \max_\xi q(\xi)$.

PROOF. Multiplying (2.10) by $h(v)v'$ and integrating over the interval $[\xi_1, \xi_2]$, we obtain

$$
\begin{aligned}
&\int_{\xi_1}^{\xi_2} v''' h(v)v' d\xi + \int_{\xi_1}^{\xi_2} \frac{(v')^2 h(v)d\xi}{\beta^2} \\
&\qquad = \frac{\delta}{2}\left(\int_{\xi_1}^{\xi_2} h(v)v'd\xi - \int_{\xi_1}^{\xi_2} v^2 h(v)v'd\xi - \int_{\xi_1}^{\xi_2} v'h(v)q^2 d\xi\right) \\
&\qquad \geq \frac{\delta}{2}\left(\int_{v_1}^{v_2} h(v)dv - \int_{v_1}^{v_2} v^2 h(v)dv - q_0^2\int_{v_1}^{v_2} h(v)dv\right).
\end{aligned}
$$

Since $v'(\xi) \leq g(\beta)$ for any ξ, we have

$$
\begin{aligned}
&\left(1-q_0^2-\frac{2g(\beta)}{\delta\beta^2}\right)\int_{v_1}^{v_2} h(v)dv-\int_{v_1}^{v_2} v^2h(v)dv \\
&\quad\le \frac{2}{\delta}\int_{\xi_1}^{\xi_2} v'''h(v)v'd\xi \\
&\quad= -\frac{2}{\delta}\left(\int_{\xi_1}^{\xi_2}(v'')^2h(v)d\xi+\int_{\xi_1}^{\xi_2}(v')^2v''h'(v)d\xi\right) \\
&\quad\le -\frac{2}{\delta}\int_{\xi_1}^{\xi_2}(v')^2v''h'(v)d\xi. \qquad \square
\end{aligned}
$$

Lemma 4.2 *Let v be any bounded solution of (2.10), and let*

$$
\max_{\xi} q^2(\xi) < G(\beta) := 1-2\frac{g(\beta)}{\beta^2\delta}-\frac{1}{6}\left(\frac{\gamma+1}{\gamma-1}\right)^2. \tag{4.2}
$$

Let also $[\xi_1,\xi_2]$, $\xi_1<\xi_2$, be an interval such that $v'(\xi)>0$ for $\xi\in(\xi_1,\xi_2)$, $v'(\xi_1)=v'(\xi_2)=0$. Then

$$
v_1v_2>0 \quad \textit{and} \tag{4.3}
$$

$$
\max\{|v_1|,|v_2|\} > \frac{1}{\sqrt{6}}\frac{\gamma+1}{\gamma-1}, \tag{4.4}
$$

where $v_1=v(\xi_1)$, $v_2=v(\xi_2)$.

PROOF. First, consider the case when $0\le v_1<v_2$. Set

$$
h(v) := v_2 - v \quad \text{for} \quad v\in[v_1,v_2]. \tag{4.5}
$$

Note that

$$
\begin{aligned}
\int_{\xi_1}^{\xi_2}(v')^2v''h'(v)d\xi &= \frac{1}{3}\int_{\xi_1}^{\xi_2}\left((v')^3\right)'h'(v)d\xi \\
&= -\frac{1}{3}\int_{\xi_1}^{\xi_2}(v')^3h''(v)d\xi = 0.
\end{aligned}
$$

Denote

$$a = a(\beta, q_0) := 1 - q_0^2 - 2\frac{g(\beta)}{\beta^2\delta}. \tag{4.6}$$

Using Lemma 4.1 we obtain

$$a\int_{v_1}^{v_2}(v_2 - v)dv \le \int_{v_1}^{v_2} v^2(v_2 - v)dv,$$

$$a\left(v_2(v_2 - v_1) - \frac{1}{2}(v_2^2 - v_1^2)\right) \le \frac{1}{3}v_2(v_2^3 - v_1^3) - \frac{1}{4}(v_2^4 - v_1^4).$$

So, we have

$$a \le \frac{v_2^2 + 2v_1v_2 + 3v_1^2}{6}.$$

Thus, by the assumption (4.2),

$$v_2^2 + 2v_1v_2 + 3v_1^2 > \left(\frac{\gamma+1}{\gamma-1}\right)^2. \tag{4.7}$$

Therefore, $v_1 \ne 0$ by virtue of (3.2). Moreover,

$$6\max\{v_1^2, v_2^2\} \ge v_2^2 + 2v_1v_2 + 3v_1^2 > \left(\frac{\gamma+1}{\gamma-1}\right)^2,$$

which implies

$$\max\{|v_1|, |v_2|\} > \frac{1}{\sqrt{6}}\frac{\gamma+1}{\gamma-1}. \tag{4.8}$$

In the case when $v_1 < v_2 \le 0$ we define h on the interval $[v_1, v_2]$ as

$$h(v) := v - v_1. \tag{4.9}$$

As in the previous case we get that $v_2 \ne 0$ and

$$v_1^2 + 2v_1v_2 + 3v_2^2 > \left(\frac{\gamma+1}{\gamma-1}\right)^2. \tag{4.10}$$

Therefore,

$$\max\{|v_1|, |v_2|\} > \frac{1}{\sqrt{6}}\frac{\gamma+1}{\gamma-1}. \tag{4.11}$$

In the remaining case when $v_1 v_2 < 0$ we have to get a contradiction. Define the function h on the interval $[v_1, v_2]$ as follows.

$$h(v) := \begin{cases} \frac{v-v_1}{-v_1} & \text{when } v_1 \le v < 0, \\ \frac{v_2-v}{v_2} & \text{when } 0 \le v \le v_2. \end{cases} \tag{4.12}$$

Let $\xi_0 \in (\xi_1, \xi_2)$ be the point where v vanishes. Then notice that

$$\begin{aligned} \int_{\xi_1}^{\xi_2} (v')^2 v'' h'(v) d\xi &= -\frac{1}{v_1}\int_{\xi_1}^{\xi_0} (v')^2 v'' d\xi - \frac{1}{v_2}\int_{\xi_0}^{\xi_2} (v')^2 v'' d\xi \\ &= -\frac{1}{v_1}\frac{1}{3}(v'(\xi_0))^3 + \frac{1}{v_2}\frac{1}{3}(v'(\xi_0))^3 > 0. \end{aligned}$$

By the Lemma 4.1,

$$\begin{aligned} a\int_{v_1}^{0} \frac{v-v_1}{-v_1} dv + a\int_{0}^{v_2} \frac{v_2-v}{v_2} dv &< \int_{v_1}^{0} \frac{v^2(v-v_1)}{-v_1} dv + \int_{0}^{v_2} \frac{v^2(v_2-v)}{v_2} dv, \\ \frac{a}{2}\left(\frac{v_1^2}{-v_1} + \frac{v_2^2}{v_2}\right) &< \frac{1}{-v_1}\left(-\frac{v_1^4}{4} + v_1\frac{v_1^3}{3}\right) + \frac{1}{v_2}\left(-\frac{v_2^4}{4} + v_2\frac{v_2^3}{3}\right), \\ \frac{a}{2}(|v_1| + v_2) &< \frac{1}{12}(|v_1|^3 + v_2^3). \end{aligned}$$

Thus,

$$a < \frac{1}{6}\frac{|v_1|^3 + v_2^3}{|v_1| + v_2}.$$

So, by virtue of (4.2),

$$\frac{|v_1|^3 + v_2^3}{|v_1| + v_2} > \left(\frac{\gamma+1}{\gamma-1}\right)^2. \tag{4.13}$$

Finally,

$$|v_1|\left(v_1^2-\left(\frac{\gamma+1}{\gamma-1}\right)^2\right)+v_2\left(v_2^2-\left(\frac{\gamma+1}{\gamma-1}\right)^2\right)>0,$$

which contradicts (3.2). $\square$

Theorem 4.3 *Let v be any periodic solution of (2.10) with zero mean. Then the function q from equation (2.10) must satisfy*

$$\max_{\xi} q^2(\xi) \geq G(\beta), \tag{4.14}$$

for G defined in Lemma 4.2.

PROOF. Suppose, to the contrary, that

$$\max_{\xi} q^2(\xi) < G(\beta).$$

Since v has zero mean, but is not identical zero (since q^2 can not reach one), there exists ξ_0 with the property that $v(\xi_0)=0$ and that for any $\epsilon>0$ there exists $\xi_\epsilon>\xi_0$ such that $\xi_\epsilon-\xi_0<\epsilon$ and $v'(\xi_\epsilon)>0$. Given $\epsilon>0$, consider the maximal interval $(\xi_\epsilon^{\mathrm{L}},\xi_\epsilon^{\mathrm{R}})$ such that $\xi_\epsilon\in(\xi_\epsilon^{\mathrm{L}},\xi_\epsilon^{\mathrm{R}})$ and $v'(\xi)>0$, $\xi\in(\xi_\epsilon^{\mathrm{L}},\xi_\epsilon^{\mathrm{R}})$. If $\xi_0\in[\xi_\epsilon^{\mathrm{L}},\xi_\epsilon^{\mathrm{R}}]$ then we get $v(\xi_\epsilon^{\mathrm{L}})v(\xi_\epsilon^{\mathrm{R}})\leq 0$ which contradicts the previous lemma. Thus, $\xi_0<\xi_\epsilon^{\mathrm{L}}$. Choose now $\xi_*\in[\xi_0,\xi_\epsilon^{\mathrm{L}})$ such that $v'(\xi_*)>0$. Consider again the maximal interval $(\xi_*^{\mathrm{L}},\xi_*^{\mathrm{R}})$ such that $\xi_*\in(\xi_*^{\mathrm{L}},\xi_*^{\mathrm{R}})$ and $v'(\xi)>0$, $\xi\in(\xi_*^{\mathrm{L}},\xi_*^{\mathrm{R}})$. Notice that

$$|\xi_*^{\mathrm{R}}-\xi_0|<\epsilon.$$

As above, $\xi_0<\xi_*^{\mathrm{L}}$ by virtue of the previous lemma. Moreover, that lemma also implies that

$$v(\xi_*^{\mathrm{R}})>\frac{1}{\sqrt{6}}\frac{\gamma+1}{\gamma-1}.$$

Since this is valid for any ϵ, we have destroyed the continuity of v at ξ_0. $\square$

5 Explicit Estimates

Theorem 4.3 can be stated in a more explicit way. For this consider the following equation:

$$q_0^2 = G(\beta). \tag{5.1}$$

Note that the right hand side is bounded

$$G(\beta) < q_*^2 := \lim_{\tau\to\infty} G(\tau), \quad \forall\beta > 0.$$

For $q_0^2 < q_*^2$ we can solve (5.1) numerically. Let us define $\beta_0(q_0^2)$ as the largest positive solution of (5.1). The graph of $\beta_0(q_0^2)$ is shown in Figure 1. Then Theorem 4.3 has the following corollary:

Theorem 5.1 *If there exists a periodic solution of (2.10) with zero mean and* $q_0^2 := \max q(\xi)^2 < q_*^2 \approx 0.68$, *then*

$$\beta \le \beta_0(q_0^2). \tag{5.2}$$

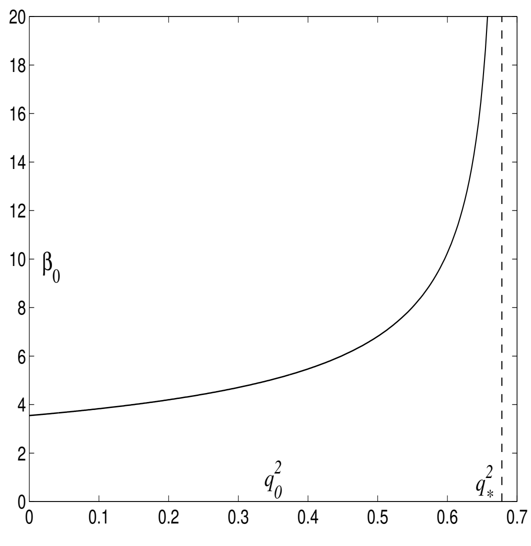

Fig. 1. Graph of $\beta_0(q_0^2)$

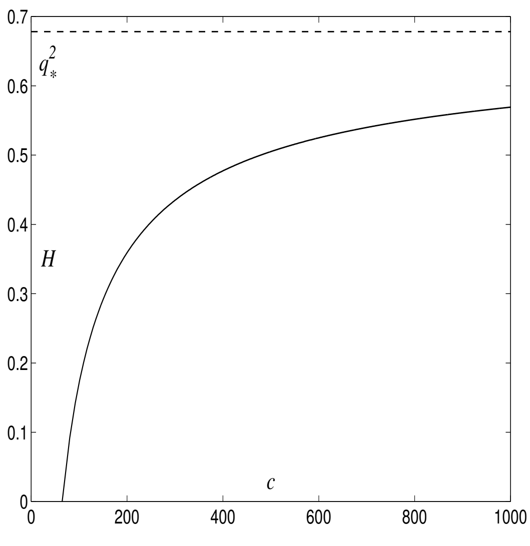

Fig. 2. Graph of $H(c)$

Let us reformulate this theorem in terms of equation (2.7). Recall that β was defined as a function of c in the following way: $\beta(c) := \sqrt{\frac{3}{4}b^2 + 1}$, where b is the positive solution of $b^3 + b = c$. Then equality (5.1) can be written as

$$q_0^2 = H(c) := G(\beta(c)). \tag{5.3}$$

From Theorem 4.3 we see that

$$\max_x q(x)^2 \geq H(c).$$

We solve (5.3) numerically for c and define $C(q_0^2)$ to be the largest positive solution. Let also

$$K(q_0^2) := \frac{\gamma(\beta(C(q_0^2))) + 1}{\gamma(\beta(C(q_0^2))) - 1} C(q_0^2). \tag{5.4}$$

The graphs of $C(q_0^2)$ and $K(q_0^2)$ are shown in Figures 3 and 4 respectively. Then using Theorem 3.2 and the definition (2.9) for q we get the following corollary:

Corollary 5.2 *If there exists a periodic solution of (2.7) with zero mean, then*

$$\frac{\max_x p(x)^2}{c^2} \geq H(c) \quad \textit{and} \quad |u(x)| \leq \frac{\gamma+1}{\gamma-1} c < \frac{7}{5} c, \quad \forall x. \tag{5.5}$$

In particular, if $\frac{\max_x p(x)^2}{c^2} \leq q_0^2 < q_*^2 \approx 0.68$, *then*

$$c \leq C(q_0^2) \quad \textit{and} \quad |u(x)| \leq K(q_0^2), \quad \forall x \in \mathbb{R}. \tag{5.6}$$

The graphs of $H(c)$, $C(q_0^2)$, *and* $K(q_0^2)$ *are shown in Figures 2, 3, and 4.*

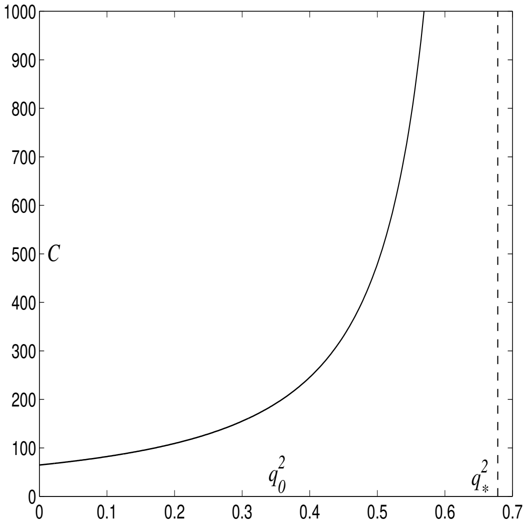

Fig. 3. Graph of $C(q_0^2)$

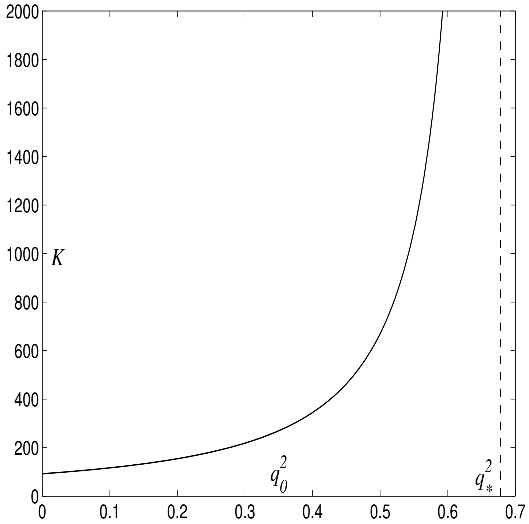

Fig. 4. Graph of $K(q_0^2)$

Consider (2.7) with $p = 0$:

$$u''' + u' + \frac{1}{2}u^2 = \frac{1}{2}c^2. \tag{5.7}$$

As a particular case of Corollary 5.2, we obtain an explicit estimate for the Michelson constant K_{M}. For this, define $C := C(0)$, $K := K(0)$. Then we have

Theorem 5.3 *Let u be any periodic solution of (5.7) with zero mean. Then*

$$c \leq C \approx 64.7 \tag{5.8}$$

and

$$|u(x)| \leq K_{\mathrm{M}} \leq K \approx 92.2, \quad \text{for all } x \in \mathbb{R}. \tag{5.9}$$

Choosing $c = \sqrt{F_{\max}}$, the theorem can be reformulated in terms of F. Namely, let u be a periodic solution of (1.9) and let $F_{\max}$ and $F_{\min}$ be defined as in (2.4). Then Theorem 4.3 has also the following consequence:

Corollary 5.4 *If there exists a periodic solution u of (1.9) with zero mean, then*

$$\frac{F_{\max} - F_{\min}}{F_{\max}} \geq H\left(\sqrt{F_{\max}}\right) \quad \text{and} \quad |u(x)| \leq \frac{\gamma+1}{\gamma-1}\sqrt{F_{\max}} < \frac{7}{5}c, \quad \forall x. \tag{5.10}$$

In particular, if $\frac{F_{\max}-F_{\min}}{F_{\max}} \leq q_0^2 < q_*^2 \approx 0.68$, then

$$F_{\max} \leq C(q_0^2)^2 \quad \text{and} \quad |u(x)| \leq K(q_0^2), \quad \forall x \in \mathbb{R}.$$

6 Universal Bounds on Averages of Solutions

In this section we apply the results obtained thus far to time averages of periodic solutions of the non-stationary Kuramoto-Sivashinsky equation (1.4). In order to define the average, we have to use a functional Lim which is an extension of the ordinary limit to the Banach space $B(0,\infty)$ of all bounded functions on $(0,\infty)$. Thanks to the Hahn-Banach theorem, there exists a functional Lim satisfying the following conditions:

(1) $|\mathrm{Lim}_{t\to\infty} f(t)| \leq \|f\|_\infty$.

(2) $\mathrm{Lim}_{t\to\infty} f(t) = \lim_{t\to\infty} f(t)$ if this limit exists in the classical case.

Let $u(x,t)$ be a solution of (1.4) satisfying the conditions (1.5), (1.6).

Denote by $\bar{u}(x)$ the time average

$$\bar{u}(x) := \mathrm{Lim}_{t\to\infty} \frac{1}{t} \int_0^t u(x,\tau)d\tau \tag{6.1}$$

As in [1] one can prove that there exists an invariant probability measure μ on $\mathcal{A}_L$ such that

$$\bar{u}(x) = \int u(x)\mu(du) \tag{6.2}$$

Henceforth we let an upper bar denote the average with respect to such an invariant probability measure μ. It is then easy to show using equation (1.4) that $\bar{u}$ satisfies

$$\bar{u}'''' + \bar{u}'' + \left(\frac{1}{2}\overline{u^2}\right)' = 0, \tag{6.3}$$

where

$$\overline{u^2}(x) = \int u^2(x)\mu(du).$$

Set $\tilde{u}(x) = u(x) - \bar{u}(x)$ and let $p(x)$ be the positive square root of

$$p^2(x) = \overline{\tilde{u}^2(x)} = \int \tilde{u}^2(x)\mu(du).$$

Since $\overline{u^2} = \bar{u}^2 + p^2$, we have

$$\bar{u}'''' + \bar{u}'' + \bar{u}\bar{u}' + \frac{1}{2}(p^2)' = 0. \tag{6.4}$$

Integrating this equation, we get

$$\bar{u}''' + \bar{u}' + \frac{1}{2}\bar{u}^2 + \frac{1}{2}p^2 = c_1.$$

Since u is periodic, $\bar{u}$ is periodic also. Thus, after integrating one more time over the period, we get

$$c_1 L \geq 0.$$

Denote $c = \sqrt{2c_1}$. So, we have

$$\bar{u}''' + \bar{u}' + \frac{1}{2}\bar{u}^2 + \frac{1}{2}p^2 = \frac{1}{2}c^2. \tag{6.5}$$

Thus, $\bar{u}(x)$ satisfies equation (2.7) with

$$c^2 = \frac{1}{L}\int_0^L \overline{|u(x)|^2}dx \quad \text{and} \quad p^2(x) = \overline{|u(x) - \bar{u}(x)|^2}. \tag{6.6}$$

So, Corollary 5.2 implies

Corollary 6.1 *Let $u(x,t)$ be any periodic solution of (1.4) with zero mean. Then*

(1)
$$|\bar{u}(x)| \leq \frac{\gamma+1}{\gamma-1}\left[\frac{1}{L}\int_0^L \overline{|u(x)|^2}dx\right]^{\frac{1}{2}}, \quad \forall x \in \mathbb{R}.$$

(2) Moreover, if

$$\overline{|u(x) - \bar{u}(x)|^2} \leq q_0^2 \frac{1}{L}\int_0^L \overline{|u(x)|^2}dx, \forall x \in \mathbb{R}, \tag{6.7}$$

for some $q_0 < q_*$, then

$$\frac{1}{L}\int_0^L \overline{|u(x)|^2}dx \leq C(q_0^2)^2 \quad \textit{and} \quad |\bar{u}(x)| \leq K(q_0^2), \quad \forall x \in \mathbb{R},$$

where $C(\cdot)$ and $K(\cdot)$ are the functions in the Corollary 5.2.

For example, in the particular case of (6.7) when

$$\overline{|u(x) - \bar{u}(x)|^2} < 0.5\frac{1}{L}\int_0^L \overline{|u(x)|^2}dx, \quad \forall x,$$

we have

$$\frac{1}{L}\int_0^L \overline{|u(x)|^2}dx \leq 235425 \quad \text{and} \quad |\bar{u}(x)| \leq 678.7, \quad \text{for all } x \in \mathbb{R}.$$

Acknowledgements

We would like to thank Prof. Michael S. Jolly for helpful discussions and constructive comments.This work was supported in part by the National Science Foundation Grant Number DMS-9706903.

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