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HETEROCLINIC CYCLES AND MODULATED TRAVELLING WAVES IN SYSTEMS WITH O(2) SYMMETRY

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We analyze unfoldings of a codimension two, steady-state/steady-state modal interaction possessing O(2) symmetry. At the degenerate bifurcation point there are two zero eigenvalues, each of multiplicity two. The spatial wavenumbers of the critical modes k_i are assumed to satisfy $k_2 = 2k_1$. We base our analysis on a detailed study of the third order truncation of the resulting equivariant normal form, which is a four-dimensional vector field. We find that heteroclinic cycles and modulated travelling waves exist for open sets of parameter values near the codimension two bifurcation point. We provide conditions on parameters which guarantee existence and uniqueness of such solutions and we investigate their stability types. We argue that such motions will be prevalent in continuum systems having the symmetry of translation and reflection with respect to one (or more) spatial directions.

1. Introduction

This paper extends the analysis of Dangelmayr [1] and Dangelmayr and Armbruster [2]. Related work, in which the spatial wavenumbers k_i satisfy $k_2 = k_1$ rather than $k_2 = 2k_1$, as in this paper, has been done by Guckenheimer [3] and Dangelmayr and Knobloch [4]; however, the nature of the present system and the methods and results obtained are very different in character. In particular, our linearized system is diagonalizable while the 1 : 1 systems referred to have a nilpotent linear component. Independently of our work, Jones and Proctor [5] and Proctor and Jones [6] have obtained many similar results in the context of convection problems.

Our interest in these systems was motivated by the study of turbulent fluid boundary layers in the wall region (Aubry et al. [7]). Numerical investigations of models for the dynamics of fluctuations in the boundary layer reveal the presence of intermittent solutions ("bursts") that are persistent over a range of parameter values in the model and that correspond to heteroclinic cycles in the model equations. Since homoclinic and heteroclinic trajectories are destroyed by generic perturbations we wish to determine whether the intermittent solutions of the model are accidental or an essential feature. Our conclusion is that the heteroclinic cycles present in this model arise as a natural feature in the context of evolution equations which are translation and reflection invariant with respect to a spatial direction. Heteroclinic cycles also occur in simpler situations (Guckenheimer and Holmes [8]). In this paper we describe the dynamical systems analysis which underlies this conclusion.

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The vector fields we study are defined on $\mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2 = \mathbb{C} \times \mathbb{C}$ by the complex equations

$$\begin{aligned} \dot{z}_1 &= \bar{z}_1 z_2 + z_1(\mu_1 + e_{11}|z_1|^2 + e_{12}|z_2|^2), \\ \dot{z}_2 &= \pm z_1^2 + z_2(\mu_2 + e_{21}|z_1|^2 + e_{22}|z_2|^2), \end{aligned} \tag{1.1}$$

where μ_j and e_{jk} are real parameters.

These equations are preserved (equivariant) with respect to the symmetry operations $(z_1, z_2) \to (e^{i\theta}z_1, e^{2i\theta}z_2)$ and $(z_1, z_2) \to (\bar{z}_1, \bar{z}_2)$. The main body of this paper gives a thorough analysis of the dynamics of this system. The derivation of these equations from a translation and reflection invariant system of partial differential equations yields a context within which they occur naturally in applications. We summarize this derivation here, but the discussion of the dynamics of (1.1) is independent of this derivation, apart from motivating the nomenclature we use for different solutions.

Consider an evolution equation of the form $u_t = F(u)$ with $u(x, \cdot)$ dependent upon a spatial variable x (and perhaps other variables as well). We assume that the equation is symmetric with respect to translations and reflections in x and that periodic boundary conditions in x of period D are imposed. For example, $F(u)$ might be a nonlinear differential operator with constant coefficients in which each term involves an even number of differentiations of x (e.g. $F(u) = -u_{xxxx} - u_{xx} - \frac{1}{2}(u_x^2)$ the Kuramoto–Sivashinsky operator). Under these conditions, the translations together with the reflections induce an action of O(2), the 2×2 orthogonal group. If $F(0) = 0$, then the linearization of F at this trivial solution will have subspaces that are invariant under a representation of the symmetry group. In concrete terms, u can be expanded in a Fourier series, $\sum_{k=-\infty}^{\infty} a_k \exp(2\pi kxi/D)$ in the variable x, and the linearization of F preserves the space of objects associated to each Fourier mode $\exp(\pm 2\pi kxi/D)$. Typically, one can expect that bifurcation in a family $u_t = F_\lambda(u)$ will occur in the simplest way compatible with the symmetry. For non-zero wavenumbers with respect to x, the symmetry group acts in such a way that the simplest bifurcation will have a zero eigenvalue of multiplicity two. If there is a second parameter in the problem that varies (e.g., the length scale D), then for certain parameter values, it may happen that there is simultaneous marginal stability of modes with wavenumbers k and l. A reduced description of the dynamics associated with such a "codimension two" bifurcation leads to the study of equations of the form

$$\begin{aligned} \dot{z}_1 &= f_1(z_1, z_2), \\ \dot{z}_2 &= f_2(z_1, z_2), \end{aligned} \tag{1.2}$$

which are equivariant with respect to a (k, l) representation of O(2). Explicitly

$$\begin{aligned} \dot{z}_1 &= e^{-ik\theta} f_1(e^{ik\theta}z_1, e^{il\theta}z_2), \\ \dot{z}_2 &= e^{-il\theta} f_2(e^{ik\theta}z_1, e^{il\theta}z_2), \end{aligned} \tag{1.3}$$

and

$$\begin{aligned} \dot{\bar{z}}_1 &= \bar{f}_1(\bar{z}_1, \bar{z}_2), \\ \dot{\bar{z}}_2 &= \bar{f}_2(\bar{z}_1, \bar{z}_2). \end{aligned}$$

The most general equations of this form can be written in the form

$$\begin{aligned} \dot{z}_1 &= P_1(E_1, E_2, M)z_1 + Q_1(E_1, E_2, M)\bar{z}_1^{l-1}z_2^k, \\ \dot{z}_2 &= P_2(E_1, E_2, M)z_2 + Q_2(E_1, E_2, M)z_1^l\bar{z}_2^{k-1}, \end{aligned} \tag{1.4}$$

with

$$E_j = |z_j|^2, \quad j = 1,2 \quad \text{and} \quad M = z_1^l \bar{z}_2^k + \bar{z}_1^l z_2^k.$$

In this paper, we restrict attention to the case $k = 1$, $l = 2$ corresponding to simultaneous bifurcation of modes whose periods in x are D and $D/2$. For (partial) studies of the general case, see Dangelmayr [1] and Dangelmayr and Armbruster [2]. Near a point of (1,2) codimension two bifurcation, we seek to understand the nonlinear dynamics of small amplitude solutions. For systems with well-behaved operators $F(u)$, center manifold theory (Carr [9], Guckenheimer and Holmes [10]) gives the basis for a reduction to a four-dimensional system of the form described above. The system (1.1) then represents the first few terms in the Taylor expansion of the reduced system at the point of codimension two bifurcation: i.e., after rescaling it is the most general vector field, up to cubic terms, compatible with the group action. Regarding (μ_1, μ_2) as parameters in (1.1), dynamical features of this system which persist under symmetric perturbations of (1.1) can be expected to be general features of (1,2) bifurcation of the system $u_t = F(u)$. In particular, the persistent stable homoclinic cycles that we find represent solutions which linger near a 2-mode equilibrium, then undergo an event after which the 2-mode equilibrates again in a position which is shifted by half a period, or $D/4$.

The phase portraits of (1.1) near the origin fall into a small number of classes giving different stability diagrams as (μ_1, μ_2) are varied. Much of the structure of the phase portraits is related to the presence of invariant subspaces whose existence can be deduced from the symmetry properties of the vector field. Up to conjugation by the symmetry group, there are three invariant subspaces for (1.1): (1) the *real subspace* defined by $\text{Im}(z_1) = \text{Im}(z_2) = 0$ and fixed by complex conjugation regarded as an element of O(2), (2) the 2-subspace defined by $z_1 = 0$ and fixed by the action of the group element $e^{i\pi}$, and (3) the $x_2 = \text{Re}(z_2)$ axis which is the intersection of (1) and (2). The group O(2) leaves the 2-subspace invariant, but there is an S^1-family of conjugates of the real subspace. Note that two of these conjugate subspaces pass through each point of the x_2-axis, a crucial fact in understanding the homoclinic cycles of the system. In the next sections, we describe the dynamics in these invariant subspaces. Although the system (1.1) may have equilibria that do not approach the origin as $\mu_1, \mu_2 \to 0$, we use perturbation arguments to focus attention on the small amplitude behavior. In the real plane, there are also *mixed mode* steady solutions (both x_1 and x_2 are non-zero) that bifurcate to limit cycles through Hopf bifurcation. The limit cycles are called *standing waves*. This part of our work is essentially a review of Dangelmayr's results.

To understand the phase portrait of (1.1) outside its invariant subspaces, we introduce polar coordinates $z_1 = r_1 e^{i\theta_1}$, $z_2 = r_2 e^{i\theta_2}$ and observe that the equations for r_1, r_2 and $\phi = 2\theta_1 - \theta_2$ separate from the fourth variable due to the symmetry of the vector field. There is a new family of equilibria of the reduced system that yields *travelling waves*, periodic solutions that correspond, for one-dimensional partial differential equations, to waves drifting at constant speed in evolution equations such as that described above. A scaling in terms of a small parameter ε is then introduced in an effort to obtain an integrable limit from which perturbations can be analyzed. This goal is achieved by noting that the system

$$\dot{z}_1 = \bar{z}_1 z_2,$$
$$\dot{z}_2 = -z_1^2,$$

has the pair of integrals $E = |z_1|^2 + |z_2|^2 = r_1^2 + r_2^2$ and $L = (1/2i)[z_1^2\bar{z}_2 - \bar{z}_1^2 z_2] = r_1^2 r_2 \sin\phi$. Along the level curves of the reduced system in polar coordinates, we use the method of averaging to compute, to first order in ε, the average variation of E and L along a trajectory of the integrable system. We prove that there is a region of parameters (μ_1, μ_2) in which the reduced system has a unique limit cycle and hence the system (1.1) has an invariant torus. These solutions we call *modulated travelling waves*, see section 2, below. For fixed E, we follow the modulated waves lying close to various values of L from birth in a Hopf bifurcation to their termination at heteroclinic cycles. These cycles involve two distinct 2-mode equilibria that are translates of each other by $D/4$, half a 2-period. At each of these equilibria, there are two invariant "real" subspaces as noted above. The subspaces associated to each of the two equilibria in the cycle are the same, and the cycle contains one heteroclinic trajectory in each of the two subspaces. For general vector fields, the stability of a heteroclinic cycle involving nondegenerate saddles is determined by the magnitudes of the eigenvalues associated to the saddles. Here, cycles that are predicted to be unstable according to these eigenvalues are actually stable as a result of the symmetry of the system.

The rest of the paper is organized as follows. In the next section we introduce the equations to be studied, discuss the limit case noted above, and review some elementary bifurcation results from the works listed above. Section 3 contains results on non-existence and existence of heteroclinic cycles. We discuss modulated travelling waves in the limit of small amplitude $|z_j| = \mathcal{O}(\varepsilon)$ and very small parameters $|\mu_j| = \mathcal{O}(\varepsilon^2)$ in section 4 and show that the unique branch of such waves limits on the heteroclinic cycle at one end and collapses onto the branch of travelling waves in a Hopf bifurcation at the other. Stability of heteroclinic cycles and modulated travelling waves is discussed in section 5 and section 6 contains some examples of unfoldings and bifurcation diagrams for specific cases.

2. Normal form and scaling

As shown by Buzano and Russo [11] and Dangelmayr [1] (cf. Dangelmayr and Armbruster [2]) and outlined above, the normal form for an O(2) equivariant vector field involving two modes can be written in the complex form as

$$\begin{aligned} \dot{z}_1 &= z_1 p_1 + q_1 \bar{z}_1^{l-1} z_2^k, \\ \dot{z}_2 &= z_2 p_2 + q_2 z_1^l \bar{z}_2^{k-1}, \end{aligned} \tag{2.1}$$

where the p_i, q_i are smooth real functions of the invariants $|z_1|^2$, $|z_2|^2$ and $z_1^l \bar{z}_2^k + \bar{z}_1^l z_2^k$. Specializing to the case of interest $k = 1$, $l = 2$, (2.1) becomes

$$\begin{aligned} \dot{z}_1 &= z_1\left(\mu_1 + d_{11}|z_1|^2 + d_{12}|z_2|^2\right) + c_{12}\bar{z}_1 z_2 \\ \dot{z}_2 &= z_2\left(\mu_2 + d_{21}|z_1|^2 + d_{22}|z_2|^2\right) + c_{11} z_1^2 \end{aligned} + \mathcal{O}(4), \tag{2.2}$$

where $\mathcal{O}(4)$ represents terms of order $|z_j|^4$ which will not be explicitly included henceforth, c_{jk}, d_{jk} are real coefficients and μ_1, μ_2 are unfolding parameters. Assuming that $c_{12}, c_{11} \neq 0$, we can rescale, reversing time if necessary, to obtain

$$\begin{aligned} \dot{z}_1 &= \bar{z}_1 z_2 + z_1\left(\mu_1 + e_{11}|z_1|^2 + e_{12}|z_2|^2\right), \\ \dot{z}_2 &= \pm z_1^2 + z_2\left(\mu_2 + e_{21}|z_1|^2 + e_{22}|z_2|^2\right), \end{aligned} \tag{2.3}$$

where $e_{11} = d_{11}/|c_{11}c_{12}|$, $e_{12} = d_{12}/c_{12}^2$, $e_{21} = d_{21}/|c_{11}c_{12}|$ and $e_{22} = d_{22}/c_{12}^2$. (One can normalize one cubic coefficient to ± 1 by a further time scale change; moreover, by nonlinear coordinate changes, it is possible to further eliminate one of the cubic coefficients [1]. However, there is no advantage for our discussion in performing these coordinate changes.)

Letting $z_j = r_j e^{i\theta_j}$, (2.3) may conveniently be rewritten as

$$\begin{aligned} \dot{r}_1 &= r_1 r_2 \cos\phi + r_1(\mu_1 + e_{11} r_1^2 + e_{12} r_2^2), \\ \dot{r}_2 &= \pm r_1^2 \cos\phi + r_2(\mu_2 + e_{21} r_1^2 + e_{22} r_2^2), \\ \dot{\phi} &= -(2r_2 \pm r_1^2/r_2)\sin\phi, \end{aligned} \tag{2.4}$$

where $\phi = 2\theta_1 - \theta_2$. The emergence of the phase difference ϕ and reduction to three (real) dimensions is a consequence of the O(2) symmetry discussed above and we remark that this phase translation invariance is a characteristic of the full (non-truncated) O(2)-equivariant system (1.4). This is important in our discussion of quasiperiodic motions below. However, since the transformation is singular when $r_1, r_2 = 0$ it is also necessary to use the real Cartesian form

$$\begin{aligned} \dot{x}_1 &= x_1 x_2 + y_1 y_2 + x_1(\mu_1 + e_{11} r_1^2 + e_{12} r_2^2), \\ \dot{y}_1 &= x_1 y_2 - y_1 x_2 + y_1(\mu_1 + e_{11} r_1^2 + e_{12} r_2^2), \\ \dot{x}_2 &= \pm(x_1^2 - y_1^2) + x_2(\mu_2 + e_{21} r_1^2 + e_{22} r_2^2), \\ \dot{y}_2 &= \pm 2 x_1 y_1 + y_2(\mu_2 + e_{21} r_1^2 + e_{22} r_2^2), \end{aligned} \tag{2.5}$$

where $r_j^2 = x_j^2 + y_j^2$. This will be especially useful in our discussion of heteroclinic cycles in section 3. Reflection symmetry implies that the purely real system ($y_1 = y_2 = 0$) is invariant, as is any ϕ-invariant rotation of it, for example the y_1, x_2 system ($x_1 = y_2 = 0$). This will also be used in section 3.

Finally, the scaling $r_j = \varepsilon s_j$, $\mu_j = \varepsilon^2 \nu_j$ and $(\dot{\ }) = \varepsilon(\ )'$ transforms the polar system to

$$\begin{aligned} s_1' &= s_1 s_2 \cos\phi + \varepsilon s_1(\nu_1 + e_{11} s_1^2 + e_{12} s_2^2), \\ s_2' &= \pm s_1^2 \cos\phi + \varepsilon s_2(\nu_2 + e_{21} s_1^2 + e_{22} s_2^2), \\ \phi' &= -(2 s_2 \pm s_1^2/s_2)\sin\phi, \end{aligned} \tag{2.6}$$

and reveals two integrals for the limit $\varepsilon = 0$. These are

$$E = s_1^2 \mp s_2^2 \tag{2.7}$$

and

$$L = s_1^2 s_2 \sin\phi. \tag{2.8}$$

We note that the existence of these integrals was observed independently by Jones and Proctor [5] and that they also use them in their analysis. The integrals play a crucial role in our analysis of modulated travelling waves in section 4.

It is appropriate at this point to discuss various classes of solutions exhibited by eqs. (2.3), (2.4) and (2.6). It is important here to recognize that the individual phase equations

$$\dot{\theta}_1 = -r_2 \sin\phi \quad \text{and} \quad \dot{\theta}_2 = \pm(r_1^2/r_2)\sin\phi$$

have been suppressed in writing (2.4). With this in mind, *steady solutions* or fixed points of (2.3) correspond to fixed points of (2.4) or (2.6) with $\phi = 0$ or π. In addition to the trivial solution $r_1 = r_2 = 0$, we have *pure modes* ($r_1 = 0$, $r_2 \neq 0$) and *mixed modes*, $r_1, r_2 \neq 0$. Fixed points of (2.4), (2.6) with $\phi \neq 0, \pi$ correspond to *travelling waves* of (2.3) in which the phase difference remains constant, but θ_1 and θ_2 both increase or both decrease linearly with time (such solutions only occur in the '$-$' case, and also require $2r_2^2 = r_1^2$ so that $2\dot{\theta}_1 = \dot{\theta}_2$). Periodic orbits of (2.4)–(2.6) on the subspace $\phi = 0, \pi$ correspond to singly periodic *standing waves* of (2.3) while periodic orbits with $\phi \neq 0, \pi$ correspond to *doubly periodic modulated travelling waves* of (2.3). Since $\phi = 2\theta_1 - \theta_2$ is the only combination of phase variables that appears in the full normal form equations (1.4) such an isolated two-torus will not be subject to phase locking. Along a cross-section defined by $\phi =$ constant, the points $(r_1, r_2, \theta_1, \theta_2)$ and $(r_1, r_2, \theta_1 + \alpha, \theta_2 + 2\alpha)$ evolve so that they are always separated by the rotation by angles $(\alpha, 2\alpha)$ in $\mathbf{C} \times \mathbf{C}$. The cross-section can be parametrized by α and its return map is then a rigid rotation of this coordinate. Here we explicitly see how the $SO(2) \subset O(2)$-equivariance prevents phase locking; for a more general analysis, see Rand [12]. Finally, a quasiperiodic solution of (2.4)–(2.6) corresponds to a *triply periodic modulated travelling wave* of (2.3). These statements are all elementary deductions from the equations and the polar coordinate transformations.

Before starting our analysis of the dynamics, it is convenient to collect some facts about steady solutions of (2.3)–(2.5). The origin is always a steady solution with eigenvalues μ_1, μ_2, each of multiplicity 2. Other branches of nontrivial fixed points lie in subspaces that are fixed by subgroups of O(2). Rotation by π fixes the subspace $z_1 = 0$, and (2.3) reduces to the equation

$$\dot{z}_2 = z_2(\mu_2 + e_{22}|z_2|^2)$$

on this subspace. When $\mu_2/e_{22} < 0$, there are equilibria (pure modes) along the circles $|z_2| = (-\mu_2/e_{22})^{1/2}$. The eigenvalues of these equilibria are 0 and $-2\mu_2$ within the plane $z_1 = 0$ and $\mu_1 - \mu_2 e_{12}/e_{22} \pm (-\mu_2/e_{22})^{1/2}$ normal to the plane $z_1 = 0$. Note that the zero locus for the normal eigenvalues has quadratic contact with the parabola $e_{22}\mu_1^2 + \mu_2 = 0$. There is also a family of planes invariant under the reflections in O(2). These are all rotations of the "real" subspace $y_1 = y_2 = 0$. The restriction of (2.5) to the real subspace gives

$$\begin{aligned} \dot{x}_1 &= x_1 x_2 + x_1(\mu_1 + e_{11}x_1^2 + e_{12}x_2^2), \\ \dot{x}_2 &= \pm x_1^2 + x_2(\mu_2 + e_{21}x_1^2 + e_{22}x_2^2). \end{aligned} \tag{2.9}$$

The system (2.9) is still quite complicated as $\mu_1, \mu_2 \to 0$; there may be equilibria that do not tend to the origin. However, these equilibria play no role in our local analysis and by using the rescaling employed in the transformation (2.5) to (2.6), we obtain the system

$$\begin{aligned} \dot{\xi}_1 &= \xi_1\xi_2 + \varepsilon\xi_1(\nu_1 + e_{11}\xi_1^2 + e_{12}\xi_2^2), \\ \dot{\xi}_2 &= \pm\xi_1^2 + \varepsilon\xi_2(\nu_2 + e_{21}\xi_1^2 + e_{22}\xi_2^2), \end{aligned} \tag{2.10}$$

that can be studied in terms of power series in ε. Since $\varepsilon\xi_i = x_i$, phenomena associated with (2.10) are local for our problem if they are $\mathcal{O}(\varepsilon^\alpha)$ with $\alpha > -1$.

Non-zero equilibria of (2.10) satisfy either $\xi_1 = 0$ and $\xi_2^2 = -\nu_2/e_{22}$ or they satisfy $\xi_1 \neq 0$. The equilibria on the ξ_2 axis have already been discussed, while the equilibria with $\xi_1 \neq 0$ satisfy $\pm\nu_1 + (\pm\varepsilon^{-1} + \varepsilon e_{21}\nu_1 - \varepsilon e_{11}\nu_2)\xi_2 + (e_{21} \pm e_{12})\xi_2^2 + \varepsilon[e_{12}e_{21} - e_{11}e_{22}]\xi_2^3 = 0$ and hence solutions which are $\mathcal{O}(\varepsilon^\alpha)$ with $\alpha > -1$

satisfy $\xi_2 = -\varepsilon\nu_1 - \varepsilon^3(e_{12}\nu_1^2 \pm e_{11}\nu_1\nu_2) + o(\varepsilon^3)$, $\xi_1^2 = \pm\varepsilon^2\nu_1\nu_2 + o(\varepsilon^2)$. These equilibria correspond to *mixed modes* and the linearization of the equations at them has eigenvalues $\varepsilon(\nu_2/2 \pm (\nu_2^2/4 \mp \nu_1\nu_2)^{1/2}) + o(\varepsilon)$ in the real plane ($\phi = 0, \pi$). Since $\xi_1^2 \approx \pm\varepsilon^2\nu_1\nu_2 > 0$ for existence of the mixed mode, zero eigenvalues cannot occur. For pure imaginary eigenvalues ν_2 must be $o(\varepsilon)$. Further calculations yield

$$\nu_2 = -3\varepsilon^2 e_{22}\nu_1^2 + o(\varepsilon^2)$$

or

$$\mu_2 = -3e_{22}\mu_1^2 + o(\mu_1^2), \tag{2.11}$$

in terms of the original parameters. The eigenvalues in the direction normal to the real subspace are $\varepsilon(2\nu_1 + \nu_2) + o(\varepsilon)$ and 0, the latter having its eigenvector tangent to the circle of equilibria obtained by rotation of the equilibria in the real subspace $y_1 = y_2 = 0$ by the symmetry group.

Elementary bifurcation computations show that the mixed modes bifurcate from the trivial solution when $\mu_1 = 0$ and from the pure modes $r_2 = (-\mu_2/e_{22})^{1/2}$, $\phi = \pi$ (resp. $\phi = 0$) on the curves

$$\mu_1 - \mu_2 e_{12}/e_{22} \pm (-\mu_2/e_{22})^{1/2} = 0, \tag{2.12}$$

respectively.

We have already noted that *travelling waves* are equilibria of (2.4) or (2.6) with $\phi \neq 0, \pi$. Such solutions always come in pairs and do *not* correspond to equilibria of (2.5), but rather to periodic orbits that are left invariant by the symmetry group. The travelling waves must satisfy $2r_2^2 \pm r_1^2 = 0$, showing that they can only exist in the '$-$' case, when they are given by

$$r_2^2 = -(2\mu_1 + \mu_2)/(4e_{11} + 2e_{12} + 2e_{21} + e_{22}), \tag{2.13a}$$

$$r_1^2 = 2r_2^2, \tag{2.13b}$$

$$\cos\phi = \frac{\mu_2(2e_{11} + e_{12}) - \mu_1(2e_{21} + e_{22})}{[-(2\mu_1 + \mu_2)(4e_{11} + 2e_{12} + 2e_{21} + e_{22})]^{1/2}}. \tag{2.13c}$$

The stability of the travelling waves is discussed in section 5. Travelling waves bifurcate from the mixed mode when eqs. (2.13) are simultaneously satisfied with $|\cos\phi| = 1$, so that they exist in the region of parameter space defined by

$$[\mu_2(2e_{11} + e_{12}) - \mu_1(2e_{21} + e_{22})]^2 < -(2\mu_1 + \mu_2)(4e_{11} + 2e_{12} + 2e_{21} + e_{22}). \tag{2.14}$$

Alternatively, we may write the bifurcation set as

$$\begin{aligned}\mu_2 &= \left(\frac{2e_{21} + e_{22}}{2e_{11} + e_{12}}\right)\mu_1 + \frac{4e_{11} + 2(e_{12} + e_{12}) + e_{22}}{2(2e_{11} + e_{12})^2}\left[(1 - 4(2e_{11} + e_{12})\mu_1)^{1/2} - 1\right] \\ &= -2\mu_1 - (4e_{11} + 2(e_{12} + e_{21}) + e_{22})\mu_1^2 + o(\mu_1^2).\end{aligned} \tag{2.15}$$

We remark that if we treat the truncated system (2.3), (2.4) without restricting discussion to "local" solutions, with $|\mu_j|$ and $|z_j|$ small, then additional mixed modes can occur and interaction between bifurcation to travelling waves and standing waves becomes possible. Dropping the condition that $\xi_2 = -\varepsilon\nu_1 + o(\varepsilon^3)$, $\xi_1 = \pm\varepsilon^2\nu_1\nu_2 + o(\varepsilon^2)$, it is possible for the mixed mode to have a second zero eigenvalue with eigenvector normal to the zero subspace, corresponding to a bifurcation to travelling waves. The eigenvalues of the (possibly non-local) mixed mode $x_1 = \bar{x}_1$, $x_2 = \bar{x}_2$ on the real subspace are $-(2\bar{x}_2^2 \pm \bar{x}_1^2)/\bar{x}_2$, 0 and the roots of

$$\begin{aligned}
&\lambda^2 - T\lambda + D = 0,\\
&T = 2\left(e_{11}\bar{x}_1^2 + e_{22}\bar{x}_2^2\right) \mp \bar{x}_1^2/\bar{x}_2,\\
&D = 4\left(e_{11}e_{22} - e_{12}e_{21}\right)\bar{x}_1^2\bar{x}_2^2 - 2\bar{x}_1^2\left(e_{21}\bar{x}_2 \pm 2e_{12}\bar{x}_2 \pm 1 \pm e_{11}\bar{x}_1^2/\bar{x}_2\right).
\end{aligned} \tag{2.16}$$

Thus, for coincident Hopf bifurcation to standing waves and pitchfork bifurcation to travelling waves we require that (2.13a, b) and (2.15) hold and the trace T of (2.16) be zero simultaneously. Solving for $\bar{x}_1$ and $\bar{x}_2$, we find that the critical mixed mode lies at

$$\bar{x}_1 = \pm\sqrt{2\bar{x}_2}, \quad \bar{x}_2 = -1/(2e_{11} + e_{22}) \tag{2.17}$$

and the bifurcation point is given by

$$\mu_1 = \frac{e_{22} - e_{12}}{\left(2e_{11} + e_{22}\right)^2}, \quad \mu_2 = -\frac{4e_{11} + 2e_{21} + 3e_{22}}{\left(2e_{11} + e_{22}\right)^2}. \tag{2.18}$$

Since this implies that the μ_j and x_j are of O(1), it goes beyond a strict local analysis, and we will not discuss it in this paper. Proctor and Jones [6] discuss this multiple bifurcation point further.

In the following analysis we sometimes restrict ourselves to the case of $e_{ij} < 0$, so that the system is globally stable. (It is easy to see that the function $E = r_1^2 + r_2^2$ decreases on solution curves for E sufficiently large in this situation.)

3. Existence and non-existence of heteroclinic cycles

We start by showing that no heteroclinic cycles exist in the '+' case. In fact we have more:

Proposition 3.1. If c_{11} and c_{12} have the same sign then all solutions of (2.2) are asymptotic to the set $\{\phi = 2\theta_1 - \theta_2 = 0\} \cup \{\phi = \pi\}$. No travelling waves, modulated travelling waves, or heteroclinic cycles exist in this case.

Proof. In this situation the "normalized" phase equation (2.4) is $\dot{\phi} = -(2r_2 + r_1^2/r_2)\sin\phi$ and since the quantity in parenthesis is non-negative we find that $\dot{\phi} < 0$ for $\phi \in (0, \pi)$ and $\dot{\phi} > 0$ for $\phi \in (\pi, 2\pi)$ and $r_2 \neq 0$. Thus $\phi(t) \to 0 = 2\pi$ unless solutions start with $\phi(0) = \pi$. We note that the coordinate singularity at $r_2 = 0$ does not complicate matters: solutions passing through $r_2 = 0$ near $\phi = \pi$ emerge near $\phi = 0, 2\pi$, but, since $\dot{r}_2 = r_1^2\cos\phi$ for $r_2 = 0$, solutions near $\phi = 0, 2\pi$ cannot reach $r_2 = 0$ again. They must therefore go to ∞ or approach the 'pure' mode $r_2 = (-\mu_2/e_{22})^{1/2}$ or approach a mixed mode, which may be stationary or periodic. □

In contrast, in the '$-$' case we have:

Theorem 3.2. If $e_{11}, e_{22} < 0$, $e_{12} + e_{21} < 2(e_{11}e_{22})^{1/2}$, $\mu_1, \mu_2 > 0$, $\mu_1 - \mu_2 e_{12}/e_{22} - (-\mu_2/e_{22})^{1/2} < 0 < \mu_1 - \mu_2 e_{12}/e_{22} + (-\mu_2/e_{22})^{1/2}$ and no mixed modes exist, then there is a heteroclinic cycle connecting each diametrically opposite pair of points $r_2 = \bar{r}_2 = (-\mu_2/e_{22})^{1/2}$, $\theta_2 = \bar{\theta}_2$, $r_1 = 0$ and $r_2 = \bar{r}_2$, $\theta_2 = \bar{\theta}_2 + \pi$, $r_1 = 0$ on the circle of pure modes.

Proof. For $e_{22} < 0 < \mu_2$ there is a circle of pure modes $r_2 = (-\mu_2/e_{22})^{1/2}$, $r_1 = 0$. Without loss of generality we restrict our attention to the two members of this circle lying in the real subspace $y_1 = y_2 = 0$. These fixed points are $(x_1, y_1, x_2, y_2) = (0, 0, \pm(-\mu_2/e_{22})^{1/2}, 0) \stackrel{\text{def}}{=} \boldsymbol{r}^+, \boldsymbol{r}^-$ with eigenvalues $\mu_1 - \mu_2 e_{12}/e_{22} \pm (-\mu_2/e_{22})^{1/2}$, $-2\mu_2$, 0, respectively, and eigenvectors (1,0,0,0), (0,1,0,0), (0,0,1,0) and (0,0,0,1) for $\boldsymbol{r}^+$ and (0,1,0,0), (1,0,0,0), (0,0,1,0), (0,0,0,1) for $\boldsymbol{r}^-$. Under the hypotheses of the theorem, the first two eigenvalues are positive and negative, respectively, and the third is negative. To establish the existence of a heteroclinic cycle it is necessary to show that the (one-dimensional) unstable manifold of $\boldsymbol{r}^+, W^{\mathrm{u}}(\boldsymbol{r}^+)$, intersects the two-dimensional stable manifold of $\boldsymbol{r}^-, W^{\mathrm{s}}(\boldsymbol{r}^-)$, and that $W^{\mathrm{u}}(\boldsymbol{r}^-)$ intersects $W^{\mathrm{s}}(\boldsymbol{r}^+)$. However, since the Cartesian equations (2.5) are invariant under the transformation $(x_1, y_1, x_2, y_2) \to (y_1, -x_1, -x_2, -y_2)$ (or $z_j \to z_j e^{ij\pi/2}$), it suffices to prove the existence of the first of these connections: the second is obtained from the first by applying an element of the symmetry group.

The eigenvector computations above show that the unstable manifold $W^{\mathrm{u}}_{\mathrm{loc}}(\boldsymbol{r}^+)$ is tangent to the plane $y_1 = y_2 = 0$ and, since this plane is invariant for (2.5) the global manifold $W^{\mathrm{u}}(\boldsymbol{r}^+)$ similarly lies in this plane. Our problem therefore reduces to showing that the real planar system

$$\begin{aligned} \dot{x}_1 &= x_1\left(\mu_1 + e_{11}x_1^2 + e_{12}x_2^2 + x_2\right), \\ \dot{x}_2 &= x_2\left(\mu_2 + e_{21}x_1^2 + e_{22}x_2^2\right) - x_1^2, \end{aligned} \tag{3.1}$$

has an orbit connecting the saddle $(x_1, x_2) = (0, +(-\mu_2/e_{22})^{1/2})$ to the sink $(0, -(-\mu_2/e_{22})^{1/2})$. Furthermore, since (3.1) is invariant under $x_1 \to -x_1$ we may confine our attention to the positive half plane $H = \{(x_1, x_2) | x_1 > 0\}$. Now the unstable manifold $W^{\mathrm{u}}(\boldsymbol{r}^+)$ enters H, and since the x_2 axis is invariant, the ω limit set of $W^{\mathrm{u}}(\boldsymbol{r}^+)$ must lie in H (possibly at ∞) or on the x_2 axis itself. The nonexistence of mixed modes implies, via index arguments and the Poincaré–Bendixson theorem, that there are no limit sets such as closed loops or periodic orbits contained in H and the conditions $e_{11}, e_{22} < 0$ and $e_{12} + e_{21} < 2(e_{11}e_{22})^{1/2}$ imply that solutions cross the semicircle $E = x_1^2 + x_2^2$ inwards for E sufficiently large $(\mathrm{d}E/\mathrm{d}t = 2[\mu_1 x_1^2 + \mu_2 x_2^2 + e_{11}x_1^4 + (e_{12} + e_{21})x_1^2 x_2^2 + e_{22}x_2^4])$. Thus $\omega(W^{\mathrm{u}}(\boldsymbol{r}^+)) \subset \{x_1 = 0\}$. Now there are only three fixed points $\boldsymbol{r}^+$, $\boldsymbol{r}^-$ and (0,0) on $x_1 = 0$ and, since $x_1 = x_2 = 0$ is a source for $\mu_1, \mu_2 > 0$, we conclude that $\omega(W^{\mathrm{u}}(\boldsymbol{r}^+)) = \boldsymbol{r}^-$, as claimed. See fig. 1a. □

Remarks. The nonexistence of mixed modes is not necessary for heteroclinic connections. However, stating specific conditions for existence in the presence of mixed modes is awkward. Here is an example of the kind of statements one can make for the planar system (3.1):

Lemma 3.3. If $e_{ij} < 0\ \forall\ ij$, $\mu_2 > 0$, $\mu_1 - \mu_2 e_{12}/e_{22} - (-\mu_2/e_{22})^{1/2} < 0 < \mu_1 - \mu_2 e_{12}/e_{22} + (\mu_2/e_{22})^{1/2}$ and either $\mu_2 - \mu_1 e_{21}/e_{11} > 0$ and $|\mu_1|$ is small or $\mu_1 > 0$, then all solutions starting in the quarter plane $Q = \{(x_1, x_2) | x_1 > 0,\ x_2 \leq 0\}$ approach the sink at $(x_1, x_2) = (0, -(-\mu_2/e_{22})^{1/2})$.

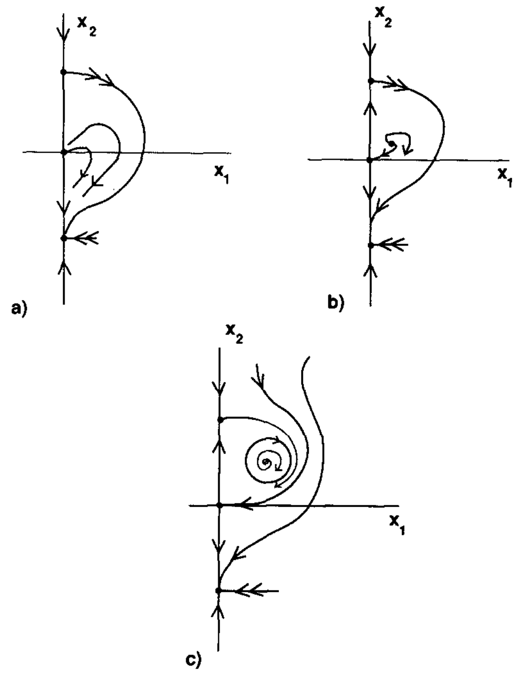

Fig. 1. Heteroclinic connections in the real subspace. (a) No mixed modes; (b) with a mixed mode; (c) a mixed mode obstructs a connection. Note the stable standing wave (periodic orbit).

Proof. Simultaneously setting $\dot{x}_1 = \dot{x}_2 = 0$ with $x_1 \neq 0$ in (3.1) leads to the conditions

$$\begin{aligned} x_1^2 &= -\left(\mu_1 + x_2 + e_{12}x_2^2\right)/e_{11} \stackrel{\text{def}}{=} f_{\mu_1}(x_2), \\ x_1^2 &= \left[\left(e_{11}e_{22} - e_{12}e_{21}\right)x_2^2 - e_{21}x_2 + \mu_2 e_{11} - \mu_1 e_{21}\right]x_2/e_{11} \\ &\stackrel{\text{def}}{=} g_{\mu_1,\mu_2}(x_2). \end{aligned} \tag{3.2}$$

While a single cubic for x_2 can clearly be derived (cf. section 2), it is convenient to consider (3.2) geometrically and interpret mixed modes as intersections of the graphs of f and g in the upper (x_2, x_1^2) half plane. The conditions of the theorem guarantee that such solutions only occur in the quarter plane $x_2 > 0$, $x_1 > 0$. Thus no fixed points exist in Q and, as in the proof above, this implies that no other limit sets exist in Q.

Since $\dot{x}_2 = -x_1^2$ on $x_2 = 0$, solutions starting on the boundary $x_2 = 0$ enter Q. This, together with the fact that solutions cannot escape to ∞ ($\mathrm{d}E/\mathrm{d}t < 0$ for large E) implies that all solutions reaching or starting in Q are asymptotic to the sink, as claimed. □

If the hypotheses of lemma 3.3 are met, then we need only show that the unstable manifold $W^u(r^+)$ meets the x_2 axis transversely at some point $x_1 > 0$ to conclude that it must therefore limit on the sink. Such a condition can be checked by numerical integration or careful construction of invariant domains. See figs. 1b, c. We will return to examples such as these in our discussion of specific unfoldings in section 6.

4. The '−' case: modulated travelling waves

We now turn to the scaled equations (2.6) in the case that c_{12} and c_{11} of (2.2) have opposite signs. The main result is:

Theorem 4.1. If c_{12} and c_{11} have opposite signs, the quantity $e = 4e_{11} + 2e_{12} + 2e_{21} + e_{22} \neq 0$ and $\varepsilon > 0$ is sufficiently small, then eq. (2.6) possesses at most one periodic orbit. The family of periodic orbits (modulated travelling waves) obtained by varying ν_1 and ν_2 limits on the travelling wave solutions $s_1^2 = 2s_2^2 = -2(2\nu_1 + \nu_2)/e$, $\phi = \pi/2, 3\pi/2$ when $\nu_2 = \nu_1[1 + 9(e_{22} - e_{12})/(e - 3(e_{22} - e_{12}))]$ and limits on the heteroclinic cycle at $\nu_2 = \nu_1 e_{22}/e_{12}$.

Proof. For $\varepsilon = 0$, the functions E and L defined by (2.7) and (2.8) are constant on solutions of (2.6). The phase space of (2.6) is the product of a circle S^1 (varying ϕ) with the positive quadrant of the (s_1, s_2) plane. Note, however, that the boundary of the phase space is not invariant under the flow, and that the points $(s_1, 0, \phi)$ and $(s_1, 0, \phi + \pi)$ should be identified so that a trajectory exiting the boundary at $(s_1, 0, \phi)$ re-enters at the point $(s_1, 0, \phi + \pi)$. The intersections of E and L are transverse except on (1) the boundary of the phase space given by $s_1 s_2 = 0$, and (2) the rays defined by $s_1 = \sqrt{2}\, s_2$ and $\cos\phi = 0$. On these rays, we have $4E^3 - 27L^2 = 0$, and each ray is surrounded by a family of closed orbits formed by the intersections of $E =$ const. and $L =$ const., see fig. 2. Note that $4E^3 - 27L^2 \geq 0$ throughout the phase space. Behavior in the region $\phi \in (\pi, 2\pi)$ is identical to that in $(0, \pi)$, with L replaced by $-L$, so henceforth we implicitly restrict our discussion to $\phi \in [0, \pi]$.

Letting $\rho = s_2^2$, unperturbed (i.e. $\varepsilon = 0$) solutions of (2.6) may be expressed in terms of elliptic integrals [13]. A simple computation shows that

$$(\rho')^2 = (2s_2 s_2')^2 = 4\left[s_1^4 s_2^2 \cos^2\phi\right] = 4\left(\rho(E-\rho)^2 - L^2\right), \tag{4.1}$$

since $E = s_1^2 + \rho$ and $L = s_1^2 s_2 \sin\phi$. Evolution equations for E and L for the perturbed solutions may

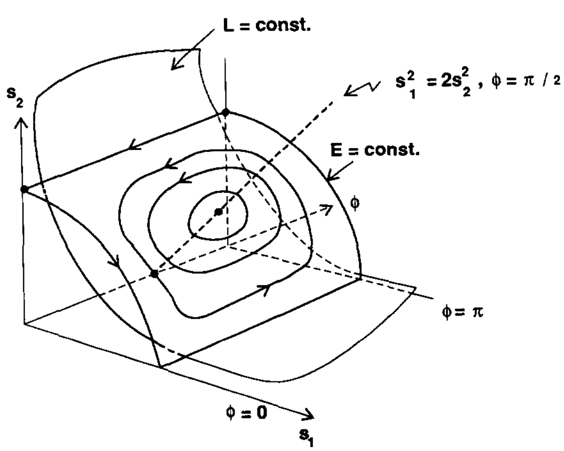

Fig. 2. Level sets of the integrals E and L, showing a family of unperturbed closed orbits on a surface $E =$ const.

then be written in terms of ρ:

$$E' = 2\varepsilon\left[(\nu_1 + e_{11}E)E + (\nu_2 - \nu_1 + (e_{12} + e_{21} - 2e_{11})E)\rho + (e_{11} + e_{22} - (e_{12} + e_{21}))\rho^2\right], \tag{4.2a}$$

$$L' = \varepsilon L\left[(2\nu_1 + \nu_2) + (2e_{11} + e_{21})E + ((2e_{12} + e_{22}) - (2e_{11} + e_{21}))\rho\right]. \tag{4.2b}$$

We wish to compute conditions under which a given solution $\rho(t)$, parametrized by E and L, will be "almost" preserved under the perturbation. More precisely, the method of averaging implies that for small $\varepsilon > 0$ periodic solutions of (2.6) will be found near the unperturbed closed orbit $E = E_0$, $L = L_0$ when the pair of functions

$$\Delta E = \int_0^{T(E_0, L_0)} E'\,\mathrm{d}t, \qquad \Delta L = \int_0^{T(E_0, L_0)} L'\,\mathrm{d}t \tag{4.3}$$

has a nondegenerate zero (here $T(E_0, L_0)$ is the period of the unperturbed closed orbit), cf. Hale [14], Guckenheimer and Holmes [10, chapter 4].

Using (4.1) and (4.2), the system (4.3) has the form

$$\begin{bmatrix} \Delta E \\ \Delta L \end{bmatrix} = 2 \begin{bmatrix} 2\displaystyle\int_b^c \frac{\alpha + \beta\rho + \gamma\rho^2}{(P(\rho))^{1/2}}\,\mathrm{d}\rho \\ \displaystyle\int_b^c \frac{L(\delta + \lambda\rho)}{(P(\rho))^{1/2}}\,\mathrm{d}\rho \end{bmatrix}, \tag{4.4}$$

where $b = b(E, L)$ and $c = c(E, L)$ are the middle and smallest roots of the cubic $P(\rho) = \rho^3 - 2E\rho^2 + E^2\rho - L^2$ and $\alpha, \beta, \gamma, \delta, \lambda$ are functions of E alone given by the coefficients in (4.2):

$$\alpha = (\nu_1 + e_{11}E)E,$$
$$\beta = (\nu_2 - \nu_1) + (e_{12} + e_{21} - 2e_{11})E,$$
$$\gamma = (e_{11} + e_{22}) - (e_{12} + e_{21}),$$
$$\delta = (2\nu_1 + \nu_2) + (2e_{11} + e_{21})E,$$
$$\lambda = (2e_{12} + e_{22}) - (2e_{11} + e_{21}).$$

We now wish to use classic reduction procedures for elliptic integrals to find the zeros and the Jacobian derivative of the mapping $F(E, L) = (\Delta E, \Delta L)$. Because we are integrating around closed curves in (4.3), the integral of any exact differential will be zero. In particular,

$$0 = \int \mathrm{d}\left((P(\rho))^{1/2}\right) = \frac{1}{2}\int \frac{P'(\rho)}{(P(\rho))^{1/2}}\,\mathrm{d}\rho = \frac{1}{2}\int \frac{3\rho^2 - 4E\rho + E^2}{(P(\rho))^{1/2}}\,\mathrm{d}\rho$$

Thus

$$\int \frac{\rho^2}{(P(\rho))^{1/2}}\,\mathrm{d}\rho = \frac{4}{3}\int \frac{E\rho}{(P(\rho))^{1/2}}\,\mathrm{d}\rho - \frac{1}{3}\int \frac{E^2}{(P(\rho))^{1/2}}\,\mathrm{d}\rho. \tag{4.5}$$

Substituting (4.5) into (4.4) and writing $I_i = \int(\rho^i/(P(\rho))^{1/2})\,d\rho$, (4.4) becomes

$$\begin{bmatrix}\Delta E\\ \Delta L\end{bmatrix} = 2\begin{bmatrix}2(\alpha - \frac{1}{3}E^2\gamma) & 2(\beta + \frac{4}{3}E\gamma)\\ L\delta & L\lambda\end{bmatrix}\begin{bmatrix}I_0\\ I_1\end{bmatrix}. \tag{4.6}$$

Now (4.6) has non-trivial zeros if and only if the determinant of the matrix is zero and $\begin{bmatrix}I_0\\ I_1\end{bmatrix}$ lies in the kernel of the matrix. For non-zero L, this gives an equation depending only on E, namely

$$\lambda(\alpha - \tfrac{1}{3}E^2\gamma) - \delta(\beta + \tfrac{4}{3}E\gamma) = 0 \tag{4.7a}$$

and an equation for the ratio of the elliptic integrals (which do depend implicitly on L via the roots b, c of $P(\rho)$).

$$\frac{I_1}{I_0} = -\frac{\delta}{\lambda}. \tag{4.7b}$$

Since (4.7a) is independent of L, a solution of the system (4.7) will be non-singular provided that the partial derivatives of (4.7a) with respect to E and (4.7b) with respect to L are non-zero:

$$\frac{\partial}{\partial E}\left(\lambda(\alpha - \tfrac{1}{3}E^2)\gamma - \delta(\beta + \tfrac{4}{3}E\gamma)\right) \neq 0, \tag{4.8a}$$

$$\frac{\partial}{\partial L}\left(\frac{I_1}{I_0}\right) \neq 0. \tag{4.8b}$$

Once again, the classical reduction procedure of Legendre for elliptic integrals [13] can be used to re-express (4.8) in terms of I_0, I_1. We have

$$\frac{\partial}{\partial L}\left(\frac{I_1}{I_0}\right) = \left[\left(\frac{\partial I_1}{\partial L}\right)I_0 - I_1\left(\frac{\partial I_0}{\partial L}\right)\right]\Big/ I_0^2,$$

and define

$$J_i = \frac{\partial I_i}{\partial L} = \int \frac{\partial}{\partial L}\left(\frac{\rho^i}{\sqrt{P(\rho)}}\right)d\rho = \int -\rho^i(P(\rho))^{-3/2}\,d\rho.$$

Now

$$\begin{aligned} I_i &= \int \frac{\rho^i P(\rho)}{(P(\rho))^{-3/2}} = J_{i+3} - 2EJ_{i+2} + E^2J_{i+1} - L^2J_i,\\ 0 &= \int d\left(\frac{\rho^{i+1}}{(P(\rho))^{1/2}}\right) = (i+1)I_i - \tfrac{1}{2}\left[3J_{i+3} - 4EJ_{i+2} + E^2J_{i+1}\right], \end{aligned} \tag{4.9}$$

and

$$0=\tfrac{1}{2}\left[3J_2-4EJ_1+E^2J_0\right].$$

From (4.9) for $i=0,1$ together with the final equation, we obtain 5 equations for I_0, I_1, J_0, J_1, J_2, J_3, J_4 which can be used to express J_0 and J_1 as functions of I_0 and I_1. We finally obtain

$$\begin{bmatrix} J_0 \\ J_1 \end{bmatrix}=\frac{1}{L^2(4E^3-27L^2)}\begin{bmatrix} 9L^2-2E^3 & 2E^2 \\ 12EL^2-2E^4 & 2E^3-9L^2 \end{bmatrix}\begin{bmatrix} I_0 \\ I_1 \end{bmatrix}. \tag{4.10}$$

Using (4.10), we evaluate

$$\frac{\partial}{\partial L}\left(\frac{I_1}{I_0}\right)=(I_0J_1-I_1J_0)/I_0^2$$

as

$$\frac{\partial}{\partial L}\left(\frac{I_1}{I_0}\right)=\frac{1}{L^2(4E^3-27L^2)}\left[(12EL^2-2E^4)+2(2E^3-9L^2)\left(\frac{I_1}{I_0}\right)-2E^2\left(\frac{I_1}{I_0}\right)^2\right]. \tag{4.11}$$

The discriminant of the quadratic expression for (I_1/I_0) in (4.11) is $3L^2(27L^2-4E^3)$, and this is negative away from the boundary of the phase space and the ray of singular points given by $s_1=\sqrt{2}s_2$. We conclude that (4.11) can never be zero in the interior of the family of closed orbits of the unperturbed system and hence that (4.8b) is satisfied.

Next, we examine (4.8a). To begin with, evaluation of (4.7a), the condition for the matrix of (4.6) to be singular, yields

$$\begin{aligned}&\lambda\gamma\left(\alpha-\tfrac{1}{3}E^2\right)-\delta\left(\beta+\tfrac{4}{3}E\gamma\right)\\&=\tfrac{1}{3}\left(\nu_1-\nu_2+(e_{12}-e_{22})E\right)\left(6\nu_1+3\nu_2+(4e_{11}+2e_{12}+2e_{21}+e_{22})E\right).\end{aligned} \tag{4.12}$$

We denote the roots of (4.12) E_{M} and E_{T}, respectively, since they correspond to the desired modulated travelling waves and the travelling waves themselves. The vanishing of (4.8a) simultaneously with (4.7a) requires that (4.12) have a double zero; i.e., both factors of (4.12) must vanish simultaneously. Now the root of (4.12) with

$$E=E_{\mathrm{T}}=\frac{-3(2\nu_1+\nu_2)}{4e_{11}+2e_{12}+2e_{21}+e_{22}} \tag{4.13a}$$

yields a value for I_1/I_0 of

$$\frac{I_1}{I_0}=\frac{-(2\nu_1+\nu_2)}{4e_{11}+2e_{12}+2e_{21}+e_{22}}=\frac{E}{3}. \tag{4.13b}$$

This value of I_1/I_0 corresponds to the value for L with $L^2=4E^3/27$ realized at the travelling wave equilibrium solutions. On a nonsingular orbit of the unperturbed equation, we conclude that (4.8a) is satisfied. Therefore, the variation $\begin{pmatrix}\Delta E\\ \Delta L\end{pmatrix}$ is a nonsingular function of $\begin{pmatrix}E\\ L\end{pmatrix}$ on the nonsingular orbits of the unperturbed equation. Our analysis implicitly assumes that the quantity $e = 4e_{11} + 2e_{12} + 2e_{21} + e_{22}$, of the theorem, is non-zero. If this condition fails, then the vector field is highly degenerate: for example, there is a whole curve of equilibrium solutions that occur when $2\nu_1 + \nu_2 = 0$.

Since the derivative of the mapping obtained from (4.6–7) is triangular and nonsingular, the mean value theorem implies that, if $e \neq 0$, eq. (4.4) has at most one zero for each set of parameter values. This establishes that there is at most one modulated travelling wave solution in $\phi \in (0, \pi)$ for each set of parameter values.

To complete the proof we compute the bifurcation point at which the two roots E_M and E_T of (4.12) coincide:

$$E_M = \frac{\nu_2 - \nu_1}{e_{12} - e_{22}} = \frac{-3(2\nu_1 + \nu_2)}{4e_{11} + 2e_{12} + 2e_{21} + e_{22}} = E_T;$$

thus

$$\nu_2 = \nu_1\left[1 + \frac{9(e_{22} - e_{12})}{4e_{11} + 5e_{12} + 2e_{21} - 2e_{22}}\right]. \tag{4.14}$$

This coalescence of modulated travelling wave and travelling wave corresponds to a Hopf bifurcation, as an elementary calculation with (2.6) linearized at the approximate fixed point shows (cf. section 5 below). At the other limit we choose $E_M = s_1^2 + s_2^2 = 0 + ((-\nu_2/e_{22})^{1/2})^2 = -\nu_2/e_{22}$, the level set corresponding to the saddle points $s_2 = \pm(-\nu_2/e_{22})^{1/2} s_1 = 0$, to obtain

$$E_M = \frac{\nu_2 - \nu_1}{e_{12} - e_{22}} = -\frac{\nu_2}{e_{22}}$$

or

$$\nu_2 = \nu_1 e_{22}/e_{12}. \tag{4.15}$$

We remark that the two bifurcation equations coincide when $e_{12} = e_{22}$ or $2e_{11} + e_{21} = 2e_{12} + e_{22}$. □

5. Stability of heteroclinic cycles and modulated travelling waves

We start with a general result on asymptotic stability of the heteroclinic cycles discussed in section 3.

Proposition 5.1. If the hypotheses of theorem 3.2 or of lemma 3.3 are met, a heteroclinic cycle exists and if, in addition

$$\min\left\{2\mu_2, -\left(\mu_1 - \mu_2 e_{12}/e_{22} - (-\mu_2/e_{22})^{1/2}\right)\right\} > \mu_1 - \mu_2 e_{12}/e_{22} + (-\mu_2/e_{22})^{1/2}$$

then the heteroclinic cycle is locally asymptotically stable.

Proof. We consider the behavior of solutions arbitrarily close to the heteroclinic loop, so that the time taken for one circuit near the loop is dominated by the periods spent in the neighborhoods of the saddle points, where the linear terms are dominant. Straightforward estimates of the type performed in Silnikov [15] (cf. Guckenheimer and Holmes [10, chapters 6.1, 6.5]) then show that, if the positive eigenvalue of the saddle $\lambda_u = \mu_1 - \mu_2 e_{12}/e_{22} + (-\mu_2/e_{22})^{1/2}$ is smaller than the magnitude $|\lambda_s|$ of its weakest negative eigenvalue, then solutions entering a δ-neighborhood U of the saddle at a distance $d_1 < \delta$ from the stable manifold exit at distance

$$d_2 \leq \delta^{(1-\lambda_s/\lambda_u)} d_1^{|\lambda_s/\lambda_u|} [1 + \mathcal{O}(\delta)] \tag{5.1}$$

from the unstable manifold. Here $\lambda_s = \min\{2\mu_2, -(\mu_1 - \mu_2 e_{12}/e_{22} - (-\mu_2/e_{22})^{1/2})\}$ is the weakest stable eigenvalue and (5.1) is obtained by integrating the flow in "normal coordinates" near the saddle:

$$\begin{aligned} \dot{x}_u &= (\lambda_u + f_u(x)) x_u, \\ \dot{x}_s &= (\Lambda_s + f_s(x)) x_s. \end{aligned} \tag{5.2}$$

where Λ_s is a matrix of negative eigenvalues, the weakest being λ_s, x_u and x_s lie in the (local) stable and unstable manifolds, respectively, and $f_{u,s} = \mathcal{O}(|x|)$ are the nonlinear terms, which are $\mathcal{O}(\delta)$ in U.

Thus the rate of attraction to the cycle in the neighborhood of a saddle point is controlled by the ratio of the unstable and weakest stable eigenvalues. If $|\lambda_s/\lambda_u| > 1$ then, for sufficiently small δ, $d_2 < d_1$ and solutions in U are attracted to the cycle.

The time taken for solutions to pass through U becomes unbounded as $d_1 \to 0$, in fact

$$t \sim \frac{1}{\lambda_u} \ln\left(\frac{\delta}{d_1}\right), \tag{5.3}$$

while the time for solutions to pass from saddle to saddle is independent of d_1, d_2. Gronwall estimates then show that, while solutions may drift away from the cycle as they pass between saddles, the attraction near the saddles dominates.

The present situation is potentially complicated by the existence of a zero eigenvalue and associated center manifold. However, the fact that this is due to the O(2) group action implies that there is no motion in the direction corresponding to this eigenvector. □

We now turn to the question of stability of the travelling and modulated travelling waves. Here it is convenient to consider the scaled equations (2.6), as in section 4. The travelling wave fixed point is given by (2.13). Expressed in terms of the scaled variables $s_j = r_j/\varepsilon$ and parameters $\nu_j = \mu_j/\varepsilon^2$, we have

$$s_2^2 = -(2\nu_1 + \nu_2)/e, \qquad s_1^2 = 2s_2^2$$

and

$$\begin{aligned} s_2 \cos\phi &= -\varepsilon\left(\nu_1 + e_{11}s_1^2 + e_{12}s_2^2\right), \\ \cos\phi &= \varepsilon \frac{\nu_2(2e_{11} + e_{12}) - \nu_1(2e_{21} + e_{22})}{\left(-(2\nu_1 + \nu_2)e\right)^{1/2}}, \end{aligned} \tag{5.4}$$

where $e = 4e_{11} + 2e_{12} + 2e_{21} + e_{22}$ (assumed $\neq 0$) is the quantity occurring in theorem 4.1. We conclude that $\sin\phi = 1 + \mathcal{O}(\varepsilon^2)$. Linearizing (2.6) at this fixed point, we obtain the matrix

$$M = \begin{bmatrix} 4\varepsilon e_{11}\rho & \sqrt{2}\,\varepsilon[2e_{12}\rho - P_1] & \sqrt{2}\,\rho \\ \sqrt{2}\,\varepsilon[2e_{21}\rho + 2P_1] & 2\varepsilon[e_{22}\rho - P_1] & 2\rho \\ 2\sqrt{2} & -4 & 0 \end{bmatrix} + \mathcal{O}(\varepsilon^2), \tag{5.5}$$

where $P_1 = \nu_1 + (2e_{11} + e_{12})\rho$ and $\rho = s_2^2 = -(2\nu_1 + \nu_2)/e$. The characteristic polynomial for (5.5) is, to order ε,

$$\lambda^3 - 2\varepsilon((e_{22} - e_{12})\rho - \nu_1)\lambda^2 + 12\rho\lambda - 8\varepsilon e\rho^2 = 0,$$

giving eigenvalues to order ε:

$$\lambda_1 = -\frac{2\varepsilon}{3}(2\nu_1 + \nu_2), \qquad \lambda_{2,3} = \varepsilon\left(\frac{\nu_2 - \nu_1}{3} + (e_{22} - e_{12})\rho\right) \pm 2\sqrt{3}\,\mathrm{i}. \tag{5.6}$$

Substituting in (5.6) for ρ, we conclude that the travelling wave fixed points are stable, for sufficiently small ε, if

$$2\nu_1 + \nu_2 > 0 \quad \text{and} \quad \frac{\nu_2 - \nu_1}{3} - \frac{(2\nu_1 + \nu_2)(e_{22} - e_{12})}{e} < 0. \tag{5.7}$$

Note that the second condition becomes an equality precisely when (4.14) is satisfied, when the branch of modulated travelling waves meets the branch of travelling waves. This leads us to:

Theorem 5.2. A Hopf bifurcation to modulated travelling waves occurs from the branch of travelling waves of the scaled system (2.6) at

$$\nu_2 = \nu_1\left[1 + \frac{9(e_{22} - e_{12})}{e - 3(e_{22} - e_{12})}\right] \stackrel{\text{def}}{=} \nu_1 a. \tag{5.8}$$

Moreover, if $e_{22}, e_{12} < 0$ and $e = 4e_{11} + 2e_{12} + 2e_{21} + e_{22} < 0$, the travelling waves are stable for $\nu_2 < \nu_1 a$ and unstable saddles (with complex conjugate eigenvalues with positive real part) for $\nu_2 > \nu_1 a$. If, furthermore,

$$a = 1 + \frac{9(e_{22} - e_{12})}{e - 3(e_{22} - e_{12})} < \frac{e_{22}}{e_{12}} \left(\text{resp.} > \frac{e_{22}}{e_{12}}\right), \tag{5.9}$$

then the modulated travelling waves exist for $\nu_2/\nu_1 \in (a, e_{22}/e_{12})$ (resp. $(e_{22}/e_{12}, a)$) and are stable (resp. of saddle type) for $|\nu_2 - \nu_1 a|$ sufficiently small.

Proof. The computations of eigenvalues above allow us to check all hypotheses of the Hopf theorem except the sign of the leading nonlinear term (Guckenheimer and Holmes [10, chapter 3.4]). However, perturbation calculations of section 4 obviate the need to do this; the conditions used in the proposition determine on which side of the bifurcation point (5.8) the modulated waves occur (cf. eqs. (4.14), (4.15)) and the stability conclusions can be drawn from center manifold theory applied at the bifurcation point. □

Remark. Similar results can be given if $e>0$, but in that case, since existence of travelling waves requires that $-(2\nu_1+\nu_2)/e>0$, we have $(2\nu_1+\nu_2)<0$ and both travelling waves and modulated waves are unstable.

The bifurcation and stability results of theorem 5.2 and theorem 4.1 are given in terms of the scaled coefficients $\nu_j=\mu_j/\varepsilon^2$. In terms of μ_j, recalling that the analysis is accurate only to order $\mathcal{O}(\varepsilon)$, we have the Hopf bifurcation set

$$\mu_2=\mu_1\left[1+\frac{9(e_{22}-e_{12})}{e-3(e_{22}-e_{12})}\right]+\mathcal{O}(\varepsilon), \tag{5.10}$$

and the coalescence of the modulated travelling wave with the heteroclinic orbit at

$$\mu_2=\mu_1 e_{22}/e_{12}+\mathcal{O}(\varepsilon). \tag{5.11}$$

An interesting fact emerges here. If $-2\mu_2<\mu_1-\mu_2 e_{12}/e_{22}-(-\mu_2/e_{22})^{1/2}<0$, then the eigenvalue ratio λ_s/λ_u of the proof of proposition 5.1 is controlled by the quantity $\mu_1-\mu_2 e_{12}/e_{22}$: specifically, if $\mu_1-\mu_2 e_{12}/e_{22}<0$ (resp. >0) then $\lambda_s/\lambda_u>1$ (resp. <1). Thus, for $\mu_2>\mu_1 e_{22}/e_{12}$ we have an attracting heteroclinic cycle and for $\mu_2<\mu_1 e_{22}/e_{12}$ a non-attracting cycle. The bifurcation value $\mu_2=\mu_1 e_{22}/e_{12}$ coincides with the leading term of (5.11): the stability type of the heteroclinic cycle changes when the branch of modulated waves reaches it. But there is a complication. In the small ε limit of the scaled equations the eigenvalues of the saddle points in the heteroclinic loop are

$$-\varepsilon(-\nu_2/e_{22})^{1/2}+\varepsilon^2(\nu_1-\nu_2 e_{12}/e_{22})<-2\varepsilon^2\nu_2<0<\varepsilon(-\nu_2/e_{22})^{1/2}+\varepsilon^2(\nu_1-\nu_2 e_{12}/e_{22})$$

and consequently, even when $\nu_2>\nu_1 e_{22}/e_{12}$, we do not necessarily have asymptotic stability, since the weakest stable eigenvalue is $\mathcal{O}(\varepsilon^2)$ and the unstable eigenvalue $\mathcal{O}(\varepsilon)$.

However, the special structure of our O(2) symmetric problem comes to the rescue here: the fact that the planes $\phi=0,\pi$ (and $s_1=0$) are invariant for (2.6) is crucial. This in turn implies that the set $L=0$ is invariant for the slowly varying perturbation equations (4.2). We will use this fact to show that $L(t)\to 0$ for solutions near the heteroclinic cycle.

The family of unperturbed heteroclinic cycles for (2.6) with $\varepsilon=0$ is given by solutions of (4.1) with $L=0$:

$$\rho'=2\sqrt{\rho}\,(E-\rho)$$

and may be written as

$$\rho(t)=E\tanh^2(\sqrt{E}\,t+c) \tag{5.12}$$

for solutions based at $\rho(0)=E\tanh^2(c)$.

The following lemmas show that the heteroclinic orbit is asymptotically stable in the limit $\varepsilon\to 0$.

Lemma 5.3. If $\nu_1,\nu_2>0$ and $\nu_1-\nu_2 e_{12}/e_{22}<0$ then $L(t)\to 0$ as $t\to+\infty$ for solutions of (2.6) lying in a neighborhood of the heteroclinic cycle connecting the saddles at $s_2=(-\nu_2/e_{22})^{1/2}$, $s_1=0$, $\phi=0,\pi$.

Proof. Existence of the heteroclinic cycle is guaranteed by theorem 3.2 since in the limit $\varepsilon \to 0$ eq. (2.6) has no mixed modes for $\nu_1, \nu_2 < 0$. We set $E = -\nu_2/e_{22} + F$, the constant corresponding to the level set on which the saddle points lie. After substitution of $\rho(t)$ from (5.12), the evolution equation (4.26) for L then becomes

$$L' = \varepsilon L\Big[2(\nu_1 - \nu_2 e_{12}/e_{22}) + ((2e_{11} + e_{21}) - (2e_{12} + e_{22}))\left(\frac{-\nu_2}{e_{22}}\right)\operatorname{sech}^2\left((-\nu_2/e_{22})^{1/2} t\right)\Big] + \mathcal{O}(LF),$$

which implies that the linearized equation satisfies

$$\int_{L(t_0)}^{L(t)} \frac{\mathrm{d}L}{L} = \varepsilon\int_{t_0}^{t}(A + B\operatorname{sech}^2\sqrt{C}\,t)\,\mathrm{d}t = \varepsilon\left[A(t-t_0) + B\sqrt{C}\left(\tanh\sqrt{C}\,t - \tanh\sqrt{C}\,t_0\right)\right]$$

or

$$L(t) = L(0)\exp(\varepsilon[A(t-t_0) + D(t,t_0)]). \tag{5.13}$$

Here $A = 2(\nu_1 - \nu_2 e_{12}/e_{22})$, $B = (2e_{11} + e_{21}) - (2e_{12} + e_{22})$, $C = -\nu_2/e_{22}$ and since $D(t, t_0) = B\sqrt{C}(\tanh\sqrt{C}\,t - \tanh\sqrt{C}\,t_0)$ is uniformly bounded by $2B\sqrt{C}$ we conclude that, provided $A < 0$, $L(t) \to 0$ as $t \to \infty$. □

Lemma 5.4. If $\nu_2 > 0$ then $E(t) \to \nu_2/e_{22}$ as $t \to +\infty$ for solutions of (2.6) lying in a neighborhood of the heteroclinic cycle.

Proof. The computation proceeds much as that above. Linearizing the evolution equation (4.2a) for E about the unperturbed heteroclinic orbit given by (5.12), letting $F = \nu_2/e_{22} + E$, we obtain

$$F' = 2\varepsilon F\Big\{-\nu_2 + \left[\nu_1 + \nu_2 - \frac{2\nu_2}{e_{22}}(e_{12} + e_{21} - e_{22})\right]\operatorname{sech}^2\left((-\nu_2/e_{22})^{1/2} t\right) - \frac{2\nu_2}{e_{22}}[(e_{11} + e_{22}) - (e_{12} + e_{21})]\operatorname{sech}^4\left((-\nu_2/e_{22})^{1/2} t\right)\Big\} + \mathcal{O}(F^2),$$

which, as above, implies that, for the linear system

$$F(t) = F(0)\exp\left(2\varepsilon[-\nu_2(t-t_0) + D(t,t_0)]\right), \tag{5.14}$$

where $D(t, t_0)$ is uniformly bounded, since it contains terms of the form const. × $\tanh(-\nu_2/e_{22})^{1/2}t)$ and const. × $\tanh^3(-\nu_2/e_{22})^{1/2}t)$. We conclude that $F(t) \to 0$ as $t \to +\infty$ for $F(0)$ sufficiently small, provided $\nu_2 > 0$. This in turn implies that $E(t) \to -\nu_2/e_{22}$ and thus that nearby solutions approach the unperturbed heteroclinic cycle. □

Remark. The time spent by solutions in the neighborhood of the saddle points at $s_2 = (-\nu_2/e_{22})^{1/2}$, $s_1 = 0$, $\phi = 0, \pi$ dominates the behavior of E and L in these computations. The same feature leads to logarithmic singularities in the elliptic function computations of section 4.

These two lemmas imply that theorem 5.2 can be extended to the scaled system for ε small:

Theorem 5.5. If a heteroclinic cycle exists for the scaled equations (2.6) then for $\varepsilon \neq 0$ sufficiently small and $\nu_2 > 0$, $\nu_1 - \nu_2 e_{12}/e_{22} < 0$, the cycle is locally asymptotically stable. If, moreover

$$a = 1 + \frac{9(e_{22} - e_{12})}{e - 3(e_{22} - e_{12})} < \frac{e_{22}}{e_{12}} \left(\text{resp.} > \frac{e_{22}}{e_{12}}\right)$$

then the modulated travelling waves which exist in a neighborhood of the homoclinic cycle for $\nu_2/\nu_1 < e_{22}/e_{12}$ (resp. $> e_{22}/e_{12}$) and $|\nu_1 - \nu_2 e_{12}/e_{22}|$ small are asymptotically stable (resp. of saddle type).

Proof. The first assertion follows directly from lemmas 5.3 and 5.4. To establish the second assertion, observe that the perturbation calculations of section 4 show that the modulated travelling wave is the only limit set of (2.6) which lies in a neighborhood of the heteroclinic orbit (when $|\nu_1 - \nu_2 e_{12}/e_{22}|$ is small) and that it exists only under the conditions specified. Consider a Poincaré return map defined on a cross section Σ in (E, L) space near the cycle $E = \nu_2/e_{22}$, $L = 0$, with domain $L \in [0, \delta)$, $E \in (-\nu_2/e_{22} - \delta, -\nu_2/e_{22} + \delta)$. (Although solutions on $L = 0$ flow into the saddle point and hence do not return, we shall think of the heteroclinic cycle as a fixed point of the map 0. The computations of (5.13), (5.14) show that the heteroclinic fixed point in Σ is a sink for $\nu_2 > 0$, $\nu_1 - \nu_2 e_{12}/e_{22} < 0$ and a saddle for $\nu_2 > 0$, $\nu_1 - e_{12}/e_{22} > 0$. It then follows from index theory arguments that the additional modulated travelling wave fixed point in Σ has the stability properties claimed. □

We note that the local stability results for the modulated travelling wave of theorem 5.2 and 5.5 extend to the entire branch of waves, which are therefore stable if $a < e_{22}/e_{12}$ and unstable if $a > e_{22}/e_{12}$. This follows from the fact that the Jacobian matrix of the linearization of the slow evolution equations (4.2) for the quantities E and L has triangular form and is nonsingular, as was pointed out in section 4.

6. Examples

To illustrate the theory developed above, we will discuss two examples. In each case we fix the coefficients e_{ij} of the cubic terms and present a bifurcation set in (μ_1, μ_2) space along with a bifurcation diagram showing how nontrivial solutions of various types branch from the trivial solution and from each other, as parameter values traverse a closed path around the origin in (μ_1, μ_2) space. In the first case we also show some phase portraits to illustrate the standing and travelling waves, heteroclinic orbits, and modulated travelling waves. We start with the '$-$' case.

Example 1. "$-$" case: $e_{11} = -4$, $e_{12} = -1$, $e_{21} = -2$, $e_{22} = -2$. Here $e = 4e_{11} + 2(e_{12} + e_{21}) + e_{22} = -24 < 0$, $e_{22}/e_{12} = 2$ and $a = 1 + 9(e_{22} - e_{12})/(e - 3(e_{22} - e_{12})) = 10/7$, so that $a < e_{22}/e_{12}$. Thus, by theorems 5.2 and 5.5, the branch of modulated travelling wave is stable. Fig. 3 shows the bifurcation set, the bifurcation of mixed from pure modes, of travelling waves from mixed modes and the Hopf and homoclinic bifurcations to modulated waves being computed from eqs. (2.12), (2.14), (5.10) and (5.11), respectively (thus the Hopf bifurcation curve for modulated travelling waves is accurate only in the scaling limit $\varepsilon \to 0$ (eq. (2.6)). Note that the other Hopf bifurcation curve, for standing waves from the mixed mode fixed point (eq. (2.11)), is tangent at $\mu_1 = \mu_2 = 0$ to the curve on which the mixed modes themselves

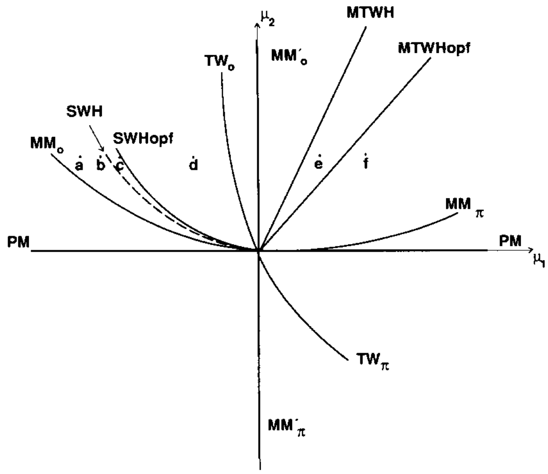

Fig. 3. Bifurcation set for eq. (1.1), '−' case: $e_{11} = -4$, $e_{12} = -1$, $e_{21} = -2$, $e_{22} = -2$. PM: bifurcation of pure mode from trivial solution; MM_0, MM_π: bifurcation of mixed modes from pure modes with $\phi = 0$ (resp. π); SWH, SWHopf: heteroclinic and Hopf bifurcation to standing waves; TW_0, TW_π bifurcation to travelling waves from mixed mode on $\phi = 0$ (resp. $\phi = \pi$); MTWH, MTWHopf: heteroclinic and Hopf bifurcation to modulated waves from travelling waves; MM_0', MM_π': bifurcation to mixed mode from trivial solution on $\phi = 0$ (resp. $\phi = \pi$)

bifurcate from the pure mode (cf. the discussion in section 2). There is a third curve tangent to these two in which the standing waves vanish in a heteroclinic loop bifurcation (cf. Guckenheimer and Holmes [10, chapter 7.4]), which we have not attempted to compute asymptotically. This curve is shown in dashes in fig. 3.

The corresponding bifurcation diagram is shown in fig. 4 and fig. 5 displays a sequence of phase portraits and two-dimensional projections of solutions for the example. The way in which the (globally) attracting heteroclinic cycles of theorem 3.2 are "released" by the heteroclinic loop bifurcation to the standing wave is made clear in figs. 5b, c, d. For these particular parameter waves the latter is unstable, and a stable mixed mode coexists with the attracting heteroclinic cycle. The parameter values corresponding to these phase points are indicated on fig. 3. Note the characteristic Lissajous figure for 1 : 2 resonance of the (x_1, x_2) projection in fig. 5f: this feature is still present in the modulated wave of fig. 5e.

We have fixed specific parameter values e_{ij} for the system illustrated here. While similar pictures are obtained for any $e_{ij} < 0$, the relative position and sequence of Hopf and heteroclinic bifurcation to standing waves and modulated travelling waves can change (cf. theorems 5.2, 5.5).

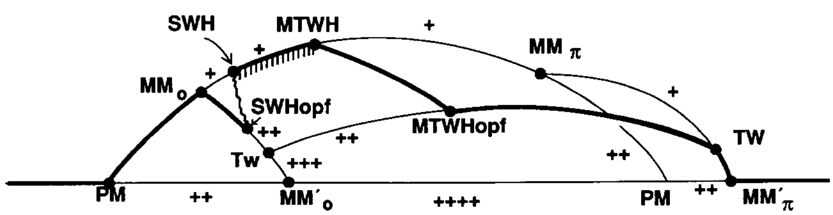

Fig. 4. Bifurcation diagram for eq. (1.1) '−' case: (μ_1, μ_2) follows a closed path around $(\mu_1, \mu_2) = (0,0)$). See fig. 3 for key. Stable solutions indicated by heavy lines, 777777 denotes attracting heteroclinic cycles and '+' denotes an eigenvalue with positive real part.

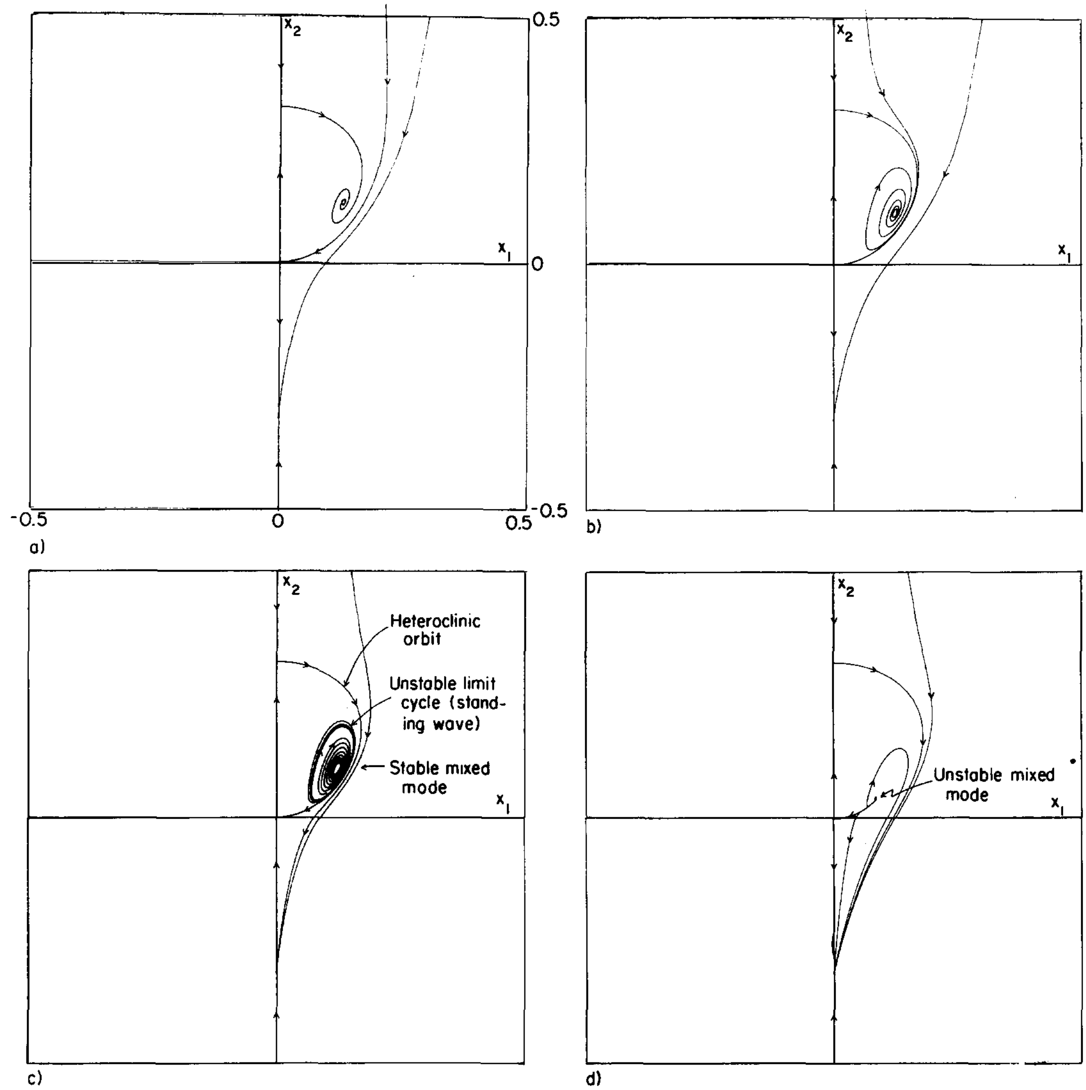

Fig. 5. Phase portraits corresponding to points a–f of fig. 3, $\mu_2 = 0.2$ all cases. (a) $\mu_1 = -0.04$; (b) $\mu_1 = -0.033$; (c) $\mu_1 = -0.030$; (d) $\mu_1 = -0.01$; (e) $\mu_1 = 0.135$; (f) $\mu_1 = 0.2$. Note unstable standing wave coexists with (stable) heteroclinic cycle in (c) and note modulated travelling wave in (e). The six panels in (e) and (f) show projections of the solutions into various planes.

Example 2. "+" case: $e_{ij} < 0$, $\forall\, ij$. In the "+" case, by proposition 3.1 there exist no travelling or modulated travelling waves and we have only to deal with pure and mixed modes and (possibly) standing waves. The bifurcation set corresponding to the choice of all cubic coefficients negative irrespective of magnitude, is shown in fig. 6. In this case the local mixed mode is always of saddle type and thus Hopf bifurcations to standing waves cannot occur. However, the existence of a *non-local* mixed mode is important, since, with all $e_{ij} < 0$, the flow is directed inwards for large $E = |z_1|^2 + |z_2|^2$ and there is always a compact attracting set. At the same time, there are parameter values (μ_1, μ_2) for which the fixed points near the origin are all unstable. This apparent paradox is resolved by the existence of a stable, non-local mixed mode, indicated in the bifurcation diagram of fig. 7.

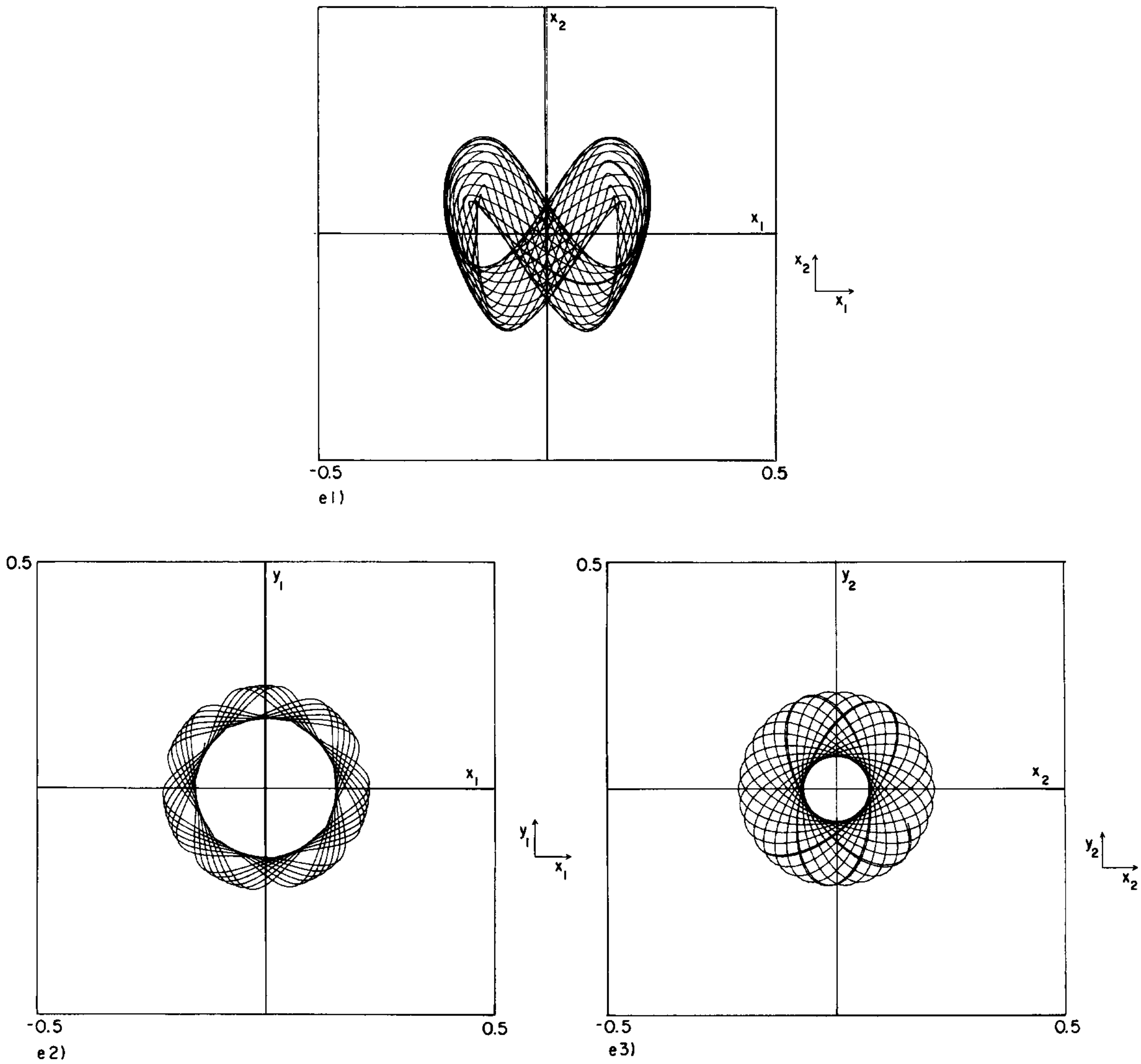

Fig. 5. Continued

In the "+" case, the precise values of the cubic coefficients e_{ij} play a less important role than in the "−" case. In view of proposition 3.1, we can restrict our attention to a study of the reduced system (2.4) with $\phi = 0$ or $\phi = \pi$. Collectively, this is equivalent to studying the purely real system in Cartesian coordinates:

$$\begin{aligned} \dot{x}_1 &= x_1 x_2 + x_1\left(\mu_1 + e_{11}x_1^2 + e_{12}x_2^2\right), \\ \dot{x}_2 &= x_1^2 + x_2\left(\mu_2 + e_{21} + e_{22}x_2^2\right). \end{aligned} \tag{6.1}$$

Local bifurcation analysis via center manifold theory or Liapunov–Schmidt reduction at the trivial

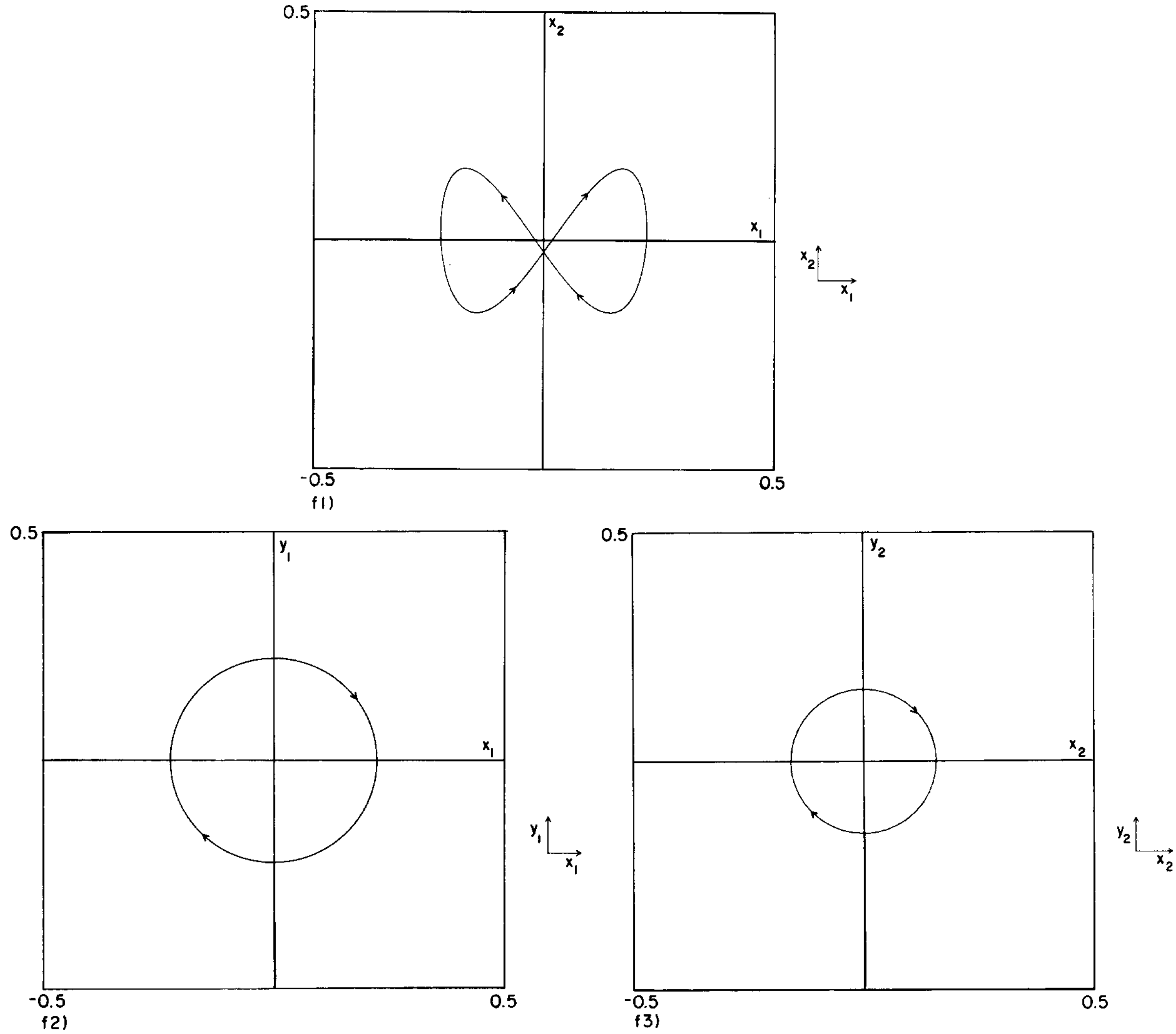

Fig. 5. Continued.

solution $x_1 = x_2 = 0$ leads to the bifurcation equations

$$\dot{\xi} = \mu_1 \xi + \left(e_{11} - \frac{1}{\mu_2} \right) \xi^3 + \mathcal{O}(|\xi|^4) \tag{6.2}$$

in the case $\mu_1 \approx 0$, and

$$\dot{\xi} = \mu_2 \xi + e_{22} \xi^3 + \mathcal{O}(|\xi|^4) \tag{6.3}$$

in the case $\mu_2 \approx 0$. (cf. Guckenheimer and Holmes [10, chapter 3]). Eq. (6.2) describes bifurcations to mixed modes from (0, 0) and eq. (6.3) describes bifurcations to pure modes from (0, 0). Similar equations

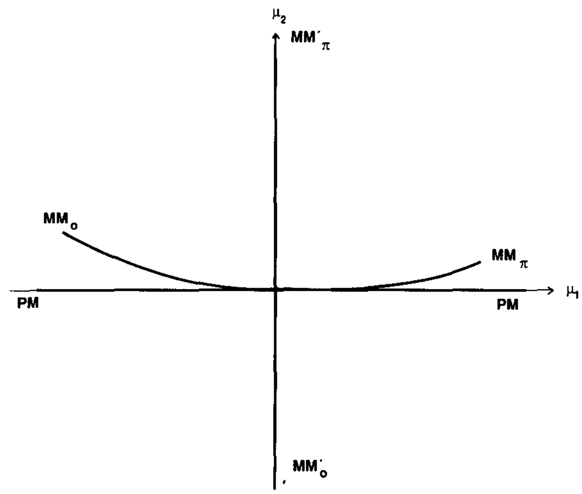

Fig. 6. Bifurcation set for eq. (1.1), "+" case, $e_{ij} < 0$ $\forall ij$. PM: bifurcation to pure modes from trivial solution; MM_0, MM_π: bifurcation of mixed modes from pure modes with $\phi = 0$ (resp. $\phi = \pi$); MM_0', MM_π': bifurcation of mixed modes from trivial solution with $\phi = 0$ (resp. $\phi = \pi$).

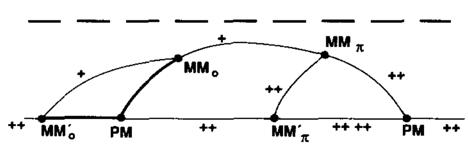

Fig. 7. Bifurcation diagram for eq. (1.1), "+" case: (μ_1, μ_2) follows a closed path around $(\mu_1, \mu_2) = (0,0)$. See fig. 6 for key. Stable solutions are indicated by heavy lines; non-local (stable) mixed mode by dashed ones; + denotes an eigenvalue with positive real part.

can be written for bifurcations to mixed modes from the pure modes $(x_1, x_2) = (0, \pm(-\mu_2/e_{22})^{1/2})$ They are

$$\dot{\xi} = \rho_\pm \xi + c_\pm \xi^3 + \mathcal{O}(|\xi|^4), \tag{6.4}$$

where

$$\rho_\pm = \mu_1 - \mu_2 e_{12}/e_{22} \pm (-\mu_2/e_{22})^{1/2}$$

and

$$c_\pm = e_{11} + \frac{1}{2\mu_2}\left(1 \pm (2e_{12} + e_{21})(-\mu_2/e_{22})^{1/2} - \frac{2\mu_2 e_{12} e_{21}}{e_{22}}\right),$$

respectively, in the cases $x_2 = (-\mu_2/e_{22})^{1/2}$ $(\phi = 0)$ and $x_2 = -(-\mu_2/e_{22})^{1/2}$ $(\phi = \pi)$.

If we are concerned only with the local behavior, so that the $|\mu_j|$ are small, then $1/\mu_2$ in (6.2) and in $c_\pm$ in (6.4) is large and the cubic coefficients in both of these equations are determined by the sign of μ_2, irrespective of the signs and relative magnitudes of e_{ij}. Of course, we require μ_2 and e_{22} to have opposite sign for the *existence* of pure modes $x_2 = \pm(\mu_2/e_{22})$). Thus, if $e_{22} < 0$, we have $\mu_2 > 0$ in (6.4) and the cubic coefficient $c_{\pm}$ is positive for small μ_2. In (6.2) the cubic coefficient is negative for $\mu_2 > 0$ and positive for $\mu_2 < 0$. Equipped with this information, the bifurcation diagram of fig. 7 can be sketched. To check the conditions for existence of local and non-local mixed mode branches, it is helpful to refer to the pair of equations analogous to (3.2) for the "+" case.

In the bifurcation diagrams, we have also indicated the stability of branches by the number of eigenvalues with positive real part (denoted by +, + +, etc.). These assignments differ from the usual ones, due to the degeneracy forced by O(2) invariance. They can be understood in the following way: every bifurcation point on the trivial branch is degenerate: i.e. two eigenvalues change sign. On the bifurcating, non-trivial branch, one of these becomes the zero eigenvalue, with eigenvector tangent to the group orbit, the other obeys the usual exchange of stability arguments. Bifurcation from non-trivial branches occurs in the usual way.

7. Conclusion

We conclude with brief remarks on the implications of our analysis of the truncated system (2.3) for the full system (1.4). We have already observed that, since the full vector field is O(2)-equivariant, the (hyperbolic) invariant tori carrying "linear" flow, found in (2.3), persist for (1.4) and that no frequency locking will occur. Perhaps more strikingly, since the heteroclinic cycles are formed from structurally stable saddle–sink connections within each two-dimensional invariant subspace $\{\phi = 2\theta_1 - \theta_2 = 0, \pi\}$, they also persist for the full problem, just as in the simpler situation considered by Guckenheimer and Holmes [8]. Thus the structure of the heteroclinic cycles and modulated traveling waves *cannot* be destroyed *unless* the O(2) symmetry is broken. The persistence of simpler structures (fixed points, periodic orbits) and codimension one local bifurcations (saddle–nodes, pitchforks, Hopfs) for the full system follows from the usual stability arguments, cf. Guckenheimer and Holmes [10].

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