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KURAMOTO-SIVASHINSKY DYNAMICS ON THE CENTER-UNSTABLE MANIFOLD*

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Abstract. This paper studies the dynamical behavior of solutions of the Kuramoto-Sivashinsky partial differential equation with periodic boundary conditions on a spatial interval $[0, h]$. The length h is the bifurcation parameter and reduction is made to a two-(complex-)dimensional system on a local center-unstable manifold near the second bifurcation point h_2 from the trivial solution. The resulting $O(2)$-equivariant system displays all the behavior found in high precision simulations of the partial differential equation near this bifurcation point. In particular, bifurcation sequences to stable traveling waves, unstable modulated traveling waves, and attracting heteroclinic cycles are reproduced qualitatively and quantitatively within 1% in the parameter range $h_2 \pm 20\%$. A clear understanding of the global dynamical behavior in this region is thus obtained.

Key words. $O(2)$-symmetry, invariant manifolds, heteroclinic cycles, modulated traveling waves

AMS(MOS) subject classifications. 34C30, 34C35, 34C40, 35B32, 58F14

1. Introduction. The Kuramoto-Sivashinsky and related model equations have been studied extensively in recent years, both in the context of inertial manifolds and finite-dimensional attractors and in numerical simulations of dynamical behavior (cf. Nicolaenko, Scheurer, and Temam [1985], [1986], Foias et al. [1985a], [1985b], Hyman and Nicolaenko [1985], [1986], Hyman, Nicolaenko, and Zaleski [1986], Kevrekidis, Nicolaenko, and Scovel [1988]). Equations of this type have been used to model flame fronts in combustion, directional solidification, and weak two-dimensional turbulence (cf. Novick-Cohen and Sivashinsky [1986], Novick-Cohen [1987], Foias, Nicolaenko, and Temam [1987]).

The specific equation we consider in this paper may be written as follows:

$$u_t + \alpha u_{xxxx} + u_{xx} + \tfrac{1}{2}(u_x)^2 = 0, \qquad 0 \leqq x \leqq h \tag{1.1a}$$

with periodic boundary conditions

$$u(0, t) = u(h, t), \qquad u_x(0, t) = u_x(h, t), \cdots. \tag{1.1b}$$

Here α is a positive viscosity parameter (it will subsequently be scaled out) and h is the domain length that will be varied as a bifurcation parameter. Equations (1.1a)-(1.1b) are invariant under translations $(x \to x + \beta)$ and reflections $(x \to -x)$ in the spatial variable. This symmetry is crucial in our subsequent analysis, as it is in those of Nicolaenko et al. However, the reader is cautioned that many of the theoretical studies (cf. Nicolaenko, Scheurer, and Temam [1985], [1986], Foias et al. [1988]) were done for the case of *even* periodic functions corresponding to Neumann boundary conditions $(u_x(0, t) = u_x(h, t) = 0)$. In particular, proof of boundedness of $\|u_x\|$, Hausdorff dimension estimates for attractors, and existence of inertial manifolds have been obtained *only* for this case. In contrast, much of the numerical work mentioned below

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was done for the fully periodic case of (1.1a)-(1.1b). Thus, while the general theory (Carr [1981], Henry [1981]) guarantees the existence of a local attracting center-unstable manifold, the fact that our calculations suggest that this manifold exists in a finite ($\mathcal{O}(1)$) neighborhood of the trivial solution and provides a good description of the global dynamics for moderate values of $h(\approx 4\pi\sqrt{\alpha})$ remains an interesting theoretical challenge (see § 6 below).

In the numerical work of Hyman and Nicolaenko [1985], Hyman, Nicolaenko, and Zaleski [1986], and Kevrekidis, Nicolaenko, and Scovel [1988], the authors varied h (or actually a nondimensional parameter involving α and h) and studied the evolution of attractors as h increases. For low values the trivial solution $u(x, t) \equiv 0$ is globally asymptotically stable and as h increases successive bifurcations to S^1-families of steady, spatially periodic solutions having one, two, three, $\cdots$ maxima occur. Only the first of these is stable. These solutions subsequently bifurcate, giving rise to traveling waves, solutions that are themselves stationary in a translating frame. Other, more complicated "pulse-like" solutions were observed, which Hyman et al. [1985], [1986] conjectured to be related to homoclinic or heteroclinic orbits to steady solutions. Modulated traveling waves (quasiperiodic solutions) were also observed. Figure 1 reproduces a part of the bifurcation diagram for low values of h due to Kevrekidis, Nicolaenko, and Scovel [1988]. We shall concentrate on the range of parameters explored in Fig. 1, but observe that the behavior in this range is repeated, with additional complications, in successively higher h-ranges. We feel that our methods may be useful in the understanding of these ranges as well.

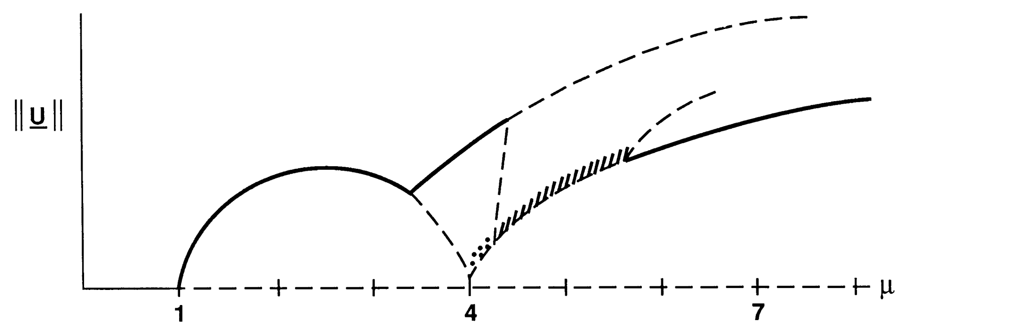

FIG. 1. *The bifurcation diagram for the Kuramoto-Sivashinsky equation obtained from numerical simulations (Nicolaenko, Scheurer, and Temam [1986]). Stable branches shown bold, attracting heteroclinic cycles hatched, nonattracting cycles dotted.*

The outline of this paper is as follows. In § 2 we use the Galerkin projection to recast equation (1.1) as an infinite or arbitrarily large system of first-order, complex, ordinary differential equations and we discuss the symmetries of this system. Section 3 briefly reviews the linear stability analysis of the trivial solution and branches bifurcating from it. Reduction to a dynamical system on a two-(complex-)dimensional center-unstable manifold near the second bifurcation point from the trivial solution is carried out in § 4. We compute a Taylor series approximation to this manifold to third order, which allows us to obtain a reduced vector field to fourth order. In § 5 we study the dynamics of this reduced system, both in third- and fourth-order truncations, using our earlier analysis of $O(2)$-equivariant systems (Armbruster, Guckenheimer, and Holmes [1988]). We show that the bifurcations and dynamical behavior of the fourth-order reduced system agree well, qualitatively and quantitatively, with simulations of the full partial differential equation (PDE) by Hyman et al. [1985], [1986] as well as simulations of 8- and 16-mode Galerkin truncations; cf. Kevrekidis, Nicolaenko, and Scovel [1988]. However, it is necessary to go to fourth order to obtain the full picture: the third-order truncation has one branch of solutions with qualitatively different properties. We conclude with a brief discussion in § 6.

At this point it is worth stressing that, while our computation of an attracting *local* center-unstable manifold and derivation of the reduced system near the second bifurcation point is rigorously justified, the most interesting behaviors we observe—heteroclinic cycles and modulated traveling waves—do not occur arbitrarily near to the bifurcation point. To show that such solutions do indeed lie in an $\mathcal{O}(1)$ (global) inertial manifold appears to be a substantial task; in particular it may be difficult to get a dimension estimate as low as the value (4) suggested by our formal extension of the local four-dimensional manifold. See the discussion in § 6.

For a general background in the methods of analysis used in §§ 3-5, see Guckenheimer and Holmes [1983]. For earlier studies of $O(2)$-equivariant vector fields of the type encountered here, see Dangelmayr [1986], Dangelmayr and Armbruster [1986], and the paper already cited (Armbruster, Guckenheimer, and Holmes [1988]). Proctor and Jones [1988] (cf. Jones and Proctor [1987]) have also performed very similar analyses of $O(2)$-equivariant systems occurring in Bénard convection studies.

2. Galerkin projection. Equation (1.1a) with boundary conditions (1.1b) defines a semiflow on the space $\mathcal{F}$ of $(2\pi/h)$-periodic functions. (We shall not concern ourselves with technical issues such as the precise functional analytic setting. See the references cited in § 1 for details.) A suitable basis for this space is provided by the Fourier coefficients $\phi_k(x)=\exp(i2\pi kx/h)_{k=-\infty}^{\infty}$; thus, we expand the dependent variable u as follows:

$$u(x,t)=\sum_{k=-\infty}^{\infty} a_k(t)\phi_k(x), \tag{2.1}$$

where the a_k are complex-valued modal coefficients. Reality of u implies that we require

$$a_{-k}=a_k^*, \tag{2.2}$$

where * denotes the complex conjugate. The inner product in $\mathcal{F}$ is given by

$$\langle f, g\rangle=\int_0^h f(x)g^*(x)\,dx. \tag{2.3}$$

Denoting the operator defined by (1.1a) as $N(u(x,t))$ and substituting (2.1), projection into the subspace spanned by the lth basis function yields

$$\langle N(\Sigma a_k\phi_k), \phi_l\rangle=0.$$

More explicitly, since $\phi_l^*=\phi_{-l}$ and $(\phi_k)_{xx}=-(2\pi k/h)^2\phi_k$, $(\phi_k)_{xxxx}=(2\pi k/h)^4\phi_k$, we have

$$\int_0^h\left\{\sum_k\left[\dot a_k\phi_k+\alpha\left(\frac{2\pi k}{h}\right)^4 a_k\phi_k-\left(\frac{2\pi k}{h}\right)^2 a_k\phi_k\right]+\frac{1}{2}\sum_k\frac{i2\pi k}{h}a_k\phi_k\sum_j\frac{i2\pi j}{h}a_j\phi_j\right\}\phi_{-l}\,dx=0. \tag{2.4}$$

Since $\int_0^h \phi_k\phi_{-l}\,dx=\delta_{kl}h$ (othogonality of Fourier modes), equation (2.4) yields

$$h\left\{\dot a_l+\alpha\left(\frac{2\pi l}{h}\right)^4 a_l-\left(\frac{2\pi l}{h}\right)^2 a_l-\frac{1}{2}\left(\frac{2\pi}{h}\right)^2\sum_j j(l-j)a_ja_{l-j}\right\}=0$$

or

$$\dot a_l=\left(\frac{2\pi l}{h}\right)^2\left(1-\alpha\left(\frac{2\pi l}{h}\right)^2\right)a_l+\frac{1}{2}\left(\frac{2\pi}{h}\right)^2\sum_j j(l-j)a_ja_{l-j}=0. \tag{2.5}$$

It is convenient to rescale time so that

$$\dot{a}_l = \frac{da_l}{dt} = \left(\frac{2\pi}{h}\right)^2 \frac{da_l}{d\tau} = \left(\frac{2\pi}{h}\right) a_l' \tag{2.6a}$$

and to define a nondimensional length

$$\mu = \frac{1}{\alpha}\left(\frac{h}{2\pi}\right)^2, \tag{2.6b}$$

which will be our bifurcation parameter. (Note that this differs from Nicolaenko's choice.) Using (2.6a), (2.6b), we may rewrite (2.5) as

$$a_l' = l^2\left(1 - \frac{l^2}{\mu}\right) a_l + \frac{1}{2}\sum_j j(l-j) a_j a_{l-j}. \tag{2.7}$$

Formally, (2.7) defines an infinite set of first-order ordinary differential equations (ODEs) as l (and j) range from $-\infty$ to $+\infty$, although generally we truncate and study the finite set obtained when $\max|l, j| = N < \infty$. The reality condition (2.2) implies that we need only consider the equations for a_l, $l \geqq 0$. Moreover, the "diagonal" linear term and particular form of the quadratic interaction leads to the decoupling of the equation for a_0, the spatial mean of u. While a_0 is driven by the other modes,

$$a_0' = -\frac{1}{2}\sum_j j^2 |a_j|^2 \tag{2.8}$$

(set $l = 0$ in (2.7)), and can become unbounded, its value does not enter the evolution equations for the coefficients a_l, $l \neq 0$. Thus we need not consider the a_0 equation explicitly (cf. Hyman et al. [1985], [1986], who remove the mean from their numerical computations to avoid overflow).

To give a feeling for the system, and for our future convenience in center manifold reduction, we display the equations obtained by truncating at $N = 4$:

$$\begin{aligned} a_1' &= \left(1 - \frac{1}{\mu}\right) a_1 - 2a_1^* a_2 - 6a_2^* a_3 - 12 a_3^* a_4, \qquad & a_2' &= 4\left(1 - \frac{4}{\mu}\right) a_2 + \frac{1}{2} a_1^2 - 3a_1^* a_3 - 8a_2^* a_4, \\ a_3' &= 9\left(1 - \frac{9}{\mu}\right) a_3 + 2a_1 a_2 - 4a_1^* a_4, & a_4' &= 16\left(1 - \frac{16}{\mu}\right) a_4 + 2a_2^2 + 3a_1 a_3. \end{aligned} \tag{2.9}$$

At this point it is important to observe how the physical symmetries of (1.1a), (1.1b) noted in § 1 appear in the system of ODEs (2.7) and in specific truncations such as (2.9). Spatial translation $x \to x + \beta$ corresponds to rotation in Fourier phase:

$$a_l \to \exp\left(\frac{i2\pi l\beta}{h}\right) a_l \tag{2.10a}$$

and reflection $x \to -x$ corresponds to complex conjugation:

$$a_l \to a_l^* = a_{-l}. \tag{2.10b}$$

It is easy to check that the vector field defined by (2.7) is invariant under the action of these group elements. In particular, the real subspace ($a_l = a_l^*$) is invariant for the flow and corresponds to the special case of Neumann boundary conditions considered in the theoretical studies referred to in § 1.

3. Linear analysis: bifurcation from $a_l \equiv 0$. Since the linear part of (2.7) is diagonal, each linearized modal equation is uncoupled and the stability and bifurcation analysis is trivial. Nonetheless, we sketch it as an introduction to the subtler analysis of the next section. There is a countable, increasing sequence of bifurcation values $\{\mu_m = m^2 | m = 1, 2, \cdots\}$ at each of which the linear operator $\alpha(\partial^4/\partial x^4) + (\partial^2/\partial x^2)$ of (1.1a) has a double zero eigenvalue and the linear part of the mth projected equation

$$a_m' = m^2\left(1 - \frac{m^2}{\mu}\right)a_m$$

vanishes. The multiplicity is a direct consequence of the $O(2)$-equivariance (Dangelmayr [1986]). The mth pair of eigenvalues of the trivial solution $a_l = 0$, $l > 0$ ($u(x, t) \equiv$ const.) are simply

$$\lambda_m = m^2\left(1 - \frac{m^2}{\mu}\right) \tag{3.1}$$

and thus, for $\mu \in (m^2, (m+1)^2)$, this solution has precisely $2m$ real, positive eigenvalues while all other eigenvalues are strictly negative (and also real).

To determine the nature of the bifurcation occurring at $\mu = \mu_m$ we seek a center manifold tangent to the eigenspace of the double zero eigenvalue: the two-dimensional subspace spanned by the complex mode a_m. Thus we seek a set of functions

$$a_l = h_l(a_m, a_m^*), \qquad l \neq m, -m, \; h_l = \mathcal{O}(|a_m|^2) \tag{3.2}$$

that are invariant for the flow. The reduced (two-dimensional) system on the center manifold is then given by

$$\begin{aligned} a_m' = {} & m^2\left(1 - \frac{m^2}{\mu}\right)a_m - 4m^2 a_{-m} h_{2m}(a_m, a_m^*) \\ & + \sum_{\substack{j=-\infty \\ j \neq m, 0, m, 2m}}^{\infty} j(m-j) f_j(a_m, a_m^*) f_{m-j}(a_m, a_m^*). \end{aligned} \tag{3.3}$$

Note that we have extracted the third-order terms from the sum.

To determine the branching behavior at the bifurcations of the trivial solution, it will suffice to truncate (3.3) at $\mathcal{O}(|a_m|^3)$, so that in studying the reduced equation for the mth mode, we need only compute the function h_{2m}. To do this, it suffices to balance the quadratic terms in the right-hand sides of the evolution equation for a_{2m}. (This is a special instance of the more general invariant manifold reduction outlined and applied in § 4.) Thus we set

$$(2m)^2\left(1 - \frac{(2m)^2}{\mu}\right)h_{2m} + \frac{1}{2}\sum_{j=-\infty}^{\infty} j(2m-j) h_j h_{2m-j} = 0, \tag{3.4}$$

where it is understood that $h_m = a_m$. Only the term $j = m$ in the sum contributes to the $\mathcal{O}(|a_m|^2)$ piece of h_{2m}; thus (3.4) leads to

$$(2m)^2\left(1 - \frac{(2m)^2}{\mu}\right)h_{2m} + \frac{1}{2} m^2 a_m^2 + \mathcal{O}(|a_m|^3) = 0,$$

or

$$h_{2m} = -\frac{1}{8}\left(1 - \frac{(2m)^2}{\mu}\right)^{-1} a_m^2 + \mathcal{O}(|a_m|^3).$$

Finally, setting $\mu = m^2$ (the bifurcation value), we obtain

$$h_{2m} = \tfrac{1}{24} a_m^2 + \mathcal{O}(|a_m|^3). \tag{3.5}$$

Substitution into (3.3) yields the reduced system

$$a_m' = m^2\left(1-\frac{m^2}{\mu}\right)a_m - \frac{m^2}{6}|a_m|^2 a_m + \mathcal{O}(|a_m|^4), \tag{3.6}$$

which is a valid approximation for $\mu \approx m^2$. Writing $\mu = m^2 + \varepsilon$ and expressing $a_m = r_m e^{i\theta m}$ in polar coordinates, we obtain the truncated amplitude and phase equations:

$$r_m' = \varepsilon r_m - \frac{m^2}{6} r_m^3 + \cdots, \qquad \theta_m' = 0. \tag{3.7}$$

It is clear that, for $\varepsilon < 0$ ($\mu < m^2$) the trivial solution $r_m = 0$ is the only fixed point locally while for $\varepsilon > 0$ ($\mu > m^2$) there is a circle of fixed points given by $r_m = \sqrt{6\varepsilon}/m$. The fact that such a circle of degenerate equilibria appears is due to the $O(2)$-equivariance. The branch of bifurcating solutions retains the symmetry induced by reflection on the x-axis. Thus the center manifold lies in the subspace defined by $a_l = a_l^*$ and θ_m' in (3.7) is identically zero, and not just zero to second order. Thus, if $(\hat{r}_m, \hat{\theta}_m)$ is an equilibrium, then so is $(\hat{r}_m, \hat{\theta}_m + \beta)$ for any β.

We conclude that we have an infinite sequence of supercritical bifurcations to invariant circles filled with fixed points. Although only the first such bifurcation yields an asymptotically stable circle (for $\mu = 1 + \varepsilon$, $\varepsilon > 0$ and small), the succession of nontrivial branches of solutions thus created acts as a kind of backbone on which the dynamical behavior of the Kuramoto-Sivashinsky equation is built. In the remainder of this paper we shall concentrate on secondary bifurcations and dynamical behavior occurring near the second bifurcation point at $\mu = 4$.

4. A center-unstable manifold near $\mu = 4$. We recall some general results on invariant manifolds for ordinary and partial differential equations (cf. Kelley [1967], Carr [1981], Henry [1981], Guckenheimer and Holmes [1983]). Consider an evolution equation of the following form:

$$\begin{aligned} \dot{x} &= A(\nu)x + f(x, y), \\ \dot{y} &= B(\nu)y + g(x, y), \qquad (x, y) \in X \times Y, \quad \nu \in \mathbb{R}, \\ \dot{\nu} &= 0, \end{aligned} \tag{4.1}$$

where A and B are linear operators depending smoothly (C^∞) on the (real) parameter ν; A having finite-dimensional domain and range; and B, defined on Y, being possibly infinite-dimensional. The third (trivial) evolution equation for ν is added for later convenience. We assume that the spectrum of $A(0)$ lies entirely in the nonnegative half plane (including the imaginary axis), while all eigenvalues of $B(0)$ have strictly negative real parts and f and g are strictly nonlinear ($\mathcal{O}(\|x\|^2, \|y\|^2)$) C^∞ functions. Thus $\nu = 0$ is a bifurcation point. In the absence of nonlinear terms, X is the center-unstable eigenspace and Y the stable eigenspace. The basic theorems state that this splitting persists, at least locally, for the nonlinear problem. Specifically, there is a $C^\gamma(\gamma < \infty)$ center-unstable, invariant manifold $\mathcal{M}$ tangent to $X \times \mathbb{R}$ at $(x, y, \nu) = (0, 0, 0)$, which can be represented locally as a graph $y = h(x, \nu)$.

Normally we seek a center manifold alone, by splitting out both stable and unstable directions, but for our application it will be important to retain both marginal *and* linearly unstable modes in the reduced model. Hence we seek a center-unstable manifold.

To explore consequences of the existence of $\mathcal{M}$ in examples we must approximate the function $h(x, \nu)$ describing it. This is done by substitution into the second component of (4.1) to obtain

$$\dot{y} = D_x h\dot{x} + D_\nu h\dot{\nu} = Bh + g(x, h),$$

or, using $\dot{\nu} = 0$ and the first component for $\dot{x}$:

$$D_x h(x, \nu)[A(\nu)x + f(x, h(x, \nu))] = B(\nu)h(x, \nu) + g(x, h(x, \nu)). \tag{4.2}$$

The function h is approximated by seeking a power series description and balancing terms of corresponding orders in the functional equation (4.2). Once h is available, we substitute into the first component of (4.1) to obtain a (finite-, low-) dimensional system describing the projection of the flow restricted to the center-unstable manifold onto the corresponding eigenspace:

$$\dot{x} = A(\nu)x + f(x, h(x, \nu)). \tag{4.3}$$

This is our reduced system, the vector field of which can (in principle) be computed to any finite order in x, ν.

Rather than continue with the general theory, we turn to the application at hand. We are interested in behavior for $\mu \approx 4$ $(m = 2)$, so we define a new parameter:

$$\nu = \frac{1}{4} - \frac{1}{\mu}. \tag{4.4}$$

At the bifurcation point, $\nu = 0$ $(\mu = 4)$, the center-unstable eigenspace is spanned by $x = (a_1, a_2)$ and the stable eigenspace by $y = (a_3, a_4, \cdots)$. Thus the center unstable manifold is given by an infinite set of functions of the following form:

$$a_l = h_l(a_1, a_1^*, a_2, a_2^*, \nu), \qquad l = 3, 4, 5, \cdots. \tag{4.5}$$

Fortunately we only need explicitly compute the leading terms of two of these functions $(l = 3, 4)$ for the order of accuracy we require, as we now show.

Our reduced system takes the following form (cf. (2.7) and (4.4)):

$$\begin{aligned} a_1' &= (\tfrac{3}{4} + \nu)a_1 - 2a_1^* a_2 - 6a_2^* h_3 - 12h_3^* h_4 - \sum_{j=4}^{\infty} j(j+1)h_j^* h_{j+1}, \\ a_2' &= 16\nu a_2 + \tfrac{1}{2}a_1^2 - 3a_1^* h_3 - 8a_2^* h_4 - \sum_{j=3}^{\infty} j(j+2)h_j^* h_{j+2}. \end{aligned} \tag{4.6}$$

We seek an approximation of (4.6) that is accurate to $\mathcal{O}(\nu|a_j|^3)$ and $\mathcal{O}(|a_j|^4)$, respectively. If we can show that $h_l = \mathcal{O}(|a_j|^3)$ for $l \geqq 5$, then the terms contained in the infinite sums of (4.6) do not enter at this order and we need only compute the leading few terms of h_3 and h_4. To see that this is the case, we consider the lowest-order terms in a_1, a_2 appearing in the components of (4.2) for $l \geqq 5$.

We have

$$\begin{aligned} &\sum_{k=1}^{2} \left(\frac{\partial h_l}{\partial a_k}[a_k'] + \frac{\partial h_l}{\partial a_k^*}[a_k']^* \right) \\ &\quad = l^2\left(1 - \frac{l^2}{4} + l^2\nu\right)h_l + \frac{1}{2}\sum_{j=1}^{l-1} j(l-j)a_j a_{l-j} - \sum_{j=1}^{\infty} j(j+l)a_j^* a_{j+l}, \end{aligned} \tag{4.7}$$

where we have rewritten the nonlinear term to include only positive indices, using (2.2). We know a priori, that

$$a_3 = h_3 = \mathcal{O}(|a_j|^2), \qquad a_4 = h_4 = \mathcal{O}(|a_j|^2); \tag{4.8}$$

thus, for $l=5$, from (4.7) we obtain

$$h_5=\mathcal{O}(|a_1h_4|,|a_2h_3|)+\mathcal{O}(|a_1h_6|,|a_2h_7|,\cdots)=\mathcal{O}(|a_j|^3). \tag{4.9}$$

(Note that, since a_1' and a_2' contain terms linear in a_1, a_2 (4.6), the left-hand side and first term on the right-hand side of (4.7) are of the same orders in a_1, a_2.) Proceeding inductively, we find that

$$h_6=\mathcal{O}(|a_1h_5|,|a_2h_4|,|h_3|^2)+\mathcal{O}(|a_1h_7|,|a_2h_8|,\cdots)=\mathcal{O}(|a_j|^3)+\cdots,$$
$$h_7=\mathcal{O}(|a_1h_6|,|a_2h_5|,|a_3h_4|)+\mathcal{O}(|a_1h_8|,|a_2h_9|,\cdots)=\mathcal{O}(|a_j|^4)+\cdots,$$

and, in general, that

$$h_l=\mathcal{O}(|a_j|^{[(l+1)/2]}), \tag{4.10}$$

where $[\gamma]$ denotes the integer part of γ. Thus $h_l=\mathcal{O}(|a_j|^3)$ for $l\geqq 5$, as claimed.

We return to computation of the Taylor series for h_3 and h_4. Writing equation (4.2) (or (4.7)) for $l=1$, 2, we obtain

$$\begin{bmatrix}\frac{\partial h_3}{\partial a_1} & \frac{\partial h_3}{\partial a_1^*} & \frac{\partial h_3}{\partial a_2} & \frac{\partial h_3}{\partial a_2^*}\\ \frac{\partial h_4}{\partial a_1} & \frac{\partial h_4}{\partial a_1^*} & \frac{\partial h_4}{\partial a_2} & \frac{\partial h_4}{\partial a_2^*}\end{bmatrix}\begin{bmatrix}(\frac{3}{4}+\nu)a_1-2a_1^*a_2-\cdots\\ (\frac{3}{4}+\nu)a_1^*-2a_1a_2^*-\cdots\\ 16\nu a_2+\frac{1}{2}a_1^2-\cdots\\ 16\nu a_2^*+\frac{1}{2}a_1^{*2}-\cdots\end{bmatrix}$$
$$=\begin{bmatrix}(\frac{45}{4}+81\nu)h_3+2a_1a_2-4a_1^*h_4-\cdots\\ (-48+256\nu)h_4+2a_2^2+3a_1h_3-\cdots\end{bmatrix}, \tag{4.11}$$

where the terms omitted are of $\mathcal{O}(|a_j|^4)$. The reader can check that the following approximation satisfies (4.11) to $\mathcal{O}(\nu|a_j|^2)$ and $\mathcal{O}(|a_j|^3)$:

$$\begin{aligned}h_3&=\tfrac{1}{6}(1+\tfrac{16}{3}\nu)a_1a_2-\tfrac{1}{162}a_1^3+\tfrac{1}{72}a_1^*a_2^2,\\ h_4&=\tfrac{1}{24}(1+\tfrac{14}{3}\nu)a_2^2+\tfrac{1}{108}a_1^2a_2.\end{aligned} \tag{4.12}$$

Note also that the h_l satisfy

$$h_l(e^{i\beta}a_1,e^{-i\beta}a_1^*,e^{i2\beta}a_2,e^{-i2\beta}a_2^*)=e^{il\beta}h_l(a_1,a_1^*,a_2,a_2^*)\quad\text{and}\quad (h_l)^*=h_{-l}, \tag{4.13}$$

so that the approximation to the center-unstable manifold is also invariant under the group action.

Finally, we substitute the functions h_3, h_4, of (4.12) into equation (4.6) to obtain the reduced system corresponding to (4.3):

$$\begin{aligned}a_1'&=(\tfrac{3}{4}+\nu)a_1-2a_1^*a_2-(1+\tfrac{16}{3}\nu)|a_2|^2a_1+\tfrac{1}{27}a_1^3a_2^*-\tfrac{1}{6}a_1^*a_2|a_2|^2+\cdots,\\ a_2'&=16\nu a_2+\tfrac{1}{2}a_1^2-\tfrac{1}{2}(1+\tfrac{16}{3}\nu)|a_1|^2a_2-\tfrac{1}{3}(1+\tfrac{14}{3}\nu)|a_2|^2a_2\\ &\quad+\tfrac{1}{54}a_1^2|a_1|^2-\tfrac{1}{24}a_1^{*2}a_2^2-\tfrac{2}{27}a_1^2|a_2|^2+\cdots.\end{aligned} \tag{4.14}$$

The errors in (4.14) are of $\mathcal{O}(\nu^2|a_j|^3)$ and $\mathcal{O}(|a_j|^5)$, respectively. As expected, (4.14) is also equivariant with respect to the group actions (2.10a), (2.10b).

5. Dynamics on the center-unstable manifold. In this section we analyze the reduced system (4.14) obtained in the preceding section. We start with a truncation at third order and apply the general results of Armbruster, Guckenheimer, and Holmes [1988] for such systems directly. We find that the resulting dynamical behavior and bifurcation sequences differ qualitatively from those for the full PDE. However, inclusion of the fourth-order terms yields behavior that agrees very well. The fact that the resulting vector field is essentially three-dimensional and is a perturbation of a completely integrable system, permits us to give a complete description of the global dynamics of this system.

5.1. Third-order truncation. We study the following system:

$$
\begin{aligned}
a_1' &= (\tfrac{3}{4}+\nu)a_1 - 2a_1^* a_2 - |a_2|^2 a_1, \\
a_2' &= 16\nu a_2 + \tfrac{1}{2}a_1^2 - \tfrac{1}{2}|a_1|^2 a_2 - \tfrac{1}{3}|a_2|^2 a_2.
\end{aligned} \tag{5.1}
$$

It is convenient to rescale and change the sign of a_2 by letting

$$\tilde{a}_2 = -2a_2, \qquad \tilde{a}_1 = a_1$$

and dropping the tildes to obtain

$$
\begin{aligned}
a_1' &= (\tfrac{3}{4}+\nu)a_1 + a_2^* a_2 - \tfrac{1}{4}|a_2|^2 a_1, \\
a_2' &= 16\nu a_2 - a_1^2 - \tfrac{1}{2}|a_1|^2 a_2 - \tfrac{1}{12}|a_2|^2 a_2,
\end{aligned} \tag{5.2}
$$

so that the coefficients of the quadratic terms are +1 and −1, respectively, as in the "−case" normal form studied by Armbruster, Guckenheimer, and Holmes [1988], henceforth referred to as [A].

It is now possible to read off a number of results directly by reference to that paper. In particular, we obtain the bifurcation diagram of Fig. 2(a). The most striking feature of this diagram is the existence of an open set of parameter values in which asymptotically stable heteroclinic cycles exist. As the analysis of [A] shows, the occurrence of such cycles in $\mathcal{O}(2)$-equivariant vectorfields is a structurally stable

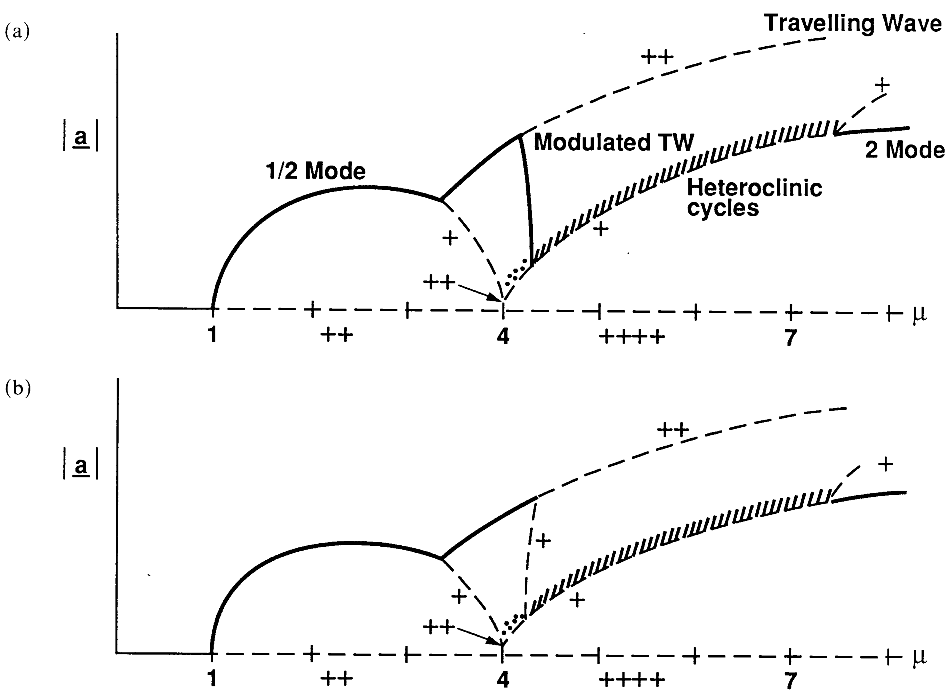

FIG. 2. *Bifurcation diagrams for the reduced system* (4.14): (a) *truncated at third-order*; (b) *including fourth-order terms. Stable branches shown bold, attracting heteroclinic cycles hatched, nonattracting cycles dotted*; + *indicates number of positive* (*unstable*) *eigenvalues.*

phenomenon (cf. Guckenheimer and Holmes [1988] for a second example of symmetry induced cycles). The asymptotic stability of these cycles changes in a global bifurcation in which the branch of modulated traveling waves merges with the cycle, but, unlike the generic, nonsymmetric case (Guckenheimer and Holmes [1983, Chap. 6.1]) the heteroclinic cycle persists on both sides of the bifurcation point.

For convenience, we summarize the relevant bifurcation equations from [A]. In [A] the normal form in the "−case" is given by

$$a_1' = a_1^* a_2 + a_1(\mu_1 + e_{11}|a_1|^2 + e_{12}|a_2|^2),\qquad a_2' = -a_1^2 + a_2(\mu_2 + e_{21}|a_1|^2 + e_{22}|a_2|^2). \tag{5.3}$$

Since we are interested only in small values of ν in (5.2), $\mu_1 = \frac{3}{4} + \nu$ will remain positive, so that the only bifurcation from the trivial solution that will concern us is that occurring on the line

$$\mu_2 = 16\nu = 0, \tag{5.4}$$

in which a circle of equilibria $|a_1| = 0$, $|a_2| = \sqrt{-\mu_2/e_{22}} = 8\sqrt{3\nu}$ are created, existing for $\nu > 0$. This is just the bifurcation at $\mu = 4$ of § 3.

There are several kinds of secondary bifurcations from this pure a_2 mode. Mixed modes for which a_1, $a_2 \neq 0$ bifurcate on the curves

$$\mu_1 - \mu_2 e_{12}/e_{22} \pm \sqrt{-\mu_2/e_{22}} = 0, \tag{5.5}$$

[A, eq. (2.12)] or, by comparing (5.2) and (5.3), for

$$\tfrac{3}{4} - 47\nu \pm 8\sqrt{3\nu} = 0.$$

The two roots are $\nu = .00218296\cdots$ and $\nu = .1166491\cdots$, giving values of $\mu = 4.03524$ and $\mu = 7.49901$ in terms of the original bifurcation parameter. Nicolaenko's values are 4.0350 and 5.639, respectively. The discrepancy in the higher values is not unexpected, since μ is no longer close to 4.

The mixed mode bifurcates to traveling waves on the curves defined by [A, eqs. (2.13)-(2.14)]:

$$[\mu_2(2e_{11} + e_{12}) - \mu_1(2e_{21} + e_{22})]^2 = -(2\mu_1 + \mu_2)(4e_{11} + 2e_{12} + 2e_{21} + e_{22}) > 0 \tag{5.6}$$

or

$$\left(\tfrac{13}{16} - \tfrac{35}{12}\nu\right)^2 = \left(\tfrac{3}{2} + 18\nu\right)\tfrac{19}{12} > 0.$$

The appropriate root is $\nu = -0.050925\cdots$ for a bifurcation value $\mu = 3.32308$. Kevrekidis, Nicolaenko, and Scovel [1988] have obtained 3.2513.

The condition on the eigenvalues of the pure mode in the range $\mu \in (4.03524, 7.49901)$ corresponding to a change in stability type of the saddle points in the heteroclinic cycle is [A, eq. (5.11)]:

$$\mu_1 = \mu_2 e_{12}/e_{22} \quad \text{or} \quad \left(\tfrac{3}{4} + \nu\right) = 16\nu\left(\tfrac{12}{4}\right) \tag{5.7}$$

giving $\nu = .0159574\cdots$ or $\mu = 4.2727\bar{3}$, compared with Kevrekidis, Nicolaenko, and Scovel's estimate of (4.20).

The computations in [A] for Hopf bifurcation to modulated traveling waves from the branch of traveling waves (which exists for $\mu > 3.32308\cdots$ in the present case) are done in the limit $\mu_j = \mathcal{O}(\varepsilon^2)$, $a_j = \mathcal{O}(\varepsilon)$ as $\varepsilon \to 0$ (cf. [A eq. (2.6), eq. (5.8)]. These yield the approximate value

$$\mu_2 = \mu_1\left[1 + \frac{9(e_{22} - e_{12})}{(4e_{11} + 2e_{12} + 2e_{21} + e_{22}) - 3(e_{22} - e_{12})}\right] + \mathcal{O}(\varepsilon) \tag{5.8}$$

(cf. [A, (5.10)]). In terms of ν, we have

$$16\nu \approx \left(\frac{3}{4}+\nu\right)\left(1+\frac{9(\frac{1}{6})}{-\frac{19}{12}-\frac{3}{6}}\right)$$

or $\nu \approx 21/1572$, yielding $\mu \approx 4.22581 \cdots$.

Referring to Theorems 5.2 and 5.5 of [A], we conclude that, since

$$\left[1+\frac{9(e_{22}-e_{12})}{(4e_{11}+2e_{12}+2e_{21}+e_{22})-3(e_{22}-e_{12})}\right]=\frac{7}{25}<\frac{1}{3}=\frac{e_{22}}{e_{12}}, \tag{5.9}$$

the modulated traveling waves on the branch connecting the Hopf bifurcation point $\mu \approx 4.22581$ to the heteroclinic stability change at $\mu \approx 4.27273$ are asymptotically stable.

The preceding calculation is unlikely to be sufficiently accurate, however, since $\mu_1=\frac{3}{4}+\nu$ is not small for $\nu \approx 0$. However, a direct calculation of the Hopf bifurcation point from the exact solution for traveling waves in (5.1) can easily be done using computer algebra.

These computations yield a value of $\nu=0.11682$ or $\mu=4.19608$. This value is still lower than the heteroclinic bifurcation at $\mu=4.27273$ (it is even lower than the approximate value of 4.22581 found above) and so the modulated waves are still found to be asymptotically stable at the level of approximation of (5.1). The corresponding Hopf and heteroclinic bifurcation values of Nicolaenko are 4.3498 and 4.20, respectively. Figure 2(a) summarizes the results obtained so far in the form of a bifurcation diagram. Also see Table 1 below.

5.2. Fourth-order truncation. To obtain a more accurate description of the reduced system we now include the terms of $\mathcal{O}(\nu|a_j|^3)$ and $\mathcal{O}(|a_j|^4)$ in (4.14). Again we rescale by letting $\tilde{a}_2=-2a_2$, $\tilde{a}_1=a_1$ and drop the tildes, so that the cubic and lower-order terms are in the standard normal form of [A]:

$$\begin{aligned} a_1' &= (\tfrac{3}{4}+\nu)a_1+a_1^*a_2-(\tfrac{1}{4}+\tfrac{4}{3}\nu)|a_2|^2a_1-\tfrac{1}{54}a_1^3a_2^*+\tfrac{1}{48}a_1^*a_2|a_2|^2,\\ a_2' &= 16\nu a_2-a_1^2-(\tfrac{1}{2}+\tfrac{8}{3}\nu)|a_1|^2a_2-(\tfrac{1}{12}+\tfrac{7}{18}\nu)|a_2|^2a_2-\tfrac{1}{27}a_1^2|a_1|^2+\tfrac{1}{48}a_1^*a_2^2+\tfrac{1}{27}a_1^2|a_2|^2. \end{aligned} \tag{5.10}$$

While the bifurcation to the pure mode still occurs at $\nu=0$, its amplitude is now given by

$$|a_2|^2=\frac{16\nu}{(\frac{1}{12}+\frac{7}{18}\nu)} \quad \text{or} \quad |a_2|=8\sqrt{\frac{3\nu}{(1+\frac{14}{3}\nu)}} \stackrel{\text{def}}{=} A. \tag{5.11}$$

This leads to correction terms in the expressions for the eigenvalues of the pure mode appearing in the bifurcation equations corresponding to (5.5) and (5.7) above. The eigenvalues are

$$\left(\frac{3}{4}+\nu\right)-\left(\frac{1}{4}+\frac{4\nu}{3}\right)A^2 \pm A\left(1+\frac{A^2}{48}\right), \tag{5.12}$$

with A given in (5.11). Bifurcation to mixed modes occurs when one of these eigenvalues is zero: the resulting equations yield the values $\nu=0.002170\cdots$ and $0.114834\cdots$ giving $\mu=4.03503\cdots$ and $7.39832\cdots$.

Note that the correction to the values from the third-order truncation are quite small (.005% and 1.34%) and that the value 7.39832 is still much too high. We certainly need additional modes to obtain a good approximation of this point, since it is so far from $\mu=4$.

When the leading parts of (5.12) cancel, giving equal and opposite eigenvalues, the heteroclinic bifurcation occurs. This gives

$$\left(\frac{3}{4}+\nu\right)=16\nu\left(\frac{1}{4}+\frac{4\nu}{3}\right)\Big/\left(\frac{1}{12}+\frac{7\nu}{18}\right) \tag{5.13}$$

or $\nu=.015799\cdots$ for a value of $\mu=4.26984\cdots$. The correction to the third-order value $\mu=4.27273$ is only .07%.

To compute the traveling wave and modulated traveling wave bifurcation points we write (5.10) in polar form, letting $a_j=r_j e^{i\theta_j}$ and defining $\phi_j=2\theta_1-\theta_2$, as in [A]:

$$\begin{aligned}
\dot r_1 &= \left(\frac{3}{4}+\nu\right)r_1+r_1r_2\left(1-\frac{r_1^2}{54}+\frac{r_2^2}{48}\right)\cos\phi-\left(\frac{1}{4}+\frac{4\nu}{3}\right)r_1r_2^2,\\
\dot r_2 &= 16\nu r_2-r_1^2\left(1+\frac{r_1^4}{27}-\frac{25r_2^2}{432}\right)\cos\phi-\left(\frac{1}{2}+\frac{8\nu}{3}\right)r_1^2r_2-\left(\frac{1}{12}+\frac{7\nu}{18}\right)r_2^3,\\
\dot\phi &= -\left(2r_2-\frac{r_1^2}{r_2}+\frac{r_2^3}{24}+\frac{23r_1^2r_2}{432}-\frac{r_1^4}{27r_2}\right)\sin\phi.
\end{aligned} \tag{5.14}$$

The traveling wave bifurcates from the mixed mode on $\phi=\pi$ when the factor mutliplying $\sin\phi$ in the third equation vanishes and the right-hand sides of the first two equations vanish with $\cos\phi=-1$. The first condition yields

$$r_1=\sqrt{2}r_2 \tag{5.15}$$

(just as in the cubic case of [A, eq. (2.13b)]), while the remaining two give fourth-order polynomials for r_2. Requiring that the appropriate (small, positive) roots of these coincide, we have an equation for ν, the bifurcation value. The procedure may easily be implemented using MACSYMA to obtain $\nu=-.06079$ or $\mu=3.21759$, a change of -3.2% from the third-order value of 3.32308, and considerably closer to Kevrekidis, Nicolaenko, and Scovel's value.

For Hopf bifurcation from the traveling wave we first solve the first equation (right-hand side $=0$) of (5.14) to obtain an expression $\cos\phi=f(r_1,r_2)$. Using (5.15), which must be satisfied for traveling waves if $\phi\neq 0$, π, we then substitute $\cos\phi=f(\sqrt{2}r_2,r_2)$ into the second equation to obtain a fifth-order polynomial of the form $r_2(\alpha_0r_2^4+\alpha_1r_2^2+\alpha_3)=0$. We pick the appropriate (small, positive) root. Assembling this information we have $(\hat r_1,\hat r_2,\hat\phi)$: the traveling wave fixed point, in terms of ν. We then compute the characteristic polynomial of the Jacobian matrix of (5.14) and obtain the condition for a pair of purely imaginary eigenvalues, also with the aid of MACSYMA. Substitution of $(\hat r_1,\hat r_2,\hat\phi)$ into this condition yields an equation for ν from which we find the bifurcation value. Taylor expansions to $\mathcal{O}(|\nu|^4)$ are used at various places. We finally obtain $\nu=0.019335\cdots$ or $\mu=4.33530\cdots$. This value represents a $+2.6\%$ change from the third-order value of 4.22581 and the bifurcation point now lies *above* the heteroclinic bifurcation point $4.26984\cdots$, in agreement with the computations of Kevrekidis, Nicolaenko, and Scovel [1988].

We summarize the various results of this section in Table 1 and on Fig. 2. Figures 2(a) and 2(b) show the third-order and fourth-order bifurcation diagrams, respectively, which should be compared with the diagram found numerically by Nicolaenko reproduced in Fig. 1. In connecting the heteroclinic and modulated traveling wave bifurcation points μ_h, μ_m, we have made the *conjecture* that the branch of modulated traveling waves is unique and well behaved in the sense that there is precisely one such wave for each $\mu\in(\mu_h,\mu_m)$. Armbruster, Guckenheimer, and Holmes [1988] have proved

TABLE 1
Bifurcation points.

Bifurcation point (μ)	Third-order truncation [exact calc]	Fourth-order truncation	Kevrekidis, Nicolaenko, and Scovel [1988]
Mode 2 from trivial	4	4	4
Mixed 1/2 mode from 2:			
#1	4.03524	4.03503	4.035 ⋯
#2	7.49901	7.39832	5.639 ⋯
Heteroclinic bifurcation	4.27273	4.26984	4.20 ?
Traveling wave from 1/2	3.32308	3.21759	3.2513
Modulated from traveling wave	4.22581 [4.19608]	4.33530	4.3498

this for the limiting case $\varepsilon \to 0$, corresponding to $|a_1|, |a_2| = \mathcal{O}(\varepsilon)$ small, but there is little hope of a general proof along those lines, since it relies on perturbation of the $\varepsilon = 0$ integrable limit system. However, we observe that the monotonicity of the modulated wave branch following from this conjecture agrees qualitatively with the computations of Kevrekidis, Nicolaenko, and Scovel.

To illustrate the results of this section, in Fig. 3 we display numerical integrations of the four-mode Galerkin truncation (2.9) and of the two-mode reduced system (4.14) on the center unstable manifold. As predicted, both systems display coexisting attractors at the parameter value selected, $\mu = 4.3$. Figure 3 indicates how well the two-mode model captures the quantitative as well as qualitative aspects of the larger system. For additional pictures of solutions of the normal form equations, see [A].

6. Discussion. We have performed a reduction of the Kuramoto-Sivashinsky partial differential equation (1.1a) with periodic boundary conditions (1.1b) to a center-unstable manifold for parameter values near that at which the second Fourier mode bifurcates. The local invariant manifold is a graph over the eigenspace of the first two modes, approximation of which as a Taylor series including terms of order three yields a reduced vector field accurate to fourth-order. At this level the correction terms involve only the third and fourth modes. Invariance under spatial translation and reflection for the PDE is reflected in $O(2)$-equivariance of this vector field. Building on earlier work of Armbruster, Guckenheimer, and Holmes [1988], [A], a fairly complete analysis of the four-dimensional reduced system is possible: the results agree remarkably well both qualitatively and quantitatively with detailed numerical simulations of the PDE using spectral methods.

One of the features of $O(2)$ symmetric vector fields that we have elucidated (within the space of symmetric perturbations) in [A] is the occurrence of structurally stable heteroclinic cycles. Such cycles had been observed by Nicolaenko [1985], [1986] in numerical simulation, but the reasons why they might be expected to occur have not been fully understood previously. However, shortly after this paper was submitted we received the preprint of Kevrekidis, Nicolaenko, and Scovel [1988] in which the authors use the $O(2)$-equivariance in the full (nonreduced, infinite-dimensional) problem to argue that heteroclinic cycles are likely to occur. We still do not know of a complete proof of the existence of such cycles and the associated modulated traveling waves in any system other than the two complex-dimensional, reduced normal form considered here and in [A]. Nonetheless, we feel that the analysis presented here and by Kevrekidis, Nicolaenko, and Scovel [1988] should alleviate any doubts that the heteroclinic cycles

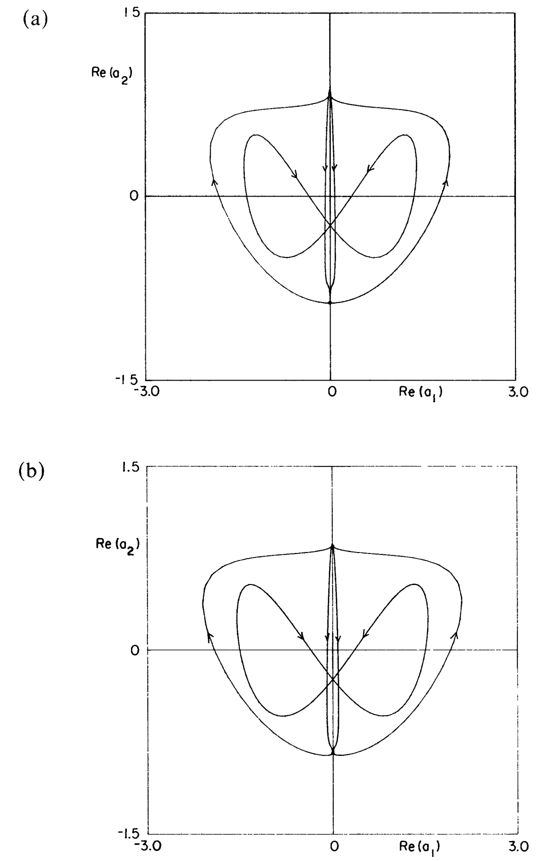

FIG. 3. *Numerical integrations of Galerkin projection and reduced system projected onto* Re (a_1) − Re (a_2) *plane.* (a) *Four-mode Galerkin projection* (2.9); (b) *two-mode fourth-order reduced system* (4.14). *Parameter value* $\mu = 4.3$. *Note coexistence of attracting heteroclinic cycle and traveling wave solution in both systems. Note the Lissajous figure characteristic of a* 1 : 2 *resonance.*

observed by those authors are numerical artifacts. Their presence is fully consistent with the dynamics to be expected from an $O(2)$-equivariant vector field.

The analysis of this paper is formal. While the existence of a local center-unstable manifold is guaranteed (cf. Carr [1981], Henry [1981]), the use of a locally valid approximation at $\mathcal{O}(1)$ distances from the bifurcating fixed point cannot be justified by the usual methods of center manifolds or bifurcation theory. We need to prove existence of a global invariant (inertial) manifold in the appropriate regions of parameter and phase space. Unfortunately, all the results on such inertial manifolds for the Kuramoto-Sivashinsky equation apply only to the case of Neumann boundary conditions $u_x(0, t) = u_x(h, t) = 0$. In particular, a first step in the existence proof relies on bounding the derivative u_x in the L^2 norm, which has only been done for the Neumann case (Foias [1987]). (In terms of our Fourier representation, this implies that

$$\sum_{j=1}^{\infty} \left(j\frac{2\pi}{h}\right)^2 |a_j|^2 < \infty;$$

we have already observed that the mean a_0 of u can become unbounded.) Boundedness of $\|u_x\|^2$ then permits construction of a global trapping region, and ultimately, of the inertial manifold (Nicolaenko, Scheurer, and Temam [1985], Foias et al. [1988]). In our context, inertial manifolds are known to exist only for the system (2.7) restricted to the real subspace $a_l = a_l^*$. This invariant subspace, spanned by Fourier cosine modes, does not contain any of the traveling or modulated traveling waves or heteroclinic cycles that dominate the asymptotically stable solutions observed numerically. It is therefore of considerable interest to extend inertial manifold results to the full periodic boundary condition case in an attempt to justify the formal calculations of this paper.

We end by remarking that rigorous results can be obtained for a modification of the Kuramoto-Sivashinsky equation (1.1a), as suggested by Sirovich and Pismen. If a term βu is added to the left-hand side, each of the projected ODEs (2.5) inherits an additional term $-\beta a_l$. After rescaling time via (2.6a) we obtain a modified linear term in (2.7).

$$\left(-\gamma + l^2\left(1-\frac{l^2}{\mu}\right)\right)a_l, \qquad \gamma = \beta\left(\frac{h}{2\pi}\right)^2.$$

The (double) eigenvalues are now given by $-\gamma + l^2(1-l^2/\mu)$ and we can choose pairs of parameter values (γ, μ) such that two pairs of eigenvalues are simultaneously zero. Specifically, for $(\gamma, \mu) = (4/5, 5)$ modes $l = 1$ and $l = 2$ simultaneously bifurcate from zero. Close to this point in the two parameter problem, a center manifold reduction and normal form analysis can be performed to yield an $O(2)$-equivariant system of the type (5.3), as treated in Armbruster, Guckenheimer, and Holmes [1988]. Since for (γ, μ) sufficiently close to $(4/5, 5)$ all the bifurcating solutions are contained in a small neighborhood of $a_l = 0$, the local bifurcation results are completely rigorous. This problem would probably be worth a detailed study. However, in this paper we have chosen to address the original problem posed by Nicolaenko and his colleagues and therefore have only a single parameter to work with.

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